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General Form of Non-Symmetric Spin Models

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Abstract. A spin model (for link invariants) is a square matrix W with non-zero complex entries which satisfies certain axioms. Recently [6] it was shown that ${}^tWW^{-1}$ is a permutation matrix (the order of this permutation matrix is called the "index" of W), and a general form was given for spin models of index 2. In the present paper, we generalize this general form to an arbitrary index m. In particular, we give a simple form of W when m is a prime number.

1 Introduction

Spin models were introduced by Vaughan Jones [7] to construct invariants of knots and links. A spin model is essentially a square matrix W with nonzero entries which satisfies two conditions (type II and type III conditions). In his definition of a spin model, Jones considered only symmetric matrices. It was generalized to non-symmetric case by Kawagoe-Munemasa-Watatani [8].

Recently, François Jaeger and the second author [6] introduced the notion of "index" of a spin model. For every spin model W, the transpose ${}^tW$ is obtained from W by a permutation of rows. Let σ denote the corresponding permutation of $X = \{1, \ldots, n\}$ (n is the size of W). Then the index m is the order of σ. In [6], it was shown that X is partitioned into m subsets $X_0, X_1, \ldots, X_{m-1}$ such that $W(x,y) = \eta^{i-j}W(y,x)$ holds for all $x \in X_i$, $y \in X_j$. Moreover, the case of $m = 2$ was deeply investigated, and a general form of spin models of index 2 was given.

In the present paper, we investigate the structure of spin models of an arbitrary index m. In Section 4, we show that W is decomposed into blocks W_{ij}, and W_{ij} splits into Kronecker product of two matrices S_{ij} and T_{ij} (Proposition 4.3). In Section 5, we give conditions on T_{ij} (Propositions 5.1 and 5.5). In Section 6, we apply this general form to some special cases (Propositions 6.1 and 6.2). In particular, we give a simple form of W when the index m is a prime number (Corollary 6.3).

2 Preliminaries

In this section, we give some basic materials concerning spin models and association schemes. For more details the reader can refer to [3, 7, 5, 6].

Let X be a finite non-empty set with n elements. We denote by $\mathrm{Mat}_X(\mathrm{C})$ the set of square matrices with complex entries whose rows and columns are indexed by X. For $W \in \mathrm{Mat}_X(\mathrm{C})$ and $x, y \in X$, the (x,y)-entry of W is denoted by $W(x,y)$.

A *type II matrix* on X is a matrix $W \in \mathrm{Mat}_X(\mathbf{C})$ with nonzero entries which satisfies the *type II condition*:

$$\sum_{x \in X} \frac{W(a,x)}{W(b,x)} = n\delta_{a,b} \qquad \text{(for all } a,\ b \in X\text{)}.$$

Let $W^- \in \mathrm{Mat}_X(\mathbf{C})$ be defined by $W^-(x,y) = W(y,x)^{-1}$. Then type II condition is written as $WW^- = nI$ (I denotes the identity matrix). Hence, if W is a type II matrix, then W is non-singular with $W^{-1} = n^{-1}W^-$. It is clear that W^{-1} and ${}^tW$ are also type II matrices.

A type II matrix W is called a *spin model* on X if W satisfies *type III condition*:

$$\sum_{x \in X} \frac{W(a,x)W(b,x)}{W(c,x)} = D\frac{W(a,b)}{W(a,c)W(c,b)} \qquad \text{(for all } a,\ b,\ c \in X\text{)} \tag{1}$$

for some nonzero complex number D. The number D is called the *loop variable* of W. Setting $b = c$ in (1), $\sum_{x \in X} W(a,x) = DW(b,b)^{-1}$ holds, so that the diagonal entries $W(b,b)$ is a constant, which is called the *modulus* of W.

For a spin model W with loop variable D, any nonzero scalar multiple λW is a spin model with loop variable $\lambda^2 D$. Usually W is normalized so that $D^2 = n$, but we allow any nonzero value of D in this paper to simplify our arguments.

Observe that, for any spin models W_i on X_i with loop variable D_i ($i = 1, 2$), their tensor (Kronecker) product $W_1 \otimes W_2$ is a spin model with loop variable $D = D_1 D_2$. Conversely, it is not difficult to show that, if $W_1 \otimes W_2$ and W_1 are spin models, then W_2 must be a spin model.

