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On the group association scheme of $W(E_6)$

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1 Introduction

This note is based on the author's paper [11]. So the details and the proofs are in [11].

It is a natural problem to characterize (or classify) association schemes by a given set of intersection numbers. There are many contributions to this problem for P- and Q-polynomial association schemes. (See [3, Section 9], for example.) We are interested in the following problem.

Problem 1.1 *Characterize the group association scheme $\mathcal{X}(G)$ of a given finite group G by its intersection numbers among all association schemes.*

Problem 1.1 has been solved for several groups G: the alternating group A_5 and the special linear group $SL(2,5)$ in [9], the projective special linear group $PSL(2,7)$ in [10], the symmetric group S_n of degree n for every n with $n \neq 4$ in [12] and [13]. In each cases $\mathcal{X}(G)$ is characterized by its intersection numbers. The first step to characterize $\mathcal{X}(G)$ was characterizing the local structure of $\mathcal{X}(G)$, and next step was characterizing the whole structure of $\mathcal{X}(G)$. Hence, to prove Problem 1.1 for other groups, it is important to determine the local structures.

We are particularly interested in simple groups. Because, G is simple if and only if $\mathcal{X}(G)$ is primitive, and primitive association schemes play an important role in commutative association schemes, similar to the role simple groups play in finite groups. (See [2, Section II.9], [7], or [8].) The groups A_5 and $PSL(2,7)$ are the smallest and the second smallest nonabelian simple groups. We also interested in infinite families of groups.

We focus on the local structures of the group association schemes of 3-transposition groups. The symmetric group S_n is a standard example of 3-transposition groups.

In [12] N. Yamazaki and the author assumed a certain configuration of four vertices does not exist and considered an association scheme $\mathcal{X}$ having the same intersection numbers as those of $\mathcal{X}(S_n)$. First, by using a character of S_n, they showed the local structure of $\mathcal{X}$ is a strongly regular graph with certain parameters. Next, by using the classification of such graphs, they uniquely determined the local structure of $\mathcal{X}$. (See [12, Lemma 5.4].) Finally, they uniquely determined the whole structure of $\mathcal{X}$, and hence they characterized $\mathcal{X}(S_n)$.

In this note, without using characters, we shall generalize [12, Lemma 5.4]. (See Theorems 2.1 and 2.2.) As a corollary, under the non-existence assumption of a certain configuration of four vertices, the local structures of the group association schemes of the Weyl groups $W(E_6)$, $W(E_7)$, $W(E_8)$, the symmetric group S_n, the symplectic group $Sp_n(2)$ over the field of order 2, and an orthogonal group $O^{\varepsilon}_n(2)$ over the field of order 2. (See Corollary 2.5.) We note each symplectic group $Sp_n(2)$ is a simple group.

2 Definitions and Main Theorems

A *commutative association scheme* is a pair $\mathcal{X} = (X, \mathcal{G})$ of a finite set X and the collection $\mathcal{G}$ of subsets of $X \times X$ such that

$(A1)$ $1 \in \mathcal{G}$ and $\emptyset \notin \mathcal{G}$, where $1 = \{(\alpha, \alpha) : \alpha \in X\}$.

$(A2)$ $X \times X = \bigcup_{g \in \mathcal{G}} g$ and $f \cap g = \emptyset$ for every f, $g \in \mathcal{G}$ with $g \neq h$.

$(A3)$ $g^* \in \mathcal{G}$ for every $g \in \mathcal{G}$, where $g^* = \{(\alpha, \beta) : (\beta, \alpha) \in g\}$.

$(A4)$ $|\{\gamma \in X : (\alpha, \gamma) \in g, (\gamma, \beta) \in h\}| = p^f_{gh}$ for every $f, g, h \in \mathcal{G}$ and for every $(\alpha, \beta) \in f$.

$(A5)$ $p^f_{gh} = p^f_{hg}$ for every $f, g, h \in \mathcal{G}$.

The non-negative integers $\{p^f_{gh}\}_{f,g,h \in \mathcal{G}}$ are called the *intersection numbers* of $\mathcal{X}$.

For every vertex α, for every relations f, g, $h \in \mathcal{G}$, and for every subset $\{r_1, \ldots, r_l\} \subseteq \mathcal{G}$, let

$$\begin{aligned} \alpha g &= \{\beta \in X : (\alpha, \beta) \in g\}, \\ \alpha(\bigcup_{i=1}^{l} r_i) &= \bigcup_{i=1}^{l} \alpha r_i, \quad \text{and} \\ (\bigcup_{i=1}^{l} r_i)^* &= \bigcup_{i=1}^{l} r_i^*. \end{aligned}$$

In this note we assume the following hypothesis.

