On Several Properties of Spiral-like Functions

Yūsuke Okuyama*
Department of Mathematics, Graduate School of Science
Kyoto University, Kyoto 606-8502, Japan
E-mail: okuyama@kusm.kyoto-u.ac.jp

March 17, 1999

1991 Mathematics Subject Classification. 30C45

Keywords and phrases. spiral-like function, pre-Schwarzian derivative, Schwarzian derivative, growth estimate, strongly normalized univalent function

1 Introduction

We consider an analytic function \(f\) on the unit disk \(\mathbb{D}\) normalized so that \(f(0) = f'(0) - 1 = 0\). For a constant \(\beta \in (-\pi/2, \pi/2)\), such a function \(f\) is called \(\beta\)-spiral-like if \(f\) is univalent on \(\mathbb{D}\) and for any \(z \in \mathbb{D}\), the \(\beta\)-logarithmic spiral \(\{f(z)\exp(-e^{i\beta}t); t \geq 0\}\) is contained in \(f(\mathbb{D})\). It is equivalent to the analytic condition that \(\Re(e^{-i\beta}zf'(z)/f(z)) > 0\) in \(\mathbb{D}\). We denote by \(SP(\beta)\) the set of \(\beta\)-spiral-like functions. We call \(f_\beta(z):=z(1-z)^{-2e^{i\beta}\cos\beta}\in SP(\beta)\) the \(\beta\)-spiral Koebe function. Note that \(SP(0)\) is the set of starlike functions and that \(f_0(z) = z(1-z)^{-2}\) is the Koebe function. The \(\beta\)-spiral Koebe function conformally maps the unit disk onto the complement of the \(\beta\)-logarithmic spiral \(\{f_\beta(-e^{-2i\beta})\exp(-e^{i\beta}t); t \leq 0\}\) in \(\mathbb{C}\). For the known results about these classes of the functions, see, for example, [1].

2 Norm estimates

For a locally univalent holomorphic function \(f\), we define

\[
T_f = \frac{f''}{f'} \quad \text{and} \quad S_f = (T_f)' - \frac{1}{2}(T_f)^2,
\]

*Partially supported by JSPS Research Fellowships for Young Scientists
which are said to be the pre-Schwarzian derivative (or nonlinearity) and the Schwarzian derivative of \( f \), respectively. For a locally univalent function \( f \) in \( \mathbb{D} \), we define the norms of \( T_f \) and \( S_f \) by

\[
||T_f||_1 = \sup_{z \in \mathbb{D}} (1 - |z|^2) |T_f(z)| \quad \text{and} \quad ||S_f||_2 = \sup_{z \in \mathbb{D}} (1 - |z|^2)^2 |S_f(z)|,
\]

respectively.

As well as \( ||S_f||_2 \), the norm \( ||T_f||_1 \) has a significant meaning in the theory of Teichmüller spaces. For example, see [8], [2] and [13]. We shall give the best possible estimate of the norms of pre-Schwarzian derivatives for the class \( SP(\beta) \).

**Main Theorem 1 ([9]).** For any \( f \in SP(\beta) \), where \( \beta \in (-\pi/2, \pi/2) \), we have the following.

1) In the case \( |\beta| \leq \pi/3 \), we have

\[
||T_f||_1 \leq ||T_{f_{\beta}}||_1 = 2|2 + e^{2i\beta}|. \quad (1)
\]

II) In the case \( |\beta| > \pi/3 \), we have \( ||T_f||_1 \leq ||T_{f_{\beta}}||_1 \), where

\[
||T_{f_{\beta}}||_1 = \max_{0 \leq m \leq \frac{4\beta}{3}} 2m \cos \beta \left( 1 + \sqrt{\frac{m^2 + 4 - 4m \sin |\beta|}{m^2 + 1 - 2m \sin |\beta|}} \right) \quad \text{and} \quad (2)
\]

\[
2|2 + e^{2i\beta}| < ||T_{f_{\beta}}||_1 < 2 \left( 1 + \frac{4}{3} \sin 2|\beta| \right). \quad (3)
\]

In particular, \( ||T_{f_{\beta}}||_1 \rightarrow 2 \) as \( |\beta| \rightarrow \pi/2 \).

In both cases, the equality \( ||T_f||_1 = ||T_{f_{\beta}}||_1 \) holds if and only if \( f \) is a rotation of the \( \beta \)-spiral Koebe function, i.e., \( f(z) = (1/\epsilon)f_{\beta}(\epsilon z) \) for some \( |\epsilon| = 1 \).

