数理解析研究所講究録
1112巻 1999年 92-97

Partial Sums of Certain Analytic Functions

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Abstract. The object of the present paper is to consider of starlikeness and convexity of partial sums of certain analytic functions in the open unit disk

1 Introduction

Let A denote the class of functions $f(z)$ of the form

(1) $$f(z)=z+\sum_{k-2}^{\infty}a_kz^k$$

which are analytic in the open unit disk $U=\{z\in\mathbb{C}:|z|<1\}$. Let$S^*(\alpha)$ be the subclass of A consisting of functions $f(z)$ which satisfy

(2) $$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\}>\alpha \qquad (z\in U)$$

for some $\alpha(0\leq\alpha<1)$. A function $f(z)$ in $S^*(\alpha)$ is said to be starlike of order α in U. Furthermore, let $K(\alpha)$ denote the subclass of A consisting of all functions $f(z)$ which satisfy

(3) $$\mathrm{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\}>\alpha \qquad (z\in U)$$

for some $\alpha(0\leq\alpha<1)$. A function $f(z)$ belonging to $K(\alpha)$ is said to be convex of oredr α in U. We note that $f(z)\in S^*(\alpha)$ if and only if $zf'(z)\in K(\alpha)$ and denote by $S^*(0)\equiv S^*$ and $K(0)\equiv K$. For $f(z)\in A$, we introduce the partial sum of $f(z)$ by

(4) $$f_n(z)=z+\sum_{k=2}^{n}a_kz^k.$$

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*Mathematics Subject Classification*1991:30C45
*Key Words and Phrases:*Partial sum, Starlike of order α, Convex of order α

Remark 1. *It is well-known that*
(i) $f(z)=\frac{z}{(1-z)^2}=z+\sum_{k=2}^{\infty}kz^k$ *is the extremal function for the class* S^*. *But* $f_2(z)=z+2z^2\notin S^*$.
(ii) $f(z)=\frac{z}{1-z}=z+\sum_{k=2}^{\infty}z^k$ *is the extremal function for the class* K. *But* $f_2(z)=z+z^2\notin K$.

For the partial sums $f_n(z)$ of $f(z)\in S^*$, Szegö [2] showed
([2])

Theorem 1. *(i)* $f(z)\in S^*$ *implies that* $f_n(z)\in S^*$ *for* $|z|<\frac{1}{4}$. *The result is sharp.*
(ii) $f(z)\in S^*$ *implies that* $f_n(z)\in K$ *for* $|z|<\frac{1}{8}$. *The result is sharp.*

Further, Padmanabhan [1] proved

Theorem 2. *If* $f(z)$ *is 2-valently starlike in* U, *then* $f_n(z)$ *is 2-valently starlike for* $|z|<\frac{1}{6}$. *The result is sharp.*

2 Function $F_n(z)$

Let us define the function $F_n(z)$ which is the partial sum of $f(z)\in A$ by

(5)
$$F_n(z)=z+a_nz^n.$$

Theorem 3. *The function* $F_n(z)$ *satisfies*

(6)
$$\frac{1-n|a_n|r^{n-1}}{1-|a_n|r^{n-1}}\leq \mathrm{Re}\left\{\frac{zF_n'(z)}{F_n(z)}\right\}\leq\frac{1+n|a_n|r^{n-1}}{1+|a_n|r^{n-1}}$$

for $0\leq r<\sqrt[n-1]{\frac{1}{|a_n|}}\leq 1$. *Therefore* $F_n(z)\in S^*(\alpha)$ *for* $0\leq r<\sqrt[n-1]{\frac{1-\alpha}{(n-\alpha)|a_n|}}\leq 1$.

Proof. Note that

(7)
$$\frac{zF_n'(z)}{F_n(z)}=\frac{z+na_nz^n}{z+a_nz^n}=n-\frac{n-1}{1+a_nz^{n-1}}.$$

It follows from (7) that

(8)
$$\mathrm{Re}\left\{\frac{zF_n'(z)}{F_n(z)}\right\}=n-(n-1)\frac{1+|a_n|r^{n-1}cos\theta}{1+|a_n|^2r^{2(n-1)}+2|a_n|r^{n-1}cos\theta}.$$

Since the righthand side of (8) is increasing for $cos\theta$ if $|a_n|<1$,we obtain (6). Further, we also see that

(9)
$$\mathrm{Re}\left\{\frac{zF_n'(z)}{F_n(z)}\right\}\geq\frac{1-n|a_n|r^{n-1}}{1-|a_n|r^{n-1}}>\alpha$$

for $0\leq r<\sqrt[n-1]{\frac{1-\alpha}{(n-\alpha)|a_n|}}\leq 1$. This completes the proof of the theorem.

