<table>
<thead>
<tr>
<th>Title</th>
<th>Partial Sums of Certain Analytic Functions</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>Owa, Shigeyoshi</td>
</tr>
<tr>
<td>Citation</td>
<td>数理解析研究所講究録 (1999), 1112: 92-97</td>
</tr>
<tr>
<td>Issue Date</td>
<td>1999-09</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/63357">http://hdl.handle.net/2433/63357</a></td>
</tr>
<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
</tr>
</tbody>
</table>

Kyoto University
Partial Sums of Certain Analytic Functions

SHIGEYOSHI OWA

Abstract. The object of the present paper is to consider of starlikeness and convexity of partial sums of certain analytic functions in the open unit disk

1 Introduction

Let $A$ denote the class of functions $f(z)$ of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k$$

which are analytic in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$. Let $S^*(\alpha)$ be the subclass of $A$ consisting of functions $f(z)$ which satisfy

$$\Re \left\{ \frac{zf'(z)}{f(z)} \right\} > \alpha \quad (z \in U)$$

for some $\alpha(0 \leq \alpha < 1)$. A function $f(z)$ in $S^*(\alpha)$ is said to be starlike of order $\alpha$ in $U$. Furthermore, let $K(\alpha)$ denote the subclass of $A$ consisting of all functions $f(z)$ which satisfy

$$\Re \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > \alpha \quad (z \in U)$$

for some $\alpha(0 \leq \alpha < 1)$. A function $f(z)$ belonging to $K(\alpha)$ is said to be convex of order $\alpha$ in $U$. We note that $f(z) \in S^*(\alpha)$ if and only if $zf'(z) \in K(\alpha)$ and denote by $S^*(0) \equiv S^*$ and $K(0) \equiv K$. For $f(z) \in A$, we introduce the partial sum of $f(z)$ by

$$f_n(z) = z + \sum_{k=2}^{n} a_k z^k.$$ 

Mathematics Subject Classification1991:30C45

Key Words and Phrases:Partial sum, Starlike of order $\alpha$, Convex of order $\alpha$
Remark 1. It is well-known that
(i) \( f(z) = \frac{z}{(1-z)^2} = z + \sum_{k=2}^{\infty} k z^k \) is the extremal function for the class \( S^* \). But \( f_2(z) = z + 2z^2 \notin S^* \).
(ii) \( f(z) = \frac{z}{1-z} = z + \sum_{k=2}^{\infty} z^k \) is the extremal function for the class \( K \). But \( f_2(z) = z + z^2 \notin K \).

For the partial sums \( f_n(z) \) of \( f(z) \in S^* \), Szegö [2] showed

\[ (2) \]

Theorem 1. (i) \( f(z) \in S^* \) implies that \( f_n(z) \in S^* \) for \( |z| < \frac{1}{4} \). The result is sharp.
(ii) \( f(z) \in S^* \) implies that \( f_n(z) \in K \) for \( |z| < \frac{1}{8} \). The result is sharp.

Further, Padmanabhan [1] proved

Theorem 2. If \( f(z) \) is 2-valently starlike in \( U \), then \( f_n(z) \) is 2-valently starlike for \( |z| < \frac{1}{6} \). The result is sharp.

2 Function \( F_n(z) \)

Let us define the function \( F_n(z) \) which is the partial sum of \( f(z) \in A \) by

\[ (5) \]

\( F_n(z) = z + a_n z^n \).

Theorem 3. The function \( F_n(z) \) satisfies

\[ (6) \]

\[ \frac{1 - n |a_n| r^{n-1}}{1 - |a_n| r^{n-1}} \leq \text{Re} \left\{ \frac{z F'_n(z)}{F_n(z)} \right\} \leq \frac{1 + n |a_n| r^{n-1}}{1 + |a_n| r^{n-1}} \]

for \( 0 \leq r < \sqrt[n-1]{\frac{1}{|a_n|}} \leq 1 \). Therefore \( F_n(z) \in S^*(\alpha) \) for \( 0 \leq r < \sqrt[n-1]{\frac{1-\alpha}{(n-\alpha)|a_n|}} \leq 1 \).

Proof. Note that

\[ (7) \]

\[ \frac{z F'_n(z)}{F_n(z)} = \frac{z + na_n z^n}{z + a_n z^n} = n - \frac{n - 1}{1 + a_n z^{n-1}}. \]

It follows from (7) that

\[ (8) \]

\[ \text{Re} \left\{ \frac{z F'_n(z)}{F_n(z)} \right\} = n - (n - 1) \frac{1 + |a_n| r^{n-1} \cos \theta}{1 + |a_n|^2 r^{2(n-1)} + 2 |a_n| r^{n-1} \cos \theta}. \]

Since the righthand side of (8) is increasing for \( \cos \theta \) if \( |a_n| < 1 \), we obtain (6).