The proof of Main Theorem 1 is in [9]. From the proof, if \( |\beta| \leq \pi/3 \), the function \( (1 - |z|^2)|T_{f_{\beta}}(z)| \) does not attain its supremum in \( \mathbb{D} \). However if \( |\beta| > \pi/3 \), it does since

\[
\max_{\partial \mathbb{D} \ni z_0} \lim_{\mathbb{D} \ni z \rightarrow z_0} \sup_{z_0} (1 - |z|^2)|T_{f_{\beta}}(z)| = 2|2 + e^{2i\beta}| < ||T_{f_{\beta}}||_1.
\]

This phenomenon of phase transition seems to be quite interesting.

**Remark.** Clearly, the \( \beta \)-spiral Koebe function \( f_{\beta} \) converges to \( id_{\mathbb{D}} \) (which is bounded) locally uniformly on \( \mathbb{D} \) as \( |\beta| \rightarrow \pi/2 \) but does not converge to it with respect to the norm \( ||\cdot||_1 \) since \( \lim_{|\beta| \rightarrow \pi/2} ||T_{f_{\beta}}||_1 = 2 \). On the other hand, it is known that a normalized analytic function \( f \) is bounded if \( ||T_f||_1 < 2 \). In fact, the value 2 is the least one of the norms of unbounded normalized analytic functions.
We would also like to mention the related works about norm estimates of pre-Schwarzian derivatives in other classes by Shinji Yamashita [11] and Toshiyuki Sugawa [10].

**Theorem 2.1.** Let $0 \leq \alpha < 1$ and $f$ be a normalized analytic function.

If $f$ is starlike of order $\alpha$, i.e., $\Re(zf'(z)/f(z)) > \alpha$, then $\|T_f\|_1 \leq 6 - 4\alpha$.

If $f$ is convex of order $\alpha$, i.e., $\Re(1+zf''(z)/f'(z)) > \alpha$, then $\|T_f\|_1 \leq 4(1-\alpha)$.

If $f$ is strongly starlike of order $\alpha$, i.e., $\arg(zf'(z)/f(z)) < \pi\alpha/2$, then $\|T_f\|_1 \leq M(\alpha) + 2\alpha$, where $M(\alpha)$ is a specified constant depending only on $\alpha$ satisfying $2\alpha < M(\alpha) < 2\alpha(1+\alpha)$.

All of the bounds are sharp.

On the other hand, we also obtain the estimate of the norms of Schwarzian derivatives of $\beta$-spiral-like functions.

**Main Theorem 2 ([9]).** Assume $|\beta| < \pi/2$. For any $f \in SP(\beta)$, $\|S_f\|_2 \leq \|S_{f_\beta}\|_2 = 6$.

**Proof.** From direct calculation, it follows that

$$S_{f_\beta} = (T_{f_\beta})' - \frac{1}{2}(T_{f_\beta})^2$$

$$= -c\frac{e^{2i\beta}(e^{2i\beta}-1)z^2 + 4(e^{2i\beta}-1)z + 6}{2(1-z)^2(1+ze^{2i\beta})^2}$$

and that

$$(1-|z|^2)^2|S_{f_\beta}(z)| = |c|\frac{(1-|z|^2)^2|e^{2i\beta}(e^{2i\beta}-1)z^2 + 4(e^{2i\beta}-1)z + 6|}{2|1-z|^2|1+ze^{2i\beta}|^2}.$$

We can easily see that $(1-|z|^2)^2|S_{f_\beta}(z)| \rightarrow 6$ as $z \rightarrow -e^{-2i\beta}$ radially. By the Kraus-Nehari theorem, we obtain $\|S_{f_\beta}\|_2 = 6$ and the extremality of $f_\beta$ in $SP(\beta)$ for any $|\beta| < \pi/2$. 

\hfill \Box

3 Order estimates of the coefficients

Knowing the norm $\|T_f\|_1$ enables us to estimate the growth of coefficients of $f$. For example, the following holds.

**Theorem 3.1 (cf. [7]).** Let $(3/2) < \lambda \leq 3$. For a normalized analytic function $f(z) = z + a_2z^2 + a_3z^3 + \cdots$ such that $\|T_f\|_1 \leq 2\lambda$, it holds that $a_n = O(n^{\lambda-2})$ as $n \rightarrow +\infty$. This order estimate is best possible.
However the sharp estimate of coefficients of $f \in \text{SP}(\beta)$ has been already obtained by Zamorski [12] in 1960. We would like to remark that we can derive the sharp growth estimate of coefficients of $f \in \text{SP}(\beta)$ from this.

