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<td>Giga, Yoshikazu; Sato, Moto-Hiko</td>
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Semicontinuous solutions for Hamilton-Jacobi equations

with general Hamiltonians

Yoshikazu Giga

Moto-Hiko Sato

1. Introduction

We consider the initial value problem for the Hamilton-Jacobi equation of form

\[ u_t + H(x, u_x) = 0 \quad \text{in } \mathbb{R}^n \times (0, T), \]
\[ u(0, x) = u_0(x), \quad x \in \mathbb{R}^n, \]

where \( u_t = \partial u/\partial t \) and \( u_x = (\partial_{x_1} u, \cdots, \partial_{x_n} u) \), \( \partial_{x_i} u = \partial u/\partial x_i; \infty \geq T > 0 \) is a fixed number. Our main goal is to find a suitable notion of solution when \( u_0 \) is discontinuous. The theory of viscosity solutions initiated by Crandall and Lions \([CL]\) yields the global solvability of the initial value problem by extending the notion of solutions when \( u_0 \) is continuous (cf. [E, Chap.10], [L], [B]). In fact, if initial data \( u_0 \) is bounded, uniformly continuous, it is well-known \([CL], [L]\) that the initial value problem \( (1a)-(1b) \) admits a unique global (uniformly) continuous viscosity solutions when \( H \) is enough regular, for example \( H \) satisfies the Lipschitz conditions

\[ |H(x, p) - H(x, q)| \leq C|p - q| \]
\[ |H(x, p) - H(y, p)| \leq C(1 + |p|)|x - y|. \]

We only refer to [B], [L] and [CIL] for the basic theory of viscosity solutions. The notion of viscosity solution has been extended to semicontinuous functions. This

*Department of Mathematics, Hokkaido University, Sapporo 060-0810, Japan. Partly supported by Ministry of Education, Science, Sports and Culture through grant 10304010 for scientific research

**Muroran Institute of Technology, 27-1 Mizumoto, Muroran 050-8585 Japan

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is very important to prove the existence of solutions without appealing hard estimates. Such a method is first introduced by [I]. However, if \( u_0 \) is, for example, upper semicontinuous, a classical semicontinuous viscosity solution may not be unique.

Recently to overcome this inconvenience, Barron and Jensen [BJ] introduced another notion of viscosity solutions for semicontinuous functions when the Hamiltonian \( H = H(x, p) \) is concave in \( p \) and proved the existence and the uniqueness of their solution for (1a), (1b) for bounded (from above), upper semicontinuous initial data \( u_0 \). Their solution is now called a bilateral solution [BD]. For later development of the theory as well as other approaches we refer to [BD] and references cited there. However, their theory is limited for concave \( H \). (In [BJ] \( H \) is assumed to be convex but they consider the terminal value problem which is easily transformed to the initial value problem with concave Hamiltonian by setting \( T - t \) by \( t \).)

In this paper we introduce a new notion of a solution which is unique for a given initial upper semicontinuous initial data. For (1a), (1b) we consider auxiliary problem

\[
\psi_t - \psi_y H(x, -\psi_x/\psi_y) = 0 \quad \text{in} \quad \mathbb{R}^{n+1} \times (0, T),
\]

(3a)

\[
\psi(0, x, y) = \psi_0(x, y), \quad (x, y) \in \mathbb{R}^n \times \mathbb{R}.
\]

(3b)

The equation (3a) is called the level set equation for the evolution of the graph of \( u \) of (1a). In fact, if a level set of a solution \( \psi \) of (3a) is given as the graph of a function \( v = v(t, x) \), then \( v \) must solve (1a). For given upper semicontinuous initial data \( u_0 : \mathbb{R}^n \to \mathbb{R} \cup \{-\infty\} \), shortly \( u_0 \in USC(\mathbb{R}^n) \), we take

\[
\psi_0(x, y) = -\min\{\text{dist}((x, y), K_0), 1\},
\]

(4)

where

\[
K_0 = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}; \; y \leq u_0(x)\}.
\]

(5)

We solve (3a), (3b) and set

\[
\bar{u}(t, x) = \sup\{y \in \mathbb{R}; \; \psi(t, x, y) \geq 0\},
\]

(6)
where $\psi$ is the continuous viscosity solution of (3a), (3b). We call $\bar{u}$ an $L$-solution of (1a), (1b). Such a solution uniquely exists globally in time under suitable condition on $H$.

**Theorem 1.** Assume that the recession function

$$H_\infty(x, p) = \lim_{\lambda \downarrow 0} \lambda H(x, p/\lambda), \quad x \in \mathbb{R}^n, \quad p \in \mathbb{R}^n \quad (7)$$

exists and that $H$ satisfies (2a), (2b). Then there exists a global unique $L$-solution for an arbitrary $u_0 \in USC(\mathbb{R}^n)$.