A *(class d) association scheme* on X is a partition of $X \times X$ with nonempty relations $R_0, R_1, \ldots, R_d$, where $R_0 = \{(x,x) \mid x \in X\}$ which satisfy the following conditions:

(i) For every i in $\{0, 1, \ldots, d\}$, there exists i' in $\{0, 1, \ldots, d\}$ such that $R_{i'} = \{(y,x) \mid (x,y) \in R_i\}$.

(ii) There exist integers p_{ij}^k ($i,\ j,\ k \in \{0, 1, \ldots, d\}$) such that for every $(x,y) \in R_k$, there are precisely p_{ij}^k elements z such that $(x,z) \in R_i$ and $(z,y) \in R_j$.

(iii) $p_{ij}^k = p_{ji}^k$ for every $i,\ j$ in $\{0, 1, \ldots, d\}$.

Let A_i denote the adjacency matrix of the relation R_i, so $A_i \in \mathrm{Mat}_X(\mathbf{C})$ is a $\{0,1\}$-matrix whose (x,y)-entry is equal to 1 if and only if $(x,y) \in R_i$. Clearly $A_0 = I$, $A_i \circ A_j = \delta_{i,j}A_i$ (entry-wise product), $\sum_{i=0}^d A_i = J$ (all 1's matrix), and $A_iA_j = \sum_{k=0}^d p_{ij}^k A_k$ hold. The linear span $\mathcal{A}$ of $\{A_0, A_1, \ldots, A_d\}$ becomes a subalgebra of $\mathrm{Mat}_X(\mathbf{C})$, called the *Bose-Mesner algebra* of the association scheme. Observe that $\mathcal{A}$ is closed under entry-wise product, $\mathcal{A}$ is closed under transposition $A \mapsto {}^tA$, and $\mathcal{A}$ contains I, J.

3 Associated Permutation

Let W be a spin model on X. Then there exists an association scheme $R_0, \ldots, R_d$ on X such that the corresponding Bose-Mesner algebra $\mathcal{A}$ contains W ([5] Theorem 11). In [6], it was shown that ${}^tWW^{-1} = A_s$ (the adjacency matrix of R_s) for some $s \in \{0, 1, \ldots, d\}$, and moreover A_s is a permutation matrix ([6] Proposition 2). Let σ denote the corresponding permutation on X, so that $A_s(x, y) = 1$ if $y = \sigma(x)$ and $A_s(x, y) = 0$ otherwise. The order m of σ is called the *index* of W.

Observe that $m = 1$ if and only if W is symmetric. Also observe that, for two spin models W_i of index m_i $(i = 1, 2)$, the index of $W_1 \otimes W_2$ is equal to the least common multiple of m_1 and m_2. In particular, tensor product of a spin model of index m with any symmetric spin model has index m.

Lemma 3. 1 *(i)* $W(x, \sigma(x)) = W(y, \sigma(y))$ $(x,\ y \in X)$.

(ii) $W(y, x) = W(\sigma(x), y)$ $(x,\ y \in X)$.

(iii) Every orbit of σ *has length* m.

Lemma 3. 2 *There is a partition* $X = X_0 \cup \cdots \cup X_{m-1}$ *such that (for all* $i,\ j \in \{0, \ldots, m-1\}$*)*

$$W(x, y) = \eta^{i-j} W(y, x) \quad (\text{for all } x \in X_i,\ y \in X_j),$$

where η *denotes a primitive* m*-root of unity. Moreover, for every* i, $\sigma(X_i) = X_j$ *holds for some* j.

We fix a primitive m-root of unity η, and let $X_0, \ldots, X_{m-1}$ be the partition of X given in Lemma 3. 2 . We identify the index set $\{0, 1, \ldots, m-1\}$ with $\mathbf{Z}_m = \mathbf{Z}/m\,\mathbf{Z}$. By Lemma 3. 2 , there is a permutation π on $\mathbf{Z}_m$ such that $\sigma(X_i) = X_{\pi(i)}$ $(i \in \mathbf{Z}_m)$. Let t denote the order of π, and set $k = m/t$.

