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ON THE STRONGLY STARLIKENESS OF MULTIVALENTLY CONVEX FUNCTIONS OF ORDER α

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Let $\mathcal{A}(p)$ denote the class of functions $f(z)=z^p+\sum_{n=p+1}^{\infty}a_nz^n$ which are analytic in the open unit disc $\mathcal{E}=\{z:|z|<1\}$. A function $f(z)\in\mathcal{A}(p)$ is called to be p-valently starlike if and only if the inequality

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\}>0$$

holds for $z\in\mathcal{E}$. A function $f(z)\in\mathcal{A}(p)$ is called p-valently convex of order α $(0\leq\alpha<p)$ if and only if the inequality

$$1+\operatorname{Re}\left\{\frac{zf''(z)}{f'(z)}\right\}>\alpha$$

holds for $z\in\mathcal{E}$. We denote by $\mathcal{C}(p,\alpha)$ the family of such functions. A function $f(z)\in\mathcal{A}(p)$ is said to be strongly starlike of order α $(0<\alpha\leq 1)$ if and only if the inequality

$$\left|\arg\left\{\frac{zf'(z)}{f(z)}\right\}\right|<\frac{\pi}{2}\alpha$$

holds for $z\in\mathcal{E}$. We also denote by $\mathcal{STS}(p,\alpha)$ the family of functions which are strongly starlike of order α. From the definition, it follows that if $f(z)\in\mathcal{STS}(p,\alpha)$, then we have

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\}>0 \qquad \text{in} \qquad \mathcal{E}$$

or $f(z)$ is p-valently starlike in $\mathcal{E}$ and therefore $f(z)$ is p-valent in $\mathcal{E}$ [1, Lemma 7].

Nunokawa [2,3] proved the following theorems.

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Theorem A. [2] *If* $f(z) \in \mathcal{A}(p)$ *satisfies*

$$1 + \operatorname{Re}\left\{\frac{zf''(z)}{f'(z)}\right\} < p + \frac{\alpha}{2}$$

where $0 < \alpha \leq 1$, *then* $f(z) \in \mathcal{STS}(p, \alpha)$.

Theorem B. [3] *If* $f(z) \in \mathcal{A}(1)$ *satisfies*

$$\left|\arg\left\{1 + \frac{zf''(z)}{f'(z)}\right\}\right| < \frac{\pi}{2}\alpha(\beta) \qquad in \qquad \mathcal{E},$$

then we have

$$\left|\arg\left\{\frac{zf'(z)}{f(z)}\right\}\right| < \frac{\pi}{2}\beta \qquad in \qquad \mathcal{E},$$

where

$$\alpha(\beta) = \beta + \frac{2}{\pi}\mathrm{Tan}^{-1}\left\{\frac{\beta q(\beta)\sin\frac{\pi}{2}(1-\beta)}{p(\beta) + \beta q(\beta)\cos\frac{\pi}{2}(1-\beta)}\right\}$$

$$p(\beta) = (1+\beta)^{\frac{1+\beta}{2}} \qquad and \qquad q(\beta) = (1-\beta)^{\frac{\beta-1}{2}}.$$

It is the purpose of the present paper to prove that if

$$f(z) \in \mathcal{C}\left(1, 1 - \frac{\alpha}{2}\right),$$

then $f(z) \in \mathcal{STS}\,(1, \alpha)$.

In this paper, we need the following lemma.

Lemma 1. *If* $f(z) \in \mathcal{A}(1)$ *be starlike with respect to the origin in* $\mathcal{E}$. *Let* $C(r, \theta) = \{f(te^{i\theta}) : 0 \leq t \leq r < 1\}$ *and* $\mathcal{T}(r, \theta)$ *be the total variation of* $\arg f(te^{i\theta})$ *on* $C(r, \theta)$, *so that*

$$\mathcal{T}(r, \theta) = \int_0^r \left|\frac{\partial}{\partial t}\arg\{f(te^{i\theta})\}\right| dt.$$

Then we have

$$\mathcal{T}(r, \theta) < \pi.$$

We owe this lemma to Sheil-Small [6, Theorem 1].

Main Theorem. *Let* $f(z) \in \mathcal{A}(1)$ *and*

(1) $$1 + \operatorname{Re}\left\{\frac{zf''(z)}{f'(z)}\right\} > 1 - \frac{\alpha}{2} \quad in \quad \mathcal{E},$$

where $0 < \alpha \leq 1$. *Then we have*

$$\left|\arg\left\{\frac{zf'(z)}{f(z)}\right\}\right| < \frac{\pi}{2}\alpha \quad in \quad \mathcal{E},$$

or $f(z)$ *is strongly starlike of order* α *in* $\mathcal{E}$.

Proof. Let us put

(2) $$\frac{2}{\alpha}\left\{1 + \frac{zf''(z)}{f'(z)} - 1 + \frac{\alpha}{2}\right\} = \frac{zg'(z)}{g(z)}$$

where $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$. From the assumption (1), we have

$$\operatorname{Re}\left\{\frac{zg'(z)}{g(z)}\right\} > 0 \quad \text{in} \quad \mathcal{E}.$$

This shows that $g(z)$ is starlike and univalent in $\mathcal{E}$. With an easy calculation (see e.g. [4]), the equality (2) gives us that

$$f'(z) = \left\{\frac{g(z)}{z}\right\}^{\frac{\alpha}{2}}.$$

Since

$$f'(z) \neq 0 \quad \text{in} \quad 0 < |z| < 1,$$

we easily have

(3) $$\begin{aligned}\frac{f(z)}{zf'(z)} &= \int_0^1 \frac{f'(tz)}{f'(z)}dt \\ &= \int_0^1 t^{-\frac{\alpha}{2}}\left\{\frac{g(tre^{i\theta})}{g(re^{i\theta})}\right\}^{\frac{\alpha}{2}} dt\end{aligned}$$

where $z = re^{i\theta}$ and $0 < r < 1$. Since $g(z)$ is starlike in $\mathcal{E}$, from Lemma 1, we have

(4) $$-\pi < \arg\left\{g(tre^{i\theta})\right\} - \arg\left\{g(re^{i\theta})\right\} < \pi$$

for $0 < t \leq 1$. Putting

$$\xi = \left\{\frac{g(tre^{i\theta})}{g(re^{i\theta})}\right\}^{\frac{\alpha}{2}},$$

we have

$$\arg s = \frac{\alpha}{2} \arg \left\{ \frac{g(tre^{i\theta})}{g(re^{i\theta})} \right\}. \tag{5}$$

From (4) and (5), s lies in convex sector

$$\left\{ s : |\arg s| \leq \frac{\pi}{2}\alpha \right\}$$

and the same is true of its integral mean of (3), (see e.g. [5, Lemma 1]). Therefore we have

$$\left| \arg \left\{ \frac{f(z)}{zf'(z)} \right\} \right| < \frac{\pi}{2}\alpha \qquad \text{in} \qquad \mathcal{E},$$

or

$$\left| \arg \left\{ \frac{zf'(z)}{f(z)} \right\} \right| < \frac{\pi}{2}\alpha \qquad \text{in} \qquad \mathcal{E}.$$

This shows that

$$\mathrm{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} > 0 \qquad \text{in} \qquad \mathcal{E},$$

which completes the proof of our main theorem.

Remark. This result is sharp for the case $\alpha \to 0$ and $\alpha = 1$.

(a) For the case $\alpha \to 0$, let us put $f(z) = z$, then $f(z)$ is a convex function of order $1 - \dfrac{\alpha}{2} \to 1$ and $f(z)$ is a strongly starlike function of order $\alpha \to 0$.

(b) For the case $\alpha = 1$, let us put

$$1 + \frac{zf''(z)}{f'(z)} = \frac{1}{1-z}. \tag{6}$$

Then we have

$$1 + \mathrm{Re} \left\{ \frac{zf''(z)}{f'(z)} \right\} > \frac{1}{2} \qquad \text{in} \qquad \mathcal{E},$$

and therefore $f(z)$ is a convex function of order $1/2$. From (5), we easily have

$$f'(z) = \frac{1}{1-z} \qquad \text{and} \qquad f(z) = \log \left\{ \frac{1}{1-z} \right\}.$$

Putting $|z| = 1, z = e^{i\theta}, 0 \leq \theta < 2\pi$, then it follows that

$$\frac{z}{1-z} = -\frac{1}{2} + i \frac{\cos \dfrac{\theta}{2}}{2 \sin \dfrac{\theta}{2}}$$

and

$$\log\left\{\frac{1}{1-z}\right\} = \log\left|\frac{1}{2} + i\frac{\cos\frac{\theta}{2}}{2\sin\frac{\theta}{2}}\right| + i\arg\left\{\frac{1}{2} + i\frac{\cos\frac{\theta}{2}}{2\sin\frac{\theta}{2}}\right\}.$$

$$\lim_{\substack{\theta\to+0\\ z=e^{i\theta}}} \arg\frac{zf'(z)}{f(z)} = \lim_{\substack{\theta\to+0\\ z=e^{i\theta}}} \arg\left\{\frac{\frac{z}{1-z}}{\log\frac{1}{1-z}}\right\}$$

$$= \lim_{\substack{\theta\to+0\\ z=e^{i\theta}}} \arg\left\{-\frac{1}{2} + i\frac{\cos\frac{\theta}{2}}{2\sin\frac{\theta}{2}}\right\}$$

$$-\lim_{\substack{\theta\to+0\\ z=e^{i\theta}}} \arg\left\{\log\left|\frac{1}{2} + i\frac{\cos\frac{\theta}{2}}{2\sin\frac{\theta}{2}}\right| + i\arg\left(\frac{1}{2} + i\frac{\cos\frac{\theta}{2}}{2\sin\frac{\theta}{2}}\right)\right\} = \frac{\pi}{2}.$$

The above shows that the main theorem is sharp for the case $\alpha \to 0$ and $\alpha = 1$.

Applying the same method as the above and [2], we can obtain the following result.

Theorem C. *If* $f(z) \in \mathcal{A}(p)$ *and satisfies*

$$p - \frac{\alpha}{2} < 1 + \mathrm{Re}\left\{\frac{zf''(z)}{f'(z)}\right\} \qquad in \qquad \mathcal{E}$$

where $0 < \alpha \leq 1$, *then* $f(z) \in \mathcal{STS}(p, \alpha)$.

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