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Kyoto University
Multiple Existence of Entire Solutions for
Semilinear Elliptic problems on $R^N$

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1. Introduction. Our purpose in this talk is to show the multiple existence of entire solutions of the problem

\[(P)\quad -\Delta u + u = g(x, u), \quad u \in H^1(R^N)\]

where $N \geq 2$ and $g : R^N \times R \to R$ is a continuous function with superlinear growth and $g(x, 0) = 0$ on $R^N$.

We fix $p$ such that $p > 1$ when $N = 2$ and $1 < p < (N + 2)/(N - 2)$ when $N \geq 3$. It is well known that the problem

\[(P_0)\quad -\Delta u + u = |u|^{p-1}u, \quad u \in H^{1,2}(R^N)\]

has a unique positive solution $u$ up to translation. The positive solution $u$ is characterized as the ground state soluiton. That is if we consider a functional $I$ defined by

$$I(u) = \int_{R_N} \frac{1}{2} |\nabla u|^2 \, dx - \frac{1}{p+1} \int_{R_N} |u|^{p+1} \, dx$$

for $u \in H^1(R^N)$, then $c = I(u)$ is the minimal positive critical level of $I$. The existence of positive entire solution of problem

\[(P_Q)\quad \begin{cases} 
- \Delta u + u = Q(x) \, |u|^{p-1} u, & x \in R^N \\
  u \in H^1(R^N) 
\end{cases}\]
has been studied by several authors. Here $Q(x)$ satisfies $Q(x) \to 1$ as $|x| \to \infty$. In case that $Q(x) \geq 1$ in $R^N$, the existence of a solution of $P_Q$ was established by Lions using the concentrate compactness method. Lions's result was improved by Zhu and Cao. The case that $Q(x) \mid t \mid^{p-1} t$ is replaced by a more general function $g(x, t)$, the existence of positive solutions is proved by the author.

To attack this kind of problem, one can take the advantage of variational structure of problem $P_Q$. That is the solutions of problem $(P_Q)$ is characterized as critical points of functional $I_Q$ defined by

$$I_Q(u) = \int_{R^N} \frac{1}{2} |\nabla u|^2 \, dx - \frac{1}{p+1} \int_{R^N} Q(x) |u|^{p+1} \, dx, \quad u \in H^1(R^N).$$

As in case that $Q(x) \equiv 1$, we can obtain a positive solution as a ground state solution. In this talk, we consider the case that $g \in C^2(R^N, R)$ satisfies the following conditions:

$(g1)$ There exists $0 < \theta < 1/2$ such that

$$\theta g(x, t)t \geq G(x, t) = \int_0^t g(x, s)ds > 0 \quad \text{for all } x \in R^N \text{ and } t > 0;$$

$(g2)$

$$\lim_{|x| \to \infty} g(x, t)/|t|^{p-1} t = 1$$

uniformly on closed bounded subsets of $(0, \infty)$

$(g3)$ there exists $\rho > 0$ such that

$$|g(x, t) - |t|^{p-1} t| \leq \rho |t|^{p-1} t \quad \text{for all } x \in R^N \text{ and } t \in R;$$

We can now state our main result.

**Theorem 1.** Assume that $(g1)$ and $(g2)$ hold. Then there exists a positive number $\rho_0$ such that if $(g3)$ hold with $0 < \rho < \rho_0$, then problem $(P)$ possesses at least two nontrivial solutions.
We next impose the following conditions on $g$.

\begin{align*}
\text{(g4)} \quad g(x, t) &= -g(x, -t) \quad \text{for all } x \in \mathbb{R}^N \text{ and } t \in \mathbb{R}.
\end{align*}

\begin{align*}
\text{(g5)} \quad \text{there exist positive numbers } a, C \text{ such that } a < 1 \text{ and }
g(x, t) / |t|^p t &\geq 1 + Ce^{-a|x|} \quad \text{for all } x \in \mathbb{R}^N \text{ and } t \neq 0.
\end{align*}

\begin{theorem}
Assume that (g1)(g2), (g4) and (g5) hold. Then there exists a positive number $\rho_0$ such that if (g2) hold with $0 < \rho < \rho_0$, then problem (P) possesses at least two pairs of nontrivial solutions.
\end{theorem}

To get a sign changing solution of (P), we impose the following condition instead of (g5).