Further, we also see that

\[ (9) \]

\[ \text{Re} \left\{ \frac{z F'_n(z)}{F_n(z)} \right\} \geq \frac{1 - n |a_n| r^{n-1}}{1 - |a_n| r^{n-1}} > \alpha \]

for \( 0 \leq r < \sqrt[n-1]{\frac{1-\alpha}{(n-\alpha)|a_n|}} \leq 1 \). This completes the proof of the theorem.
Next we derive

**Theorem 4.** The function $F_n(z)$ satisfies

\[(10) \quad \frac{1 - n^2 |a_n| r^{n-1}}{1 - n |a_n| r^{n-1}} \leq \text{Re} \left\{ 1 + \frac{z F''_n(z)}{F'_n(z)} \right\} \leq \frac{1 + n^2 |a_n| r^{n-1}}{1 + n |a_n| r^{n-1}}\]

for $0 \leq r < \sqrt{\frac{1}{n|a_n|}}$. Therefore, $F_n(z) \in K$ for $0 \leq r < \sqrt{\frac{1 - \alpha}{n(n - \alpha)|a_n|}} \leq 1$.

**Proof.** Noting that

\[(11) \quad 1 + \frac{z F''_n(z)}{F'_n(z)} = n - \frac{n - 1}{1 + n a_n z^{n-1}},\]

we have

\[(12) \quad \text{Re} \left\{ 1 + \frac{z F''_n(z)}{F'_n(z)} \right\} = n - (n - 1) \frac{1 + n |a_n| r^{n-1} \cos \theta}{1 + n^2 |a_n|^2 r^{2(n-1)} + 2 n |a_n| r^{n-1} \cos \theta'},\]

which derives (10).

By virtue of the above theorems, we have

**Conjecture 1.** For the partial sum $f_n(z)$ of $f(z)$ belonging to the class $A$,

(i) $f_n(z) \in S^*(\alpha)$ for $0 \leq r < \sqrt{\frac{1 - \alpha}{(n - \alpha)|a_n|}} \leq 1$,

and (ii) $f_n(z) \in K(\alpha)$ for $0 \leq r < \sqrt{\frac{1 - \alpha}{n(n - \alpha)|a_n|}} \leq 1$.

**3 The partial sums of certain analytic functions**

In this section, we consider the partial sums of functions $f(z) = \frac{z}{1 - z}$ and $f(z) = \frac{z}{(1 - z)^2}$.

**Theorem 5.** Let $f_3(z) = z + z^2 + z^3$ be the partial sum of $f(z) = \frac{z}{1 - z}$ which is the extremal function of the class $K$. Then $f_3(z) \in S^*(\frac{828}{961})$ for $0 \leq r < \beta$ ($\frac{1}{7} < \beta < \frac{1}{6}$), where $\beta$ is the positive root of

\[(13) \quad x^4 - 8x^3 + 9x^2 - 8x + 1 = 0 \quad (0 < x < \frac{1}{\sqrt{3}}).\]

**Proof.** We consider $\alpha$ such that

\[(14) \quad \text{Re} \left\{ \frac{zf_3'(z)}{f_3(z)} \right\} = \text{Re} \left\{ 3 - \frac{2 + z}{1 + z^2 + z^3} \right\} > \alpha\]
for $0 \leq r < \beta$. This implies that

$$\operatorname{Re}\left\{\frac{2+z}{1+z^2+z^3}\right\} = 1 + \frac{(1-r^2)(1 + r^2 + r \cos \theta)}{1 - r^2 + r^4 + 4r^2 \cos^2 \theta + 2r(1 + r^2) \cos \theta} < 3 - \alpha, \tag{15}$$

that is, that

$$\operatorname{Re}\left\{\frac{(1-r^2)(1 + r^2 + r \cos \theta)}{1 - r^2 + r^4 + 4r^2 \cos^2 \theta + 2r(1 + r^2) \cos \theta}\right\} < 2 - \alpha. \tag{16}$$

Let the function $g(t)$ be given by

$$g(t) = \frac{(1-r^2)(1 + r^2 + rt)}{1 - r^2 + r^4 + 4r^2 t^2 + 2r(1 + r^2)t} \quad (t = \cos \theta). \tag{17}$$

Then we have

$$g'(t) = \frac{r(r+1)(r-1)(1+5r^2+r^4+4r^2t^2+8r(1+r^2)t)}{(1 - r^2 + r^4 + 4r^2 t^2 + 2r(1 + r^2)t)^2}. \tag{18}$$

