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<th>CLOSE-TO-CONVEXITY, STARLIKENESS AND CONVEXITY</th>
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<tbody>
<tr>
<td>Author(s)</td>
<td>Owa, Shigeyoshi; Nunokawa, Mamoru; Saitoh, Hitoshi; Srivastava, H.M.</td>
</tr>
<tr>
<td>Citation</td>
<td>数理解析研究所講究録 (1999), 1112: 98-105</td>
</tr>
<tr>
<td>Issue Date</td>
<td>1999-09</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/63356">http://hdl.handle.net/2433/63356</a></td>
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<td>Right</td>
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<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
</tr>
</tbody>
</table>

Kyoto University
CLOSE-TO-CONVEXITY, STARLIKENESS AND CONVEXITY

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Abstract

The main object of the present paper is to derive several sufficient conditions for close-to-convexity, starlikeness and convexity of certain (normalized) analytic functions. Relevant connections of some of the results obtained in this paper with those in earlier works are also provided.

1991 Mathematics Subject Classification. Primary 30C45.

Key words and phrases. Analytic functions, starlike functions, close-to-convex functions, convex functions, subordination principle, univalent functions.
CLOSE-TO-CONVEXITY, STARLIKENESS AND CONVEXITY

1. Introduction and Definitions

Let $A$ denote the class of functions $f$ normalized by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$  \hspace{1cm} (1.1)

which are analytic in the open unit disk

$$\mathcal{U} := \{ z : z \in \mathbb{C} \text{ and } |z| < 1 \}.$$

Also let $S^*(\alpha)$, $\mathcal{K}(\alpha)$, and $C(\alpha)$ denote the subclasses of $A$ consisting of functions which are, respectively, starlike, convex close-to-convex of order $\alpha$ in $\mathcal{U}$ $(0 \leq \alpha \leq 1)$. Thus we have (see, for details, Duren [1] and Goodman [2]; see also Srivastava and Owa [6])

$$S^*(\alpha) := \{ f : f \in A \text{ and } \Re\left( \frac{zf'(z)}{f(z)} \right) > \alpha \ (z \in \mathcal{U}; 0 \leq \alpha < 1) \},$$  \hspace{1cm} (1.2)

$$\mathcal{K}(\alpha) := \{ f : f \in A \text{ and } \Re\left( 1 + \frac{zf''(z)}{f(z)} \right) > \alpha \ (z \in \mathcal{U}; 0 \leq \alpha < 1) \},$$  \hspace{1cm} (1.3)

and

$$C(\alpha) := \{ f : f \in A \text{ and } \Re\left( \frac{f'(z)}{g(z)} \right) > \alpha \ (z \in \mathcal{U}; 0 \leq \alpha < 1; g \in \mathcal{K}) \},$$  \hspace{1cm} (1.4)

where, for convenience,

$$S^* := S^*(0), \quad \mathcal{K} := \mathcal{K}(0), \quad \text{and} \quad C := C(0).$$  \hspace{1cm} (1.5)

Next, with a view to recalling the principle of subordination between analytic functions, let the functions $f$ and $g$ be analytic in $\mathcal{U}$. Then we say that the function $f$ is subordinate to $g$ if there exists a function $h$, analytic in $\mathcal{U}$, with

$$h(0) = 0 \quad \text{and} \quad |h(z)| < 1 \ (z \in \mathcal{U}),$$  \hspace{1cm} (1.6)

such that

$$f(z) = g(h(z)) \ (z \in \mathcal{U}).$$  \hspace{1cm} (1.7)

We denote this subordination by

$$f(z) \prec g(z).$$  \hspace{1cm} (1.8)

In particular, if the function $g$ is univalent in $\mathcal{U}$, the subordination (1.8) is equivalent to (cf. [1, p. 190])

$$f(0) = g(0) \quad \text{and} \quad f(\mathcal{U}) \subset g(\mathcal{U}).$$  \hspace{1cm} (1.9)

Recently, R. Singh and S. Singh [5] proved several interesting results involving univalence and starlikeness of functions $f \in A$. In our attempt here to generalize these results of Singh and Singh [5], we are led naturally to several sufficient conditions for close-to-convexity, starlikeness, and convexity of functions $f \in A$.

The following lemma (popularly known as Jack's lemma) will be required in our present investigation.
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Lemma 1 (cf. Jack [3]; see also Miller and Mocanu [4]). Let the (non-constant) function $w(z)$ be analytic in $\mathcal{U}$ with $w(0) = 0$. If $|w(z)|$ attains its maximum value on the circle $|z| = r < 1$ at a point $z_0 \in \mathcal{U}$, then

$$z_0 w'(z_0) = cw(z_0),$$

where $c$ is a real number and $c \geq 1$.

2. SUFFICIENT CONDITIONS FOR CLOSE-TO-CONVEXITY

Our first result (Theorem 1 below) provides a sufficient condition for close-to-convexity of functions $f \in A$.

Theorem 1. Let the function $f \in A$ satisfy the inequality:

$$\Re \left( 1 + \frac{zf''(z)}{f'(z)} \right) > \frac{1 + 3\alpha}{2(1 + \alpha)} \quad (z \in \mathcal{U}; 0 \leq \alpha < 1).$$

(2.1)

Then

$$\Re \{f'(z)\} > \frac{1 - \alpha}{2} \quad (z \in \mathcal{U}; 0 \leq \alpha < 1)$$

or, equivalently,

$$f \in C \left( \frac{1 - \alpha}{2} \right) \quad (0 \leq \alpha < 1).$$

(2.3)

Proof. We begin by defining a function $w$ by

$$f'(z) = \frac{1 + \alpha w(z)}{1 + w(z)} \quad (w(z) \neq -1; z \in \mathcal{U}; 0 \leq \alpha < 1).$$

(2.4)

Then, clearly, $w$ is analytic in $\mathcal{U}$ with $w(0) = 0$. We also find from (2.4) that

$$1 + \frac{zf''(z)}{f'(z)} = 1 + \frac{\alpha zw'(z)}{1 + \alpha w(z)} - \frac{zw'(z)}{1 + w(z)} \quad (z \in \mathcal{U}).$$

(2.5)

Suppose now that there exists a point $z_0 \in \mathcal{U}$ such that

$$|w(z_0)| = 1 \quad \text{and} \quad |w(z)| < 1 \quad \text{when} \quad |z| < |z_0|.$$  

(2.6)

Then, by applying Lemma 1, we have

$$z_0 w'(z_0) = cw(z_0) \quad \left( c \geq 1; w(z_0) = e^{i\theta}; \theta \in \mathbb{R} \right).$$

(2.7)
Thus we find from (2.5) and (2.7) that

\[
\Re \left( 1 + \frac{z_0 f''(z_0)}{f'(z_0)} \right) = 1 + \Re \left( \frac{c \alpha e^{i\theta}}{1 + \alpha e^{i\theta}} \right) - \Re \left( \frac{c e^{i\theta}}{1 + e^{i\theta}} \right)
\]

\[
= 1 + \frac{c \alpha (\alpha + \cos \theta)}{1 + \alpha^2 + 2\alpha \cos \theta} - \frac{c}{2}
\]

\[
\leq \frac{1 + 3\alpha}{2(1 + \alpha)} \quad (z_0 \in \mathcal{U}; 0 \leq \alpha < 1),
\]

which obviously contradicts our hypothesis (2.1). It follows that

\[|w(z)| < 1 \quad (z \in \mathcal{U}),\]

that is, that

\[\left| \frac{1 - f'(z)}{f'(z) + \alpha} \right| < 1 \quad (z \in \mathcal{U}; 0 \leq \alpha < 1). \tag{2.8}\]

This evidently completes the proof of Theorem 1.

**Theorem 2.** If the function \( f \in A \) satisfies the inequality:

\[
\Re \left( 1 + \frac{z f''(z)}{f'(z)} \right) < \frac{3 + 2\alpha}{2 + \alpha} \quad (z \in \mathcal{U}; 0 \leq \alpha < 1), \tag{2.9}\]

then

\[|f'(z) - 1| < 1 + \alpha \quad (z \in \mathcal{U}; 0 \leq \alpha < 1). \tag{2.10}\]

**Proof.** Our proof of Theorem 2, also based upon Lemma 1, is much akin to that of Theorem 1. Indeed, in place of the definition (2.4), here we let the function \( w \) be given by

\[f'(z) = (1 + \alpha)w(z) + 1 \quad (z \in \mathcal{U}; 0 \leq \alpha < 1). \tag{2.11}\]

The details may be omitted.

**Remark 1.** Since the inequality (2.10) implies that

\[
\Re \{f'(z)\} > -\alpha \quad (z \in \mathcal{U}; 0 \leq \alpha < 1), \tag{2.12}\]

by setting \( \alpha = 0 \) in Theorem 2, we readily obtain

**Corollary 1** (Singh and Singh [5, p. 311, Corollary 2]). If the function \( f \in A \) satisfies the inequality:

\[
\Re \left( 1 + \frac{zf''(z)}{f'(z)} \right) < \frac{3}{2} \quad (z \in \mathcal{U}), \tag{2.13}\]

then

\[|f'(z) - 1| < 1 \quad (z \in \mathcal{U}), \tag{2.14}\]

that is, \( f \in C \).