\begin{align*}
\text{(g5')} \quad \text{there exist positive numbers } a, C \text{ such that } a < 1 \text{ and }
g(x, t) / |t|^p t &\geq 1 + C|x|^N \quad \text{for all } x \in \mathbb{R}^N \text{ and } t \neq 0.
\end{align*}

\begin{theorem}
Assume that (g1)(g2), (g4) and (g5') hold. Then there exists a positive number $\rho_0$ such that if (g2) hold with $0 < \rho < \rho_0$, then problem (P) possesses at least two pairs of nontrivial solutions. Moreover (P) possesses at least one pair of sign changing solutions.
\end{theorem}

2. Preliminaries.

We put $H = H^1(\mathbb{R}^N)$ and
\[ \|z\|^2 = |\nabla z|_2^2 + |z|_2^2 \text{ for } z \in H. \]

For each $a \in \mathbb{R}$ and each functional $F : H \to \mathbb{R}$, we denote by $F_a$ the set $F_a = \{ v \in X : F(v) \leq a \}$. We call a real number $d$ a critical value of a functional $F$ if there exists a sequence $\{v_n\} \subset H$ such that $\lim_{n \to \infty} F(v_n) = d$ and $\lim_{n \to \infty} \| F'(v_n) \| = 0$.

For $z \in H$, $D \subset H$ and $x \in \mathbb{R}^N$, we denote by $z_x$ and $D_x$,
\[ z_x(y) = z(y - x) \quad \text{for } y \in \mathbb{R}^N \text{ and } D_x = \{ z_x : z \in D \}. \]
For each $x \in \mathbb{R}^N$, the function $u_x$ is a solution of $I$ with $I(u_x) = c$. It is also known that there exist no critical value of $I$ in $(0, 2c) \setminus \{c\}$.

We define a functional $J^\infty$ on $H^1(\mathbb{R}^N)$ by

$$J^\infty(v) = \int_{\mathbb{R}^N} \frac{1}{2} (|\nabla v|^2 + |v|^2) dx - \int_{\mathbb{R}^N} \int_0^{v(x)} g(x, t) dt dx dx$$

for $v \in H^1(\mathbb{R}^N)$. We put

$$M = \{v \in H \setminus \{0\} : \|v\|^2 = \int_{\mathbb{R}^N} |v|^{p+1}\}$$

Noting that

$$c = I(u) = \min \{I(v) : \|v\|^2 = \int_{\mathbb{R}^N} |v|^{p+1} dx\}, \quad (2.1)$$

we have that

$$I(v) \geq c \quad \text{on } M. \quad (2.2)$$

It is also easy to see that

$$M \cap \{\lambda v : v \in H \setminus \{0\}, \lambda \geq 0\} \text{ is a unique point,} \quad (2.3)$$

$$I(v) = \max \{I(\lambda v) : \lambda \geq 0\} \quad \text{for each } v \in M \quad (2.4)$$

and each critical point of $I$ is contained in $M$ (cf. [12]).

Let $\epsilon_0 > 0$ with $2\epsilon_0 < c$.

The following results is well known.

**Lemma 2.1.** For each $\epsilon > 0$ with $\epsilon < c$, there exists $V_\epsilon \subset M$ such that

$$I_{c+\epsilon} \cap M = V_\epsilon \cup -V_\epsilon, \quad V_\epsilon \cap -V_\epsilon = \emptyset.$$ 

Here we put

$$X_{1/2} = \{\mu v \in M, \mu \geq \frac{1}{2}\}$$
Then $M \subset \text{int}X_{1/2}$. Let $V_0, V_1$ be bounded neighborhoods of $V_{\epsilon_0} (\subset M \cap I_{c+\epsilon_0})$ such that

$$V_0 \subset \text{int}V_1 \subset X_{1/2} \quad \text{and} \quad V_1 \subset I^{-1}[\epsilon_0, c + 2\epsilon_0]$$

Then we have that

$$\delta_0 = \inf \{ \| I(v) \| : v \in V_1 \backslash V_0 \} > 0.$$

We next define a functional $J$. $\alpha(x) : H \rightarrow [0, 1]$ be a continuous function such that

$$\alpha(x) = \begin{cases} 1 & \text{for } x \in V_1^c \\ 0 & \text{for } x \in V_0 \end{cases}$$

and we put

$$J(v) = \alpha(v)I(v) + (1 - \alpha(x))J^\infty(v) \quad \text{for all } v \in H.$$ 

Then from the definition, $J \equiv J^\infty$ on $V_0$ and $J \equiv I$ on $V_1^c$.