Letting

$$h(t) = 1 + 5r^2 + r^4 + 4r^2 t^2 + 8r(1 + r^2)t, \tag{19}$$

we see that

(i) $h(t) < 0 \implies g'(t) > 0,$

(ii) $h(t) > 0 \implies g'(t) < 0,$

and

(iii) $h(t) = 0$ for $t = \frac{-2(1+r^2) \pm \sqrt{3(1+r^2+r^4)}}{2r}.$

If we write

$$t_1 = \frac{-2(1+r^2) + \sqrt{3(1+r^2+r^4)}}{2r} < 0,$$

then $0 \leq r \leq \beta$ implies that $t_1 \leq -1$, so that, $h(t) \geq 0$. This gives us that

$$g(t) \leq g(-1) = \frac{1 - r + r^3 - r^4}{1 - 2r + 3r^2 - 2r^3 + r^4} = \frac{g_1(r)}{g_2(r)}. \tag{20}$$

It is easy to check that $g_1(r)$ is decreasing for $r \in (0, \frac{1}{\sqrt{3}})$. Therefore

$$\frac{8 - 2\sqrt{3}}{9} = g_1\left(\frac{1}{\sqrt{3}}\right) < g_1(r) \leq g_1(0) = 1. \tag{21}$$

Also, $g_2(r)$ is decreasing for $r \in (0, \beta)$, because $g_2'(0) = -2 < 0$ and $g_2\left(\frac{1}{6}\right) = \frac{-31}{27} < 0$. This gives that

$$\frac{961}{1296} = g_2\left(\frac{1}{6}\right) < g_2(r) \leq 1. \tag{22}$$
Consequently, we conclude that

\[(23) \quad g(t) \leq g(-1) = \frac{g_1(r)}{g_2(r)} < \frac{1296}{961} = 2 - \alpha,\]

that is, that \(\alpha = \frac{626}{961} = 0.651 \ldots\). Thus we have that

\[(24) \quad \text{Re} \left\{ \frac{zf_3''(z)}{f_3'(z)} \right\} > \alpha \quad (\alpha = \frac{626}{961})\]

for \(0 \leq r < \beta\).

Finally, we obtain

**Theorem 6.** Let \(f_3(z) = z + 2z^2 + 3z^3\) be the partial sum of the Keobe function \(f(z) = \frac{z}{(1-z)^2}\) which is the extremal function for the class \(S^*\). Then \(f_3(z) \in K(\frac{3191}{15876})\) for \(0 \leq r < \beta\) \((\frac{1}{14} < \beta < \frac{1}{13})\), where \(\beta\) is the positive root of

\[(25) \quad 81x^4 - 162x^3 + 72x^2 - 18x + 1 = 0 \quad (0 \leq x < \frac{1}{3}).\]

**Proof.** Since

\[(26) \quad \text{Re} \left\{ 1 + \frac{zf_3''(z)}{f_3'(z)} \right\} = \text{Re} \left\{ 3 - \frac{2(1+2z)}{1+4z+9z^2} \right\} > \alpha\]

implies that

\[(27) \quad \text{Re} \left\{ \frac{1 * 2z}{1 + 4z + 9z^2} \right\} = \frac{1}{2} + \frac{4r(1 - 9r^2) \cos \theta \ast 1 - 81r^4}{2(1 - 2r^2 + 81r^4 + 8r(1 + 9r^2) \cos \theta + 36r^2 \cos^2 \theta)} < \frac{3 - \alpha}{2},\]

we have to check that

\[(28) \quad \frac{(1 - 9r^2)(1 + 9r^2 + 4r \cos \theta)}{1 - 2r^2 + 81r^4 + 8r(1 + 9r^2) \cos \theta + 36r^2 \cos^2 \theta} < 2 - \alpha.\]

If we let

\[(29) \quad h(t) = \frac{(1 - 9r^2)(1 + 9r^2 + 4rt)}{1 - 2r^2 + 81r^4 + 8r(1 + 9r^2)t + 36r^2t^2},\]

then we have

\[h(t) \leq h(-1) = \frac{1 - 4r + 36r^3 - 81r^4}{1 - 8r + 34r^2 - 72r^3 + 81r^4} = \frac{g_1(r)}{g_2(r)},\]

Noting that \(0 < g_1(r) < 1\), and \(g_2(r) > g_2(\frac{1}{13}) = \frac{15876}{28561}\), we have

\[h(t) \leq h(-1) < \frac{1}{g_2(r)} < \frac{28561}{15876} = 2 - \alpha,\]

which implies that \(\alpha = \frac{3191}{15876} = 0.200 \ldots\).
References


Shigeyoshi Owa  
*Department of Mathematics*  
*Kinki University*  
*Higashi-Osaka, Osaka 577-8502*  
*Japan*