Lemma 3. 3 $\pi(i) - i = \pi(j) - j$ *for all* $i,\ j \in \mathbf{Z}_m$.

Lemma 3. 4 *There exists an automorphism* φ *of the additive group* $\mathbf{Z}_m$ *such that* $\pi(\varphi(i)) = \varphi(i + k)$ *for all* $i \in \mathbf{Z}_m$. *Moreover,* $W(x, y) = (\eta^{\varphi(1)})^{i-j} W(y, x)$ *for every* $x \in X_{\varphi(i)}$, $y \in X_{\varphi(j)}$.

Thus, by reordering the indices $\{0, 1, \ldots, m-1\}$ by φ, and by replacing η with $\eta^{\varphi(1)}$, we may assume that

$$\pi(i) = i + k \qquad (i \in \mathbf{Z}_m).$$

4 General Form of W

We use the notation of the previous section. We also use the notation:

$$\gamma_k(\ell, i) = \eta^{-\ell i - (k/2)\ell(\ell-1)}. \tag{2}$$

Proposition 4. 1 *Let* $i,\ j \in \mathbf{Z}_m$ *and* $x \in X_i$, $y \in X_j$. *Then for* $\ell,\ \ell' \in \mathbf{Z}$,

$$W(\sigma^{\ell}(x), \sigma^{\ell'}(y)) = \gamma_k(\ell - \ell',\ i - j)\, W(x, y). \tag{3}$$

Lemma 4. 2 *If* m *is even, then* k *is even.*

For $i \in \mathbf{Z}_m$, set

$$\Delta_i = \bigcup_{h=0}^{t-1} X_{i+hk}.$$

Observe that $|\Delta_i| = t(n/m) = tn/(kt) = n/k$, and that

$$X = \bigcup_{i=0}^{k-1} \Delta_i,$$

Since $\sigma(\Delta_i) = \Delta_i$, Δ_i is partitioned into σ-orbits Y_α^i:

$$\Delta_i = \bigcup_{\alpha=1}^{r} Y_\alpha^i \qquad (i = 0, \ldots, k-1),$$

where $r = |\Delta_i|/m = n/(mk)$. Observe that $|Y_\alpha^i| = m$ and $|Y_\alpha^i \cap X_i| = k$. We choose representative elements

$$y_\alpha^i \in Y_\alpha^i \cap X_i \qquad (i = 0, \ldots, k-1,\ \alpha = 1, \ldots, r).$$

Then

$$X = \{\sigma^\ell(y_\alpha^i) \mid i = 0, \ldots, k-1,\ \alpha = 1, \ldots, r,\ \ell = 0, \ldots, m-1\},$$

and

$$W(\sigma^\ell(y_\alpha^i), \sigma^{\ell'}(y_\beta^j)) = \gamma_k(\ell - \ell', i - j)\, W(y_\alpha^i, y_\beta^j)$$

for ℓ, $\ell' \in \mathbf{Z}_m$, i, $j = 0, \ldots, k-1$ and α, $\beta = 1, \ldots, r$.

We define square matrices T_{ij} of size r and S_{ij} of size m $(i, j = 0, \ldots, k-1)$ by

$$T_{ij}(\alpha, \beta) = W(y_\alpha^i, y_\beta^j) \qquad (\alpha, \beta = 1, \ldots, r),$$

$$S_{ij}(\ell, \ell') = \gamma_k(\ell - \ell', i - j) \qquad (\ell, \ell' = 0, \ldots, m-1).$$

For subsets A, B of X, let $W|_{A\times B}$ denote the restriction (submatrix) of W on $A \times B$. For two matrices S, T, we denote the Kronecker product by $S \otimes T$.

Proposition 4. 3 *For* i, $j = 0, \ldots k-1$,

$$W|_{Y_\alpha^i \times Y_\beta^j} = T_{ij}(\alpha, \beta)\, S_{ij} \qquad (\alpha, \beta = 1, \ldots, r),$$

and

$$W|_{\Delta_i \times \Delta_j} = S_{ij} \otimes T_{ij}.$$

Thus W decomposes into blocks $W_{ij} = W|_{\Delta_i \times \Delta_j}$ $(i, j = 0, \ldots, k-1)$, and each block has the form $W_{ij} = S_{ij} \otimes T_{ij}$ $(i, j = 0, \ldots, k-1)$.

5 Type II and Type III conditions

Let m, k, t, r be positive integers with $m = kt$.

Let T_{ij} $(i, j = 0, \dots, k-1)$ be any matrices of size r with nonzero entries, and let S_{ij} $(i, j = 0, \dots, k-1)$ be the matrix of size m defined by

$$S_{ij}(\ell, \ell') = \gamma_k(\ell - \ell', i - j) \qquad (\ell, \ell' = 0, \dots, m-1),$$

where γ_k is defined by (2) for a primitive m-root of unity η. Now set

$$W_{ij} = S_{ij} \otimes T_{ij} \qquad (i, j = 0, \dots, k-1),$$

and let W be the matrix of size $n = kmr$ whose (i, j) block is W_{ij} $(i, j = 0, \dots, k-1)$. We index the rows and the columns of W by the set:

$$X = \{[i, \ell, \alpha] \mid 0 \le i \le k-1,\ 0 \le \ell \le m-1,\ 1 \le \alpha \le r\},$$

so that

$$W([i, \ell, \alpha],\ [j, \ell', \beta]) = S_{ij}(\ell, \ell') T_{ij}(\alpha, \beta).$$

Proposition 5. 1 *W is a type II matrix if and only if T_{ij} is a type II matrix for all i, $j \in \{0, \dots, k-1\}$.*

Lemma 5. 2 *Assume k is even when m is even. Then the matrix W satisfies the type III condition* (1) *if and only if the following equation holds for all i_1, i_2, $i_3 \in \{0, \dots, k-1\}$ and for all α_1, α_2, $\alpha_3 \in \{1, \dots, r\}$:*

$$\sum_{i=0}^{k-1} \left(\sum_{\ell=0}^{m-1} \eta^{-k\ell} \gamma_k(\ell, i - i_1 - i_2 + i_3) \right) \left(\sum_{\alpha=1}^{r} \frac{T_{i_1,i}(\alpha_1, \alpha) T_{i_2,i}(\alpha_2, \alpha)}{T_{i_3,i}(\alpha_3, \alpha)} \right) = D \frac{T_{i_1,i_2}(\alpha_1, \alpha_2)}{T_{i_1,i_3}(\alpha_1, \alpha_3) T_{i_3,i_2}(\alpha_3, \alpha_2)}.$$

Lemma 5. 3 *For all u, s $(0 \le u \le t-1,\ 0 \le s \le k-1)$,*

$$\gamma_k(u + st, j) = ((-1)^{t-1} \eta^{-tj})^s \gamma_k(u, j).$$

Lemma 5. 4 *(i) If t is odd, then*

$$\sum_{\ell=0}^{m-1} \eta^{-k\ell} \gamma_k(\ell, j) = \begin{cases} k \sum_{u=0}^{t-1} \eta^{-uj - ku(u+1)/2} & \text{if } j \equiv 0 \pmod{k}, \\ 0 & \text{otherwise.} \end{cases}$$

(ii) If t and k are even, then

$$\sum_{\ell=0}^{m-1} \eta^{-k\ell}\gamma_k(\ell,j) = \begin{cases} k\sum_{u=0}^{t-1}\eta^{-uj-ku(u+1)/2} & \textit{if } j \equiv \frac{k}{2} \pmod{k}, \\ 0 & \textit{otherwise.} \end{cases}$$