Here we note that

$$\lim_{\rho \rightarrow 0} |I(v) - J^\infty(v)| = \lim_{\rho \rightarrow 0} \| \nabla I(v) - \nabla J^\infty(v) \| = 0 \quad \text{uniformly on } V_1. \quad (2.5)$$

Then there exists $\rho_0 > 0$ such that if $\rho \leq \rho_0$,

$$|I(v) - J(v)| < c/2 \quad \text{on } V_1$$

and

$$\| \nabla J^\infty(v) - \nabla I(v) \| < \delta_0/2 \quad \text{on } V_1.$$ 

Therefore we have that

$$\| \nabla J(v) \| > \delta_0/2 \quad \text{for all } v \in V_1 \backslash V_0.$$ 

This implies that if $\rho \leq \rho_0$,

$$\| \nabla J(v) \| < \delta_0/2 \quad \text{and} \quad 2c > J(v) > 0 \quad \text{implies that } v \in V_0$$

and therefore $J(v) = J^\infty(v)$. This implies that if we find a critical point $v$ of $J$ with $2c > J(v) > 0$, then $v$ is a critical point of $J^\infty$ in $V_0$. 
3. **Homology groups**. Our purpose in this section is to calculate homology groups $H_*(I_{c+\epsilon}, I_{c-\epsilon})$ for $0 < \epsilon < c + 2\epsilon_0$. To calculate the homology groups $H_*(I_{c+\epsilon}, I_{c-\epsilon})$, we will find subsets $K$ and $U$ of $V_0$ satisfying

(a) $K \subset \text{int}U$;

(b) $\pm K_0 = \{\pm u_x : x \in \mathbb{R}^N\} \subset K$

for some $r > 0$, where $\partial K$ denotes the boundary of $K$ in $H$;

(c) there exists $\epsilon_1 > 0$ such that $I_{c/2}$ is a strong deformation retract of $I_{c+\epsilon}\backslash K$ for $0 < \epsilon < \epsilon_1$.

For $U$ and $K$ satisfying (a), (b) and (c), we have the following lemma.

**Lemma 3.1.** Suppose that $U$ and $K$ satisfies (a), (b) and (c). Then for each $0 < \epsilon < \epsilon_1$,

$$H_*(I_{c+\epsilon}, I_{c-\epsilon}) = H_*(U \cap I_{c+\epsilon}, (U \backslash K) \cap I_{c+\epsilon})$$

We will define subsets $U$ and $K$ of $V_0$ satisfying (a), (b) and (c).

**Lemma 3.3.** For each $0 < \epsilon < c + 2\epsilon_0$,

$$I^M_{c+\epsilon} \cong \{u\} \cup \{-u\}$$

where $I^M$ is the restriction of $I$ on $M$.

We put $\widetilde{U} = I^M_{c+2\epsilon_0}$ and $\widetilde{K} = I^M_{c+\epsilon_0}$. Then it follows that

We next define $U$ and $K$. We fix positive numbers $r_1^-, r_2^-$ with $r_1^- > r_2^-$. We assume that $r_1^-$ is so small that

$$c/2 < I(v + \lambda v) \quad \text{for all } v \in \widetilde{U} \text{ and } \lambda \in \mathbb{R} \text{ with } |\lambda| \leq r_1^- . \quad (3.1)$$

By (3.4) and Lemma 3.2, there exists $\overline{\epsilon} > 0$ such that

$$I(v + \lambda v) < I(v) - \overline{\epsilon} . \quad \text{for } v \in \widetilde{U} \text{ and } r_2^- \leq |\lambda| \leq r_1^- . \quad (3.2)$$
Then by choosing $r_2^+$ small enough, we have that \( \sup \{ I(v) : v \in \widetilde{U} \} < c + \varepsilon/2 \).