Next we prove
Theorem 3. If the function \( f \in A \) satisfies the inequality:
\[
|f'(z) - 1|^\beta |zf''(z)|^\gamma < \frac{(1-\alpha)^{\beta+\gamma}}{2^{\beta+2\gamma}} \quad (z \in \mathcal{U}; 0 \leq \alpha < 1; \beta, \gamma \geq 0),
\]
then
\[
\Re \{f'(z)\} > \frac{1+\alpha}{2} \quad (z \in \mathcal{U}; 0 \leq \alpha < 1).
\]

Proof. We define the function \( w \) by
\[
f'(z) = \frac{1 + \alpha w(z)}{1 + w(z)} \quad (w(z) \neq -1; z \in \mathcal{U}; 0 \leq \alpha < 1).
\]
Then, clearly, \( w \) is analytic in \( \mathcal{U} \) with \( w(0) = 0 \). We also find from (2.17) that
\[
|f'(z) - 1|^\beta |zf''(z)|^\gamma = \frac{(1-\alpha)^{\beta+\gamma} |w(z)|^{\beta} |zw'(z)|^{\gamma}}{|1+w(z)|^{\beta+2\gamma}} \quad (z \in \mathcal{U}).
\]
Supposing now that there exists a point \( z_0 \in \mathcal{U} \) such that
\[
|w(z_0)| = 1 \quad \text{and} \quad |w(z)| < 1 \quad \text{when} \quad |z| < |z_0|,
\]
if we apply Lemma 1 just as we did in the proof of Theorem 1, we shall obtain
\[
\left| f'(z_0) - 1 \right|^\beta \left| z_0 f''(z_0) \right|^\gamma \geq \frac{(1-\alpha)^{\beta+\gamma} c^{\gamma}}{|1 + e^{i\theta}|^{\beta+2\gamma}} \quad (z_0 \in \mathcal{U}; 0 \leq \alpha < 1),
\]
which obviously contradicts our hypothesis (2.15). Thus we have
\[
|w(z)| < 1 \quad (z \in \mathcal{U}),
\]
which implies that
\[
\left| \frac{f(z) - 1}{f(z) - \alpha} \right| < 1 \quad (z \in \mathcal{U}; 0 \leq \alpha < 1),
\]
that is, that (2.16) holds true.

By letting
\[
\beta = \gamma - 1 = 0
\]
in Theorem 2, we arrive at

Corollary 2. If the function \( f \in A \) satisfies the inequality:
\[
|zf''(z)| < \frac{1-\alpha}{4} \quad (z \in \mathcal{U}; 0 \leq \alpha < 1),
\]
then
\[
\Re \{f'(z)\} > \frac{1+\alpha}{2} \quad (z \in \mathcal{U}; 0 \leq \alpha < 1).
\]
Remark 2. An analogous result (which apparently is not contained in Corollary 2) was proven earlier by Singh and Singh [5, p. 310, Corollary 1], which asserted that, if the function $f \in A$ satisfies the inequality:

$$|zf''(z)| < 1 \quad (z \in \mathcal{U}),$$

then $f \in C$.

3. Starlikeness and Convexity

In this section, we first prove the following result (Theorem 4 below), which involves the already introduced principle of subordination between analytic functions (see Section 1).

**Theorem 4.** If the function $f \in A$ satisfies the inequality:

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) < \begin{cases} \frac{5\lambda - 1}{2(\lambda + 1)} & (z \in \mathcal{U}; 1 < \lambda \leq 2) \\ \frac{\lambda + 1}{2(\lambda - 1)} & (z \in \mathcal{U}; 2 < \lambda < 3) \end{cases}$$

for some $\lambda (1 < \lambda < 3)$, then

$$zf'(z) \prec \frac{\lambda(1-z)}{\lambda-z}.$$  

(3.2)

The result is sharp for the function $f$ given by

$$f(z) = z \left(1 - \frac{z}{\lambda}\right)^{\lambda - 1}.$$  

(3.3)

**Proof.** Let us define the function $w$ by

$$zf'(z) = \frac{z[1 - w(z)]}{\lambda - w(z)} \quad (w(z) \neq \lambda; z \in \mathcal{U}; 1 < \lambda < 3).$$

(3.4)