□

Next we derive

Theorem 4. *The function $F_n(z)$ satisfies*

$$(10)\qquad \frac{1-n^2|a_n|r^{n-1}}{1-n|a_n|r^{n-1}} \leq \mathrm{Re}\left\{1+\frac{zF_n''(z)}{F_n'(z)}\right\} \leq \frac{1+n^2|a_n|r^{n-1}}{1+n|a_n|r^{n-1}}$$

for $0 \leq r < \sqrt[n-1]{\frac{1}{n|a_n|}} \leq 1$. *Therefore,* $F_n(z) \in K$ *for* $0 \leq r < \sqrt[n-1]{\frac{1-\alpha}{n(n-\alpha)|a_n|}} \leq 1$.

Proof. Noting that

$$(11)\qquad 1+\frac{zF_n''(z)}{F_n'(z)} = n - \frac{n-1}{1+na_nz^{n-1}},$$

we have

$$(12)\qquad \mathrm{Re}\left\{1+\frac{zF_n''(z)}{F_n'(z)}\right\} = n-(n-1)\frac{1+n|a_n|r^{n-1}cos\theta}{1+n^2|a_n|^2r^{2(n-1)}+2n|a_n|r^{n-1}cos\theta},$$

which derives (10).

□

By virtue of the above theorems, we have

Conjecture 1. *For the partial sum $f_n(z)$ of $f(z)$ belonging to the class A,*
(i)$f_n(z) \in S^*(\alpha)$ *for* $0 \leq r < \sqrt[n-1]{\frac{1-\alpha}{(n-\alpha)|a_n|}} \leq 1$,
and
(ii)$f_n(z) \in K(\alpha)$ *for* $0 \leq r < \sqrt[n-1]{\frac{1-\alpha}{n(n-\alpha)|a_n|}} \leq 1$.

3 The partial sums of certain analytic functions

In this section, we consider the partial sums of functions $f(z)=\frac{z}{1-z}$ and $f(z)=\frac{z}{(1-z)^2}$.

Theorem 5. *Let $f_3(z)=z+z^2+z^3$ be the partial sum of $f(z)=\frac{z}{1-z}$ which is the extremal function of the class K. Then $f_3(z) \in S^*(\frac{626}{961})$ for $0 \leq r < \beta$ $(\frac{1}{7} < \beta < \frac{1}{6})$, where β is the positive root of*

$$(13)\qquad x^4-8x^3+9x^2-8x+1=0 \qquad (0 < x < \frac{1}{\sqrt{3}}).$$

Proof. We consider α such that

$$(14)\qquad \mathrm{Re}\left\{\frac{zf_3'(z)}{f_3(z)}\right\} = \mathrm{Re}\left\{3-\frac{2+z}{1+z^2+z^3}\right\} > \alpha$$

for $0 \leq r < \beta$. This implies that

$$(15) \qquad \text{Re}\left\{\frac{2+z}{1+z^2+z^3}\right\} = 1 + \frac{(1-r^2)(1+r^2+rcos\theta)}{1-r^2 * r^4 + 4r^2cos^2\theta + 2r(1+r^2)cos\theta} < 3-\alpha,$$

that is, that

$$(16) \qquad \text{Re}\left\{\frac{(1-r^2)(1+r^2+rcos\theta}{1-r^2+r^4+4r^2cos^2\theta+2r(1+r^2cos\theta}\right\} < 2-\alpha.$$

Let the function $g(t)$ be given by

$$(17) \qquad g(t) = \frac{(1-r^2)(1+r^2+rt)}{1-r^2+r^4+4r^2t^2+2r(1+r^2)t} \qquad (t = cos\theta).$$

Then we have

$$(18) \qquad g'(t) = \frac{r(r+1)(r-1)(1+5r^2+r^4+4r^2t^2+8r(1+r^2)t)}{(1-r^2+r^4+4r^2t^2+2r(1+r^2)t)^2}.$$