Hypothesis *Let $\mathcal{X} = (X, \mathcal{G})$ be a commutative association scheme. $\mathcal{X}$ contains the relations e, f, g, r_i, s_j, t_k $(1 \leq i \leq l,\ 1 \leq j \leq m,\ 1 \leq k \leq n)$ which satisfy*

(H1) *The relations e, f, and g are symmetric, i.e., $e^* = e$, $f^* = f$, and $g^* = g$.*

(H2) *The sums $\tilde{r} = \bigcup_{i=1}^{l} r_i$, $\tilde{s} = \bigcup_{j=1}^{m} s_j$, and $\tilde{t} = \bigcup_{k=1}^{n} t_k$ of relations are symmetric, i.e., $\tilde{r}^* = \tilde{r}$, $\tilde{s}^* = \tilde{s}$, and $\tilde{t}^* = \tilde{t}$.*

The intersection numbers of $\mathcal{X}$ satisfy

(H3) *$p^f_{ee} = 2$, $p^g_{ee} \geq 2$ and $p^h_{ee} = 0$ if $h \neq 1, f, g$. In particular $p^e_{ee} = 0$.*

(H4) $p^f_{fe} \geq 0,\ p^f_{ge} = 0,$
$p^g_{fe} = 0,\ p^g_{ge} \geq 0,$
$p^{r_i}_{fe} \geq 1,\ p^{r_i}_{ge} = 0\ (1 \leq i \leq l),$
$p^{s_j}_{fe} \geq 1,\ p^{s_j}_{ge} = 1\ (1 \leq j \leq m),$
$p^{t_k}_{fe} \geq 0,\ p^{t_k}_{ge} \geq 2\ (1 \leq k \leq n),$ *and*
$p^h_{fe} = p^h_{ge} = 0$ *if* $h \neq e, f, g, r_i, s_j, t_k\ (1 \leq i \leq l,\ 1 \leq j \leq m,\ 1 \leq k \leq n).$

In the following, for every h, $h' \in \mathcal{G}$, let

$$\begin{aligned} p^h_{\tilde{r}h'} &= \sum_{i=1}^{l} p^h_{r_i h'}, \\ p^h_{\tilde{s}h'} &= \sum_{j=1}^{m} p^h_{s_j h'}, \quad \text{and} \\ p^h_{\tilde{t}h'} &= \sum_{k=1}^{n} p^h_{t_k h'}. \end{aligned}$$

We consider the graph $\Gamma = (X, e)$ with vertex set X and edge set e.

Take any quadrangle $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ in Γ. Then $(\alpha_1, \alpha_3), (\alpha_2, \alpha_4) \in f \cup g$. A quadrangle $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ is called a *skew-quadrangle* when $(\alpha_1, \alpha_3) \in f$ and $(\alpha_2, \alpha_4) \in g$.

Now consider the local structure of the graph Γ. For every vertex α, the set of the neighbors of α is αe. Take any two distinct vertices $\beta, \gamma \in \alpha e$. Then $(\beta, \gamma) \in f \cup g$. So we can construct two graphs $(\alpha e, f)$ and $(\alpha e, g)$ with vertex sets αe such that the edge set of $(\alpha e, f)$ is $f \cap (\alpha e \times \alpha e)$ and that of $(\alpha e, g)$ is $g \cap (\alpha e \times \alpha e)$.

Theorem 2.1 *Let $\mathcal{X} = (X, \mathcal{G})$ be a commutative association scheme with above hypothesis. Suppose that*

(H5) *$\mathcal{X}$ has no skew-quadrangle.*

(H6) *The following equation holds.*
$p^e_{fe} + p^f_{\tilde{t}e} = p^e_{ge} + p^f_{fe} + p^f_{\tilde{r}e} + 1.$

Then, for every vertex α, the graph $(\alpha e, f)$ is a connected, coconnected strongly regular graph with parameters $(p^1_{ee}, p^e_{fe}, p^f_{fe} + p^f_{\tilde{r}e}, p^g_{\tilde{s}e})$.

About the general theory of a strongly regular graph and related terminology, the author referred to [3, Section 1], for example.

The next theorem is a generalization of [12, Lemma 5.4].

Theorem 2.2 *Let $\mathcal{X} = (X, \mathcal{G})$ be a commutative association scheme with above hypothesis. Suppose that*

(H5) *$\mathcal{X}$ has no skew-quadrangle.*

(H6) *The following equation holds.*
$p^{e}_{fe} + p^{f}_{\tilde{t}e} = p^{e}_{ge} + p^{f}_{fe} + p^{f}_{\tilde{r}e} + 1.$

(H7) *The following two equations hold.*
$p^{g}_{ee} = 3$ *and* $p^{1}_{ee} + 2p^{g}_{\tilde{s}e} - 3p^{e}_{fe} - 3 = 0.$

For every vertex α, let $\mathcal{P} = \alpha e$ and $\mathcal{L} = \{\alpha e \cap \beta e : \beta \in \alpha g\}$. Then $(\mathcal{P}, \mathcal{L})$ is a connected, coconnected, reduced Fischer space such that the subspace generated by any pair of distinct intersecting lines is a dual affine plane of order 2.

Moreover, the collinearity graph of $(\mathcal{P}, \mathcal{L})$ is $(\alpha e, g)$.