Then by (3.2) that
\[
I(v + \lambda v) < c \quad \text{for all } v \in \widetilde{U} \text{ and } r_2^- \leq |\lambda| \leq r_1^-.
\] (3.3)

It also follows from Lemma 3.2 that

mapping \( t \to I(v + tv) \) is decreasing on \([0, 1]\) for \( v \in \widetilde{U} \). (3.4)

Now we set

\[
U = \{ v + \lambda v : v \in \widetilde{U}, |\lambda| \leq r_1^- \}, \quad K = \{ v + \lambda v : v \in \widetilde{K}, |\lambda| \leq r_2^- \}.
\]

Then it is obvious that \( U \) and \( K \) satisfies (a) and (b). Moreover we have

**Lemma 3.4.** There exists \( \epsilon_1 > 0 \) such that for each \( 0 < \epsilon < \epsilon_1 \), \( I_{c/2} \) is a strong deformation retract of \( I_{c+\epsilon} \setminus K \)

For each \( v \in \widetilde{U} \), we put

\[
U_v = \{ v + \lambda v : |\lambda| \leq r_1^- \}, \quad K_v = \begin{cases} \{ v + \lambda v : |\lambda| \leq r_2^- \} & \text{if } v \in \widetilde{K} \\ \{ \phi \} & \text{if } v \notin \widetilde{K} \end{cases}
\]

Then

**Lemma 3.6.** Let \( 0 < \epsilon < \epsilon_0 \). Then for each \( v \in \widetilde{U} \),

\[
(U_v \setminus K_v) \cap I_{c+\epsilon} \cong v + \{-r_1^-v, r_1^-v\} \cong S^0 \cong \{-1, 1\}.
\] (3.5)

**Lemma 3.7.** For \( 0 < \epsilon < \min \{ \epsilon_1, \epsilon_0 \} \),

\[
H_*(U \cap I_{c+\epsilon}, (U \setminus K) \cap I_{c+\epsilon}) = H_*(S^0 \times D^1, S^0 \times S^0) \oplus H_*(S^0 \times D^1, S^0 \times S^0).
\]

**Proof.** Let \( 0 < \epsilon < \min \{ \epsilon_1, \epsilon_0 \} \). By Lemma 3.5 and the definition, we have that

\[
U \cap I_{c+\epsilon} \cong U \cong \widetilde{U} \times D^1 \cong \{u\} \times D^1 \cup \{-u\} \times D^1
\]

On the other hand, by Lemma 3.6, we have that

\[
(U \setminus K) \cap I_{c+\epsilon} \cong \widetilde{U} \times S^0 \cong \{u\} \times S^0 \cup \{-u\} \times S^0
\]

Then the assertion follows.

By Lemma 2.1 and Lemma 3.7, we have
**Proposition 3.8.** For each $0 < \epsilon < c$

$$H_n(I_{c+\epsilon}, I_{c-\epsilon}) = \begin{cases} 2 & \text{for } n = 1 \\ 2 & \text{otherwise} \end{cases}$$

4. **Proofs of Theorem 1.** In this section, we calculate the homology groups for $J$ and prove Theorem 1. From (2.1?), we have that there exists $\rho_2 > 0$ such that for $0 < \rho < \rho_1$ sufficiently small, that

$$H_*(I_{c+\epsilon}, I_{c/2}) \cong H_*(J_{c+\epsilon}, J_{c/2}) \quad \text{for } 0 < 2\epsilon < c. \quad (4.1)$$

We will prove Theorem 1 by contradiction. That is we assume that $J$ possesses no critical point different from 0.

Here we state a direct consequence from Lions's concentrate compactness lemma.

Now assume that $\rho < \rho_0$ and we define a manifold $\mathcal{M}$ by

$$\mathcal{M} = \{v \in H \setminus \{0\} : \|v\|^2 = \int_{\mathbb{R}^N} \int_0^{v(x)} g(x, t)dt dx\}$$

It is easy to check that for each $v \in H \setminus \{0\}$, the set $\{\lambda v : \lambda \geq 0\}$ intersect to $\mathcal{M}$ at exactly one point. For each $x \in \mathbb{R}$, we define a positive number $\alpha_{+,x}$ and a negative number $\alpha_{-,x}$ by

$$\alpha_{+,x} u_x \in \mathcal{M} \quad \text{and} \quad \alpha_{-,x} u_x \in \mathcal{M}.$$  

From condition (g3), we have that

$$\lim_{|x| \to \infty} \alpha_{\pm,x} = \pm 1. \quad (4.2)$$

For $r > 0$, we put

$$K_{\pm,r} = \{\alpha_{\pm,x} u_x : x \in \mathbb{R}^N, |x| \geq r\}.$$  

Then

$$\lim_{r \to \infty} \sup \{J(v) : v \in K_{\pm,r}\} = c. \quad (4.3)$$
Lemma 4.2. For each $\epsilon > 0$ with $2\epsilon < c$, there exists $r_\epsilon > 0$ and

$$J_{c+\epsilon}^M \cong K_{+,r_\epsilon} \cup K_{-,r_\epsilon} \cong S^{N-1} \cup S^{N-1}.$$  

Now we put $\tilde{\mathcal{K}} = J_{c+\epsilon}^M$ and $\tilde{\mathcal{U}} = J_{c+2\epsilon}^M$.

Now we set

$$\mathcal{U} = \{v + \lambda v : v \in \tilde{\mathcal{U}}, \lambda \leq r_1^{-}\}, \quad \mathcal{K} = \{v + w : v \in \tilde{\mathcal{U}}, w \lambda \leq r_2^{-}\}.$$  

Then by a parallel argument as in the proof of Lemma 2.5, we can see that there exists $\bar{\epsilon}_1 > 0$ such that $J_{c/2}$ is a strong deformation retract of $J_{c_0+c+\epsilon} \setminus \mathcal{K}$ for each $0 < \epsilon < \bar{\epsilon}_1$. That is we have

$$H_*(J_{c+\epsilon}, J_{c/2}) = H_*(\mathcal{U} \cap J_{c_0+c+\epsilon}, (\mathcal{U} \setminus \mathcal{K}) \cap J_{c_0+c+\epsilon}) \quad (4.4)$$

for each $0 < \epsilon < \bar{\epsilon}_1$.

We also have

Lemma 4.3. For each $0 < \epsilon < \bar{\epsilon}_0$,

$$\mathcal{U} \cap J_{c_0+c+\epsilon} \cong \mathcal{U} \cong K_0.$$  

The proof of Lemma 4.5 is the same as that of Lemma 2.5. Then we omit the proof. As in section 2, we put

$$\mathcal{U}_v = \{v + \lambda v : |\lambda| \leq r_1^{-}\}, \quad \mathcal{K}_v = \begin{cases} \{v + \lambda v : |\lambda| \leq r_2^{-}\} & \text{if } v \in \tilde{\mathcal{K}} \\ \{\phi\} & \text{if } v \notin \tilde{\mathcal{K}}. \end{cases}$$

for each $v \in \tilde{U}$. Then by the same argument as in section 2, we have

Lemma 4.4. Let $0 < \epsilon < \bar{\epsilon}_0$. Then for each $v \in \tilde{\mathcal{U}}$,

$$(\mathcal{U}_v \setminus \mathcal{K}_v) \cap I_{c+\epsilon} \cong v + \{-r_1^{-}v, r_1^{-}v\} \cong S^0. \quad (4.5)$$

Then by using Lemma 4.5 and Lemma 4.6, we obtain
Lemma 4.7. For each $0 < \epsilon < \min\{\bar{\epsilon}_0, \bar{\epsilon}_1\}$,

$$H_*(\mathcal{U} \cap J_{c+\epsilon}, (\mathcal{U}\backslash \mathcal{K}) \cap J_{c+\epsilon})$$

$$= H_*(S^{N-1} \times D^1, S^{N-1} \times S^0) \oplus H_*(S^{N-1} \times D^1, S^{N-1} \times S^0).$$

Thus we obtain by (4.1) and Lemma 4.7 that

Proposition 4.8.

$$H_n(J_{c+\epsilon}, J_{c/2}) = \begin{cases} 2 & \text{for } n = 1 \text{ or } n = N \\ 0 & \text{otherwise} \end{cases}.$$  

We can now finish the proof of Theorem.

Proof of Theorem 1. By (4.5) and (4.0), we have that if $\rho \leq \rho_0$, then for each $0 < \epsilon < c$,

$$H_*(J_{c+\epsilon}, J_{c/2}) \cong H_*(I_{c+\epsilon}, I_{c/2}) \cong H_*(I_{c+\epsilon}, I_{c-\epsilon}). \quad (4.6)$$

But we can see from Proposition 3.8 and Proposition 4.8 that the equality does not holds. This is a contradiction. Thus we obtain that there exists at least two solutions of (P).