Letting

$$(19) \qquad h(t) = 1+5r^2+r^4+4r^2t^2+8r(1+r^2)t,$$

we see that
$(i) h(t) < 0 \quad \Rightarrow \quad g'(t) > 0,$
$(ii) h(t) > 0 \quad \Rightarrow \quad g'(t) < 0,$
and
$(iii) h(t) = 0$ for $t = \frac{-2(1+r^2) \pm \sqrt{3(1+r^2+r^4}}{2r}$.
If we write

$$t_1 = \frac{-2(1+r^2)+\sqrt{3(1+r^2+r^4}}{2r} < 0,$$

then $0 \leq r \leq \beta$ implies that $t_1 \leq -1$, so that, $h(t) \geq 0$. This gives us that

$$(20) \qquad g(t) \leq g(-1) = \frac{1-r+r^3-r^4}{1-2r+3r^2-2r^3+r^4} = \frac{g_1(r)}{g_2(r)},$$

It is easy to check that $g_1(r)$ is decreasing for r $(0 \leq r < \frac{1}{\sqrt{3}})$. Therefore

$$(21) \qquad \frac{8-2\sqrt{3}}{9} = g_1(\frac{1}{\sqrt{3}}) < g_1(r) \leq g_1(0) = 1.$$

Also, $g_2(r)$ is decreasing for r $(0 \leq r < \beta)$, because $g_2'(0) = -2 < 0$ and $g_2'(\frac{1}{6}) = -\frac{31}{27} < 0$. This gives that

$$(22) \qquad \frac{961}{1296} = g_2(\frac{1}{6}) < g_2(r) \leq 1.$$

Consequently, we conclude that

$$(23) \qquad g(t) \le g(-1) = \frac{g_1(r)}{g_2(r)} < \frac{1296}{961} = 2 - \alpha,$$

that is, that $\alpha = \frac{626}{961} = 0.651 \cdots$.
Thus we have that

$$(24) \qquad \mathrm{Re}\left\{\frac{zf_3'(z)}{f_3(z)}\right\} > \alpha \qquad (\alpha = \frac{626}{961})$$

for $0 \le r < \beta$.

□

Finally, we obtain

Theorem 6. *Let $f_3(z)=z+2z^2+3z^3$ be the partial sum of the Keobe function $f(z)=\frac{z}{(1-z)^2}$ which is the extremal function for the class S^*. Then $f_3(z) \in K(\frac{3191}{15876})$ for $0 \le r < \beta$ ($\frac{1}{14} < \beta < \frac{1}{13}$), where β is the positive root of*

$$(25) \qquad 81x^4 - 162x^3 + 72x^2 - 18x + 1 = 0 \qquad (0 \le x < \frac{1}{3}).$$

Proof. Since

$$(26) \qquad \mathrm{Re}\left\{1 + \frac{zf_3''(z)}{f_3'(z)}\right\} = \mathrm{Re}\left\{3 - \frac{2(1+2z)}{1+4z+9z^2}\right\} > \alpha$$

implies that

(27)

$$\mathrm{Re}\left\{\frac{1 * 2z}{1+4z+9z^2}\right\} = \frac{1}{2} + \frac{4r(1-9r^2)cos\theta * 1 - 81r^4}{2(1-2r^2+81r^4+8r(1+9r^2)cos\theta+36r^2cos^2\theta} < \frac{3-\alpha}{2},$$

we have to check that

$$(28) \qquad \frac{(1-9r^2)(1+9r^2+4rcos\theta)}{1-2r^2+81r^4+8r(1+9r^2)cos\theta+36r^2cos^2\theta} < 2 - \alpha.$$

If we let

$$(29) \qquad h(t) = \frac{(1-9r^2)(1+9r^2+4rt)}{1-2r^2+81r^4+8r(1+9r^2)t+36r^2t^2},$$

then we have

$$h(t) \le h(-1) = \frac{1-4r+36r^3-81r^4}{1-8r+34r^2-72r^3+81r^4} \equiv \frac{g_1(r)}{g_2(r)}.$$

Noting that $0 < g_1(r) < 1$, and $g_2(r) > g_2(\frac{1}{13}) = \frac{15876}{28561}$, we have

$$h(t) \le h(-1) < \frac{1}{g_2(r)} < \frac{28561}{15876} = 2 - \alpha,$$

which implies that $\alpha = \frac{3191}{15876} = 0.200 \cdots$.

□

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