About the general theory of a Fischer space and related terminology, the reader is referred to [1, Section 18], for example.

Let G be a finite group. A *set of 3-transpositions* of G is a set D of involutions of G such that D is the union of conjugacy classes of G, D generates G, and for all $\alpha, \beta \in D$, the order of the product $\alpha\beta$ is 1, 2, or 3. In [6] B. Fischer classified the almost simple groups generated by 3-transpositions. We state his theorem as a form of [1, p.1, Fischer's Theorem].

Theorem 2.3 (B. Fischer [6, Theorem.]) *Let D be a conjugacy class of 3-transpositions of the finite group G. Assume the center of G is trivial and the derived subgroup of G is simple. Then one of the following holds.*

(a) *$G \simeq S_n$ is the symmetric group of degree n and D is the set of transpositions of G.*

(b) *$G \simeq Sp_n(2)$ is the symplectic group of dimension n over the field of order 2 and D is the set of transvections.*

(c) *$G \simeq U_n(2)$ is the projective unitary group of dimension n over the field of order 4 and D is the set of transvections.*

(d) *$G \simeq O^{\varepsilon}_n(2)$ is an orthogonal group of dimension n over the field of order 2 and D is the set of transvections.*

(e) *$G \simeq PO^{\mu,\pi}_n(3)$ is the subgroup of an n-dimensional projective orthogonal group over the field of order 3 generated by a conjugacy class D of reflections.*

(f) *G is a Fischer group of type $M(22)$, $M(23)$, or $M(24)$, determined up to isomorphism, and D is a uniquely determined class of involutions in G.*

Let G be a finite group and $C_1 = \{\mathrm{id}\}, C_f, \ldots, C_g$ the conjugacy classes of G. Define the relation f on G by $f = \{(x, y) : yx^{-1} \in C_f\}$ and let $\mathcal{G} = \{1, f, \ldots, g\}$. Then $\mathcal{X}(G) = (G, \mathcal{G})$ is a commutative association scheme called the *group association scheme* of G. (See [2, Example II.2.1(2)].)

Consider the group association scheme $\mathcal{X}(G)$ of a 3-transposition group G in Fischer's Theorem. Let e be the relation with respect to a conjugacy class of 3-transpositions D of G. Then $\mathcal{X}(G)$ satisfies the assumption of Theorem 2.1. (See [11, Section 5].) Moreover, if we construct a partial linear space $(\mathcal{P}, \mathcal{L})$ as in Theorem 2.2, then $(\mathcal{P}, \mathcal{L})$ is a reduced, connected, coconnected Fischer space. In [4] and [5] H. Cuypers and J. Hall showed that a reduced, connected, coconnected Fischer space $(\mathcal{P}, \mathcal{L})$ is one of the Fischer spaces constructed above from the 3-transposition groups in Fischer's Theorem. (See also [1, Theorem 20.2].)

Corollary 2.4 *If an association scheme $\mathcal{X}$ has the same intersection numbers of the group association scheme $\mathcal{X}(G)$ of a 3-transposition group G in Fischer's Theorem, then the local structure of $\mathcal{X}$ is a strongly regular graph under the non-existence assumption of a skew-quadrangle. The parameters of two strongly regular graphs in the local structures of $\mathcal{X}$ and $\mathcal{X}(G)$ are the same.*

If G is one of (a), (b), or (d) in Fischer's Theorem, then $\mathcal{X}(G)$ satisfies the assumption of Theorem 2.2. (See [11, Section 5].) We also note that $\mathcal{X}(G)$ satisfies the assumption of Theorem 2.2 if G is one of the Weyl groups of type E_6, E_7, or E_8.

Corollary 2.5 *If G is one of the Weyl groups of type E_6, E_7, E_8, the symmetric groups, the symplectic groups over the field of order 2, or the orthogonal groups over the field of order 2, then the local structure of the group association scheme $\mathcal{X}(G)$ is characterized by its intersection numbers under the non-existence assumption of a skew-quadrangle.*

If G is one of (c), (e), or (f) in Fischer's Theorem, then $\mathcal{X}(G)$ does not satisfy the second equality in (H7). In fact, in the Fischer space in the local structure of $\mathcal{X}(G)$, the subspace generated by some pair of distinct intersecting lines is the affine plane of order 3.

3 Remarks

Remarks (1) When G is one of (c), (e), and (f), can we characterize the local structure $(\mathcal{P}, \mathcal{L})$ of $\mathcal{X}(G)$? More generally, by changing the condition (H7), can we prove the similar theorem which can apply to all 3-transposition groups in the Fischer's Theorem?
(2) Can we generalize the characterization for S_n to the characterization of other 3-transposition groups? When G is one of the Weyl groups $W(E_6)$, $W(E_7)$, and $W(E_8)$, we can characterize the local structure of $\mathcal{X}(S_n)$ by Corollary 2.5. Moreover, G has a similar character to the character of S_n which is useful to characterize $\mathcal{X}(S_n)$. So there is a good chance to characterize $\mathcal{X}(G)$.

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