One may relax the assumptions on $H$ (cf. Remark right before references) but in this paper we shall always assume (2a), (2b) and (7). These assumptions guarantee that the singularity at $\psi_y = 0$ in (3a) is removable if we restrict $\psi$ satisfying $\psi_y \leq 0$. Moreover, (3a), (3b) admits a unique global solution for any bounded, uniformly continuous initial data $\psi_0 = \psi_0(x, y)$ which is nonincreasing in $y$. (The monotonicity of the solution $\psi$ in $y$ is preserved for $t > 0$.)

**2. Comparison and uniqueness**

Since a solution of (3a), (3b) enjoys a comparison principle, so does an $L$-solution (1a), (1b).

**Theorem 2 (Comparison).** Let $u$ and $v$ be the $L$-solution of (1a), (1b) with initial data $u_0$ and $v_0$, respectively, where $u_0, v_0 \in USC(\mathbb{R}^n)$. If $u_0 \leq v_0$ on $\mathbb{R}^n$, then $u \leq v$ on $\mathbb{R}^n \times (0, T)$.

In the definition of an $L$-solution the specific form of $\psi_0$ given by (4) is not important.

**Theorem 3 (Uniqueness).** Assume that $\psi_0$ is a bounded uniformly continuous function such that $\{\psi_0 \geq 0\} = K_0$ and that $y \mapsto \psi_0(x, y)$ is nonincreasing. Let $\psi$ be the solution of (3a), (3b). Then

$$\bar{u}(t, x) = \sup\{y \in \mathbb{R}; \psi(t, x, y) \geq 0\}, \quad t \in (0, T), \quad x \in \mathbb{R}^n$$
agrees with the $L$-solution of (1a), (1b).

The key observation for the proof is that the set \{\psi \geq 0\} = \{(t, x, y); \psi(t, x, y) \geq 0\} depends only on $K_0$ and is independent of the choice of $\psi_0$. This is a typical uniqueness property of a level set equation. It is based on invariance of solution under the change of the dependent variable as stated below (which is slightly more general than stated in references [ESou], [ES], [CGG1], [G], [IS] since $\theta$ need not be continuous).

Lemma 4 (Invariance). Assume that $\psi$ is a subsolution (resp. supersolution) of (3a). Assume that $\theta$ is upper (resp. lower) semicontinuous and nondecreasing. Assume that $\theta \not\equiv -\infty$ (resp. $\theta \not\equiv +\infty$). Then the composite function $\theta \circ \psi$ is also a subsolution (resp. supersolution of (3a)).

If \{\psi \geq 0\} were a bounded set, a comparison principle for (3a), (3b) and Lemma 4 would yield the uniqueness of \{\psi \geq 0\} as in [ES], [CGG1], [G]. However, since \{\psi \geq 0\} is unbounded, we actually argue as in [IS] to get the uniqueness of \{\psi \geq 0\}.

3. Consistency

We shall compare other notion of solutions.

Theorem 5. Let $\overline{u}$ be the $L$-solution of (1a), (1b) with $u_0 \in USC (\mathbb{R}^n)$. Then $\overline{u}$ be a viscosity solution of (1a) provided that $\overline{u}$ does not take $\pm\infty$.

Sketch of the proof. Let $\psi$ be the solution of (3a), (3b) with $\psi_0$ in (4). By Lemma 4 the function $I^{-} \circ \psi$ is a subsolution of (3a), where $I^{-}(\sigma) = 0$ for $\sigma \geq 0$ and $I^{-}(\sigma) = -\infty$ for $\sigma < 0$. From this it is easy to see that $\overline{u}$ is a viscosity subsolution.

To prove that $\overline{u}$ is a viscosity supersolution we need to use the fact that $y \mapsto \psi(x, y)$ is nonincreasing. This implies that the lower semicontinuous envelope $(\overline{u})_*$ of
\( \overline{u} \) equals

\[
\overline{u}(t, x) = \inf\{y \in \mathbb{R}; (t, x, y) \in \overline{\{\psi < 0\}}\} \quad t \in (0, T), \ x \in \mathbb{R}^n.
\]

Since \( I^+ \circ (\psi + 1/m) \) is a supersolution of (3a) by Lemma 4, we see, by stability as \( m \to \infty \), that

\[
\Psi(t, x, y) = \begin{cases} 
\infty & \text{for } (t, x, y) \in \text{int}\{\psi \geq 0\}, \\
0 & \text{for } (t, x, y) \in \{\psi < 0\}
\end{cases}
\]
is a subsolution of (3a), where \( I^+(\sigma) = 0 \) for \( \sigma \leq 0 \) and \( I^+(\sigma) = \infty \) for \( \sigma > 0 \). Thus \( \underline{u} \) is a supersolution.

**Theorem 6.** Assume that \( u_0 \) is bounded, uniformly continuous. Then the bounded, uniformly continuous viscosity solution \( u \) of (1a), (1b) is an \( L \)-solution.

This follows from Theorem 3 by choosing \( \psi = ((y-u(t, x)) \wedge M) \vee M \) for \( M = \sup |u| \).

**Theorem 7.** Assume that \( p \mapsto H(x, p) \) is concave. Let \( \overline{u} \) be the \( L \)-solution of (3a), (3b) with \( u_0 \in \text{USC} (\mathbb{R}^n) \) and \( \sup u_0 < \infty \). Then \( \overline{u} \) is a bilateral viscosity solution with initial data \( u_0 \).