**Theorem 3.2 (Zamorski).** If $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots$ is in $\text{SP}(\beta)$ and $|\beta| < \pi/2$, then

$$|a_n| \leq \prod_{k=1}^{n-1} \left| 1 + \frac{e^{2i\beta}}{k} \right|$$

(4)

for any $n \geq 2$. The equality in (4) holds for some $n \geq 2$ if and only if $f$ is a rotation of the $\beta$-spiral Koebe function $f_\beta$.

**Remark.** This is also shown in terms of generalized spiral-like functions by C. Burniak, J. Stankiewicz and Z. Stankiewicz [4](1980).

**Corollary 3.1.** Let $|\beta| < \pi/2$ and $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots$ be a $\beta$-spiral-like function. Then it holds that

$$a_n = O(n^{\cos 2\beta}) \quad (n \to +\infty).$$

(5)

This order estimate is sharp.

**Proof.** From the inequality (4), we have that for $|\beta| < \pi/2$,

$$\log |a_n| \leq \frac{1}{2} \sum_{k=1}^{n-1} \log \left( 1 + \frac{2 \cos 2\beta}{k} + \frac{1}{k^2} \right)$$

$$= \frac{1}{2} \sum_{k=1}^{n-1} \left( \frac{2 \cos 2\beta}{k} \right) + O(1)$$

$$= \cos 2\beta \log n + O(1)$$

as $n \to +\infty$. Therefore we obtain the estimate (5).

**Remark.** In the case $|\beta| < \pi/4$, this is shown by Basgöze and Keogh in [3](1970).

4 **Strongly normalized univalent functions are not always holomorphic.**

The following is known.
Theorem 4.1. For a holomorphic function $\phi$ on a simply connected domain $A$, there exists a locally univalent meromorphic function $f$ on $A$ such that

$$S_f = \phi.$$  

The solution is unique up to postcomposition of an arbitrary Möbius transformation.

We assume $A = \mathbb{D}$. Nehari showed that if $\|\phi\|_2 = \sup_{z \in \mathbb{D}} |\phi(z)|(1 - |z|^2)^2 \leq 2$, then $f$ is univalent (meromorphic) on $\mathbb{D}$. It is well-known that if $\|\phi\|_2 = \sup_{z \in \mathbb{D}} |\phi(z)|(1 - |z|^2)^2 \leq 2$ and $f$ is the strongly normalized solution, i.e., $f(0) = f'(0) - 1 = f''(0) = 0$, then $f$ is holomorphic on $\mathbb{D}$. Since for a normalized analytic function $f(z) = z + a_2 z^2 + \cdots$, $g := f/(a_2 f + 1)$ is strongly normalized and $\|S_f\|_2 = \|S_g\|_2$, we have the following.

Proposition 4.1 ([6] and [5] Corollary 2.). If a normalized analytic function $f(z) = z + a_2 z^2 + \cdots$ satisfies $\|S_f\|_2 \leq 2$, then $f$ is univalent and $a_2 f + 1 \neq 0$ on $\mathbb{D}$.

In [5] Chuaqui and Osgood remark that a strongly normalized univalent function $f$ is not always holomorphic if $\|S_f\|_2 > 2$. Spiral-like functions are examples for this fact.

Theorem 4.2. If $|\beta|$ is sufficiently close to $\pi/2$, the $\beta$-spiral-Koebe function $f_\beta(z) = z + a_2 z^2 + \cdots$ satisfies $a_2 f_\beta(z) + 1 = 0$ for some $z \in \mathbb{D}$.

Proof. By direct calculation, we have $a_2 = f_\beta'(0)/2 = e^{2i\beta} + 1$. The $\beta$-logarithmic spiral $\{f_\beta(-e^{-2i\beta}) \exp(e^{i\beta}t); t \geq 0\}$ is the complement of $f_\beta(\mathbb{D})$ in $\mathbb{C}$. Thus $a_2 f_\beta(z) + 1 \neq 0$ on $\mathbb{D}$ if and only if this spiral contains $-1/a_2$. We can see that if $f_\beta(-e^{-2i\beta}) \exp(e^{i\beta}t) = -1/a_2$, then

$$t = e^{i\beta} \log(1 + e^{-2i\beta}).$$

and that the imaginary part of the right side of (6) tends to $-\infty$ (resp. $+\infty$) if $\beta$ tends to $+\pi/2$ (resp. $-\pi/2$).

References