Then, clearly, $w$ is analytic in $\mathcal{U}$ with $w(0) = 0$. By logarithmic differentiation of both sides of (3.4), we also find that

$$1 + \frac{zf''(z)}{f'(z)} = \frac{\lambda[1 - w(z)]}{\lambda - w(z)} - \frac{zw'(z)}{1 - w(z)} + \frac{zw'(z)}{\lambda - w(z)} \quad (z \in \mathcal{U}).$$

(3.5)

Assuming now that there exists a point $z_0 \in \mathcal{U}$ such that

$$|w(z_0)| = 1 \quad \text{and} \quad |w(z)| < 1 \quad \text{when} \quad |z| < |z_0|,$$
if we apply Lemma 1 just as we did in the proof of Theorem 1, we shall obtain

\[
\Re \left( 1 + \frac{z_0 f''(z_0)}{f'(z_0)} \right)
\]

\[
= \Re \left( \frac{\lambda (1 - e^{i\theta})}{\lambda - e^{i\theta}} \right) - \Re \left( \frac{c e^{i\theta}}{1 - e^{i\theta}} \right) + \Re \left( \frac{c e^{i\theta}}{\lambda - e^{i\theta}} \right)
\]

\[
= \frac{\lambda(\lambda + 1)(1 - \cos \theta)}{1 + \lambda^2 - 2\lambda \cos \theta} + \frac{c(\lambda \cos \theta - 1)}{1 + \lambda^2 - 2\lambda \cos \theta}
\]

\[
= \frac{\lambda + 1}{2} + \frac{(\lambda^2 - 1)(2 - \lambda)}{2(1 + \lambda^2 - 2\lambda \cos \theta)}
\]

\[
\geq \frac{\lambda + 1}{2} + \frac{\lambda^2 - 1}{2(1 + \lambda^2 - 2\lambda \cos \theta)}
\]

\[
(\forall z_0 \in U; 1 < \lambda < 3),
\]

which yields the inequality:

\[
\Re \left( 1 + \frac{z_0 f''(z_0)}{f'(z_0)} \right) \geq \begin{cases} 
\frac{5\lambda - 1}{2(\lambda + 1)} & (z_0 \in U; 1 < \lambda \leq 2) \\
\frac{\lambda + 1}{2(\lambda - 1)} & (z_0 \in U; 2 < \lambda < 3).
\end{cases}
\]

(3.6)

Since (3.6) obviously contradicts our hypothesis (3.1), we conclude that

\[
|w(z)| < 1 \quad (z \in U),
\]

that is, that

\[
\left| \frac{zf'(z)}{f(z)} - \frac{\lambda}{\lambda + 1} \right| < \frac{\lambda}{\lambda + 1} \quad (z \in U; 1 < \lambda < 3),
\]

(3.7)

which implies the subordination (3.2) asserted by Theorem 4.

Finally, for the function \( f \) given by (3.3), we have

\[
\frac{zf'(z)}{f(z)} = \frac{\lambda(1 - z)}{\lambda - z},
\]

(3.8)

which evidently completes our proof of Theorem 4.

**Remark 3.** A special case of Theorem 4 when \( \lambda = 2 \) was given earlier by Singh and Singh [5, p. 313, Theorem 6].

Lastly, since

\[
f(z) \in \mathcal{K}(\alpha) \iff zf'(z) \in \mathcal{S}^*(\alpha) \quad (0 \leq \alpha < 1),
\]

(3.9)

whose special case, when \( \alpha = 0 \), is the familiar Alexander theorem (cf., e.g., Duren [1, p. 43, Theorem 2.12]), Theorem 4 can be applied in order to deduce
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Corollary 3. If the function $f \in A$ satisfies the inequality:

$$\Re \left( \frac{2zf''(z) + z^2f'''(z)}{f'(z) + zf''(z)} \right) < \begin{cases} \frac{3(\lambda-1)}{2(\lambda+1)} & (z \in U; 1 < \lambda \leq 2) \\ \frac{3 - \lambda}{2(\lambda-1)} & (z \in U; 2 < \lambda < 3) \end{cases}$$

for some $\lambda \ (1 < \lambda < 3)$, then

$$1 + \frac{zf''(z)}{f'(z)} \prec \frac{\lambda(1-z)}{\lambda-z}$$

The result is sharp for the function $f$ given by

$$f'(z) = (1 - \frac{z}{\lambda})^{\lambda-1}$$

Acknowledgments

The present investigation was supported, in part, by the Japanese Ministry of Education, Science and Culture under Grant-in-Aid for General Scientific Research (No. 046204) and, in part, by the Natural Sciences and Engineering Research Council of Canada under Grant OGP0007353.

REFERENCES