For the proof we use the property that the bilateral solution is given as a monotone limit of continuous viscosity solution [BJ]. Thus the proof is reduced to the next lemma.

**Lemma 8.** Assume that \( u_0 \in \text{USC} (\mathbb{R}^n) \) with \( u_0 \) which is Lipschitz in \( \mathbb{R}^n \). Assume that \( u_{0 \epsilon} \geq u_{0 \epsilon'} + \epsilon - \epsilon' \) for \( \epsilon > \epsilon' > 0 \). Let \( u_{\epsilon} \) be the solution of (1a), (1b) with \( u_0 = u_{0 \epsilon} \). Then \( \lim_{\epsilon \to 0} u_{\epsilon} \) is an \( L \)-solution of (1a), (1b) (so that it agrees with \( \overline{u} \)).

The sequence \( u_{0 \epsilon} \) is easily constructed by setting \( u_{0 \epsilon} = u_0^\epsilon + \epsilon \) with sup-convolution \( u_0^\epsilon \) of \( u_0 \).

4. Right accessibility
It is not clear in what sense the initial value is attained for $L$-solutions (unless initial data is continuous.) Since the viscosity solution of (3a), (3b) with $\psi_0$ in (4) is continuous up to $t = 0$, the set $\{\psi \geq 0\}$ is closed in $[0,T) \times \mathbb{R}^n \times \mathbb{R}$ so that

$$u_0(x) \geq \lim_{t \to 0} \overline{u}(t,y).$$

However, in general it is not clear whether there is a sequence $t_m \to 0$, $y_m \to x$ such that

$$u_0(x) = \lim_{m \to \infty} \overline{u}(t_m,y_m).$$

We call this last property the right accessibility as in [CGG2]. Since $\overline{u}$ is upper semicontinuous in $[0,T) \times \mathbb{R}^n$, the property (9) is equivalent to $u_0(x) = (\overline{u}|_{(0,T) \times \mathbb{R}^n})^*(0,x)$.

We give a simple criterion for right accessibility without mentioning its proof.

**Lemma 9.** Assume that $F \in C(\mathbb{R}^N)$ is positively homogeneous of degree one. Let $A$ be a closed convex set in $\mathbb{R}^N$. Let $w$ be the $L$-solution of

$$w_t + F(w_z) = 0, \quad z \in \mathbb{R}^N, \quad t > 0; \quad w|_{t=0} = w_0.$$  

with $w_0(z) = 0$, $z \in A$ and $\sup\{w_0(z); \text{dist}(z,A) \geq \delta\} < 0$ for $\delta > 0$. Then

$$w(t,z) = \begin{cases} 0 & z \in A + tW_\alpha \\ < 0 & \text{otherwise.} \end{cases}$$

Here

$$W_\alpha = \{z \in \mathbb{R}^N; \sup_{|p|=1} (z \cdot p - \alpha(p)) \leq 0\}, \quad \alpha(p) = -F(-p).$$

The set $W_\alpha$ is often called the Wulff shape with respect to $\alpha$ if $\alpha$ is positive. The set $W_\alpha$ may be empty. For example if $F(p) = |p|$, then $W_\alpha = \emptyset$. Thus if we consider (1a), (1b) with $H(p) = |p|$ and $u_0(x) = 0$, $x = 0$; $u_0(x) = -\infty$, $x \neq 0$, then the $L$-solution $u(t,x) = -\infty$ for all $t > 0$. Thus (9) is not fulfilled.

**Theorem 10.** If $H$ is homogeneous degree of one, and independent of $x$, then an $L$-solution is right accessible for any $u_0 \in USC(\mathbb{R}^n)$ if and only if $W_\alpha \neq \emptyset$. 

Remark 11. Our results up to §3 can be generalized for more general equation

\[ u_t + H(x, u, u_x) = 0, \]

when \( H \) fulfills

(i) \( H \in C(\mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n) \) and \( H_{\infty} \) exists;

(ii) There exists a modulus \( m_1 \) that satisfies

\[ |qH(x, y - p/q) - qH(x', y', -p/q)| \leq m_1((|x - x'| + |y - y'|)(|p| + |q| + 1)); \]

(iii) For each \( C_1 > 0 \) there exists a modulus \( m_2 \) such that

\[ |qH(x, y - p/q) - q'H(x, y, -p'/q')| \leq m_2(|p - p'| + |q - q'|) \]

for all \( x \in \mathbb{R}^n, y \in \mathbb{R}, p, p' \in \mathbb{R}^n, q, q' < 0 \) satisfying \( |p|, |p'|, |q|, |q'| \leq C_1 \);

(iv) \( y \mapsto H(x, y, p) \) is nondecreasing.

A typical example of \( H \) satisfying these assumptions is \( a(x) \sqrt{b + |p|^\beta} \) and \( a \) is Lipschitz and \( 0 \leq \beta \leq 1, b \geq 0 \).

References.