Proposition 5. 5 *Assume k is even when m is even. Then the matrix W satisfies the type III condition (1) if and only if the following equation holds for all i_1, i_2, $i_3 \in \{0,\ldots,k-1\}$ and for all α_1, α_2, $\alpha_3 \in \{1,\ldots,r\}$:*

$$\left(\sum_{u=0}^{t-1}\eta^{-u(i-\hat{\imath})-ku(u+1)/2}\right)\left(\sum_{\alpha=1}^{r}\frac{T_{i_1,i}(\alpha_1,\alpha)T_{i_2,i}(\alpha_2,\alpha)}{T_{i_3,i}(\alpha_3,\alpha)}\right) = (D/k)\frac{T_{i_1,i_2}(\alpha_1,\alpha_2)}{T_{i_1,i_3}(\alpha_1,\alpha_3)T_{i_3,i_2}(\alpha_3,\alpha_2)},$$

where $\hat{\imath} = i_1 + i_2 - i_3$, and i denotes the integer in $\{0,\ldots,k-1\}$ such that

$$i \equiv \begin{cases} \hat{\imath} \pmod{k} & \textit{if t is odd,} \\ \hat{\imath} + \frac{k}{2} \pmod{k} & \textit{if t is even.} \end{cases}$$

6 Some Special Cases

We use the notation in Section 4.

Proposition 6. 1 *Suppose $k = 1$. Then m is odd, and*

$$W = S \otimes T,$$

where S is a spin model of size m and index m which is given by

$$S(\ell,\ell') = \eta^{-(1/2)(\ell-\ell')(\ell-\ell'-1)} \qquad (\ell,\ell' = 0,1,\ldots,m-1),$$

and T is a symmetric spin model of size n/m.

Proposition 6. 2 *Suppose $k = m$. Then*

$$W|_{X_i \times X_j} = S_{ij} \otimes T_{ij} \qquad (i,j = 0,1,\ldots,m-1),$$

and

$$S_{ij}(\ell,\ell') = \eta^{-(\ell-\ell')(i-j)} \qquad (\ell,\ \ell' = 0,\ldots,m-1).$$

The matrices T_{ij} are type II matrices of size $r = n/m^2$. Moreover the following equation holds for all i_1, i_2, $i_3 \in \{0,\ldots,m-1\}$ and for all α_1, α_2, $\alpha_3 \in \{1,\ldots,r\}$:

$$\sum_{\alpha=1}^{r}\frac{T_{i_1,i}(\alpha_1,\alpha)T_{i_2,i}(\alpha_2,\alpha)}{T_{i_3,i}(\alpha_3,\alpha)} = (D/m)\frac{T_{i_1,i_2}(\alpha_1,\alpha_2)}{T_{i_1,i_3}(\alpha_1,\alpha_3)T_{i_3,i_2}(\alpha_3,\alpha_2)},$$

where i denotes the integer in $\{0,\ldots,m-1\}$ such that $i \equiv i_1 + i_2 - i_3 \pmod{m}$.

Corollary 6. 3 *Let W be a spin model on X of prime index m. Then one of the following holds, where η denotes a primitive m-root of unity.*

(i) $W = S \otimes T$, *where S is a spin model of size m with*

$$S(\ell, \ell') = \eta^{-(1/2)(\ell-\ell')(\ell-\ell'-1)} \qquad (\ell, \ell' = 0, 1, \ldots, m-1),$$

and T is a symmetric spin model of size $|X|/m$.

(ii) *W decomposes into m^2 blocks W_{ij} $(i, j = 0, \ldots, m-1)$ with $W_{ij} = S_{ij} \otimes T_{ij}$, where S_{ij} are matrices of size m defined by*

$$S_{ij}(\ell, \ell') = \eta^{-(\ell-\ell')(i-j)} \qquad (\ell, \ell' = 0, 1, \ldots, m-1),$$

and T_{ij} are type II matrices of size $r = n/m^2$ which satisfy the following equation for all i_1, i_2, $i_3 \in \{0, \ldots, m-1\}$ and for all α_1, α_2, $\alpha_3 \in \{1, \ldots, r\}$:

$$\sum_{\alpha=1}^{r} \frac{T_{i_1,i}(\alpha_1, \alpha) T_{i_2,i}(\alpha_2, \alpha)}{T_{i_3,i}(\alpha_3, \alpha)} = (D/m) \frac{T_{i_1,i_2}(\alpha_1, \alpha_2)}{T_{i_1,i_3}(\alpha_1, \alpha_3) T_{i_3,i_2}(\alpha_3, \alpha_2)},$$

where i denotes the integer in $\{0, \ldots, m-1\}$ such that $i \equiv i_1 + i_2 - i_3 \pmod{m}$.

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