ON THE STRONGLY STARLIKENESS OF
MULTIVALENTLY CONVEX FUNCTIONS OF ORDER $\alpha$

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Let $A(p)$ denote the class of functions $f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n$ which are analytic in the open unit disc $\mathcal{E} = \{z : |z| < 1\}$. A function $f(z) \in A(p)$ is called to be p-valently starlike if and only if the inequality

$$\text{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} > 0$$

holds for $z \in \mathcal{E}$. A function $f(z) \in A(p)$ is called p-valently convex of order $\alpha$ ($0 \leq \alpha < p$) if and only if the inequality

$$1 + \text{Re} \left\{ \frac{zf''(z)}{f'(z)} \right\} > \alpha$$

holds for $z \in \mathcal{E}$. We denote by $C(p, \alpha)$ the family of such functions. A function $f(z) \in A(p)$ is said to be strongly starlike of order $\alpha$ ($0 < \alpha \leq 1$) if and only if the inequality

$$\left| \arg \left\{ \frac{zf'(z)}{f(z)} \right\} \right| < \frac{\pi}{2\alpha}$$

holds for $z \in \mathcal{E}$. We also denote by $STS(p, \alpha)$ the family of functions which are strongly starlike of order $\alpha$. From the definition, it follows that if $f(z) \in STS(p, \alpha)$, then we have

$$\text{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} > 0 \quad \text{in} \quad \mathcal{E}$$

or $f(z)$ is p-valently starlike in $\mathcal{E}$ and therefore $f(z)$ is p-valent in $\mathcal{E}$ [1, Lemma 7].

Nunokawa [2,3] proved the following theorems.

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Theorem A. [2] If \( f(z) \in A(p) \) satisfies
\[
1 + \Re \left\{ \frac{zf''(z)}{f'(z)} \right\} < p + \frac{\alpha}{2}
\]
where \( 0 < \alpha \leq 1 \), then \( f(z) \in STS(p, \alpha) \).

Theorem B. [3] If \( f(z) \in A(1) \) satisfies
\[
\left| \arg \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} \right| < \frac{\pi}{2} \alpha(\beta)
\]
in \( \mathcal{E} \), then we have
\[
\left| \arg \left\{ \frac{zf'(z)}{f(z)} \right\} \right| < \frac{\pi}{2} \beta
\]
in \( \mathcal{E} \), where
\[
\alpha(\beta) = \beta + \frac{2}{\pi} \tan^{-1} \left\{ \frac{\beta q(\beta) \sin \frac{\pi}{2} (1 - \beta)}{p(\beta) + \beta q(\beta) \cos \frac{\pi}{2} (1 - \beta)} \right\}
\]
\[
p(\beta) = (1 + \beta)^{\frac{\beta + \mathcal{B}}{2}} \quad \text{and} \quad q(\beta) = (1 - \beta)^{\frac{\beta - 1}{2}}.
\]

It is the purpose of the present paper to prove that if
\[
f(z) \in C \left( 1, 1 - \frac{\alpha}{2} \right),
\]
then \( f(z) \in STS(1, \alpha) \).

In this paper, we need the following lemma.

Lemma 1. If \( f(z) \in A(1) \) be starlike with respect to the origin in \( \mathcal{E} \). Let \( C(r, \theta) = \{ f(te^{i\theta}) : 0 \leq t \leq r < 1 \} \) and \( T(r, \theta) \) be the total variation of \( \arg f(te^{i\theta}) \) on \( C(r, \theta) \), so that
\[
T(r, \theta) = \int_0^r \left| \frac{\partial}{\partial t} \arg \{ f(te^{i\theta}) \} \right| dt.
\]
Then we have
\[
T(r, \theta) < \pi.
\]

We owe this lemma to Sheil-Small [6, Theorem 1].
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Main Theorem. Let $f(z) \in A(1)$ and

\[ 1 + \text{Re} \left\{ \frac{zf''(z)}{f'(z)} \right\} > 1 - \frac{\alpha}{2} \quad \text{in} \quad \mathcal{E}, \]

where $0 < \alpha \leq 1$. Then we have

\[ \left| \arg \left\{ \frac{zf'(z)}{f(z)} \right\} \right| < \frac{\pi}{2} \alpha \quad \text{in} \quad \mathcal{E}, \]

or $f(z)$ is strongly starlike of order $\alpha$ in $\mathcal{E}$.

Proof. Let us put

\[ \frac{2}{\alpha} \left\{ 1 + \frac{zf''(z)}{f(z)} - 1 + \frac{\alpha}{2} \right\} = \frac{zg'(z)}{g(z)} \]

where $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$. From the assumption (1), we have

\[ \text{Re} \left\{ \frac{zg'(z)}{g(z)} \right\} > 0 \quad \text{in} \quad \mathcal{E}. \]

This shows that $g(z)$ is starlike and univalent in $\mathcal{E}$. With an easy calculation (see e.g. [4]), the equality (2) gives us that

\[ f'(z) = \left\{ \frac{g(z)}{z} \right\}^{\frac{\alpha}{2}}. \]

Since

\[ f'(z) \neq 0 \quad \text{in} \quad 0 < |z| < 1, \]

we easily have

\[ \frac{f(z)}{zf'(z)} = \int_{0}^{1} f'(tz) \frac{dt}{f'(z)} = \int_{0}^{1} t^{-\frac{\alpha}{2}} \left\{ \frac{g(tre^{i\theta})}{g(re^{i\theta})} \right\}^{\frac{\alpha}{2}} dt \]

where $z = re^{i\theta}$ and $0 < r < 1$. Since $g(z)$ is starlike in $\mathcal{E}$, from Lemma 1, we have

\[ -\pi < \arg \left\{ g(tre^{i\theta}) \right\} - \arg \left\{ g(re^{i\theta}) \right\} < \pi \]

for $0 < t \leq 1$. Putting

\[ \xi = \left\{ \frac{g(tre^{i\theta})}{\overline{g(re^{i\theta})}} \right\}^{\frac{\alpha}{2}}, \]
we have

\[(5) \quad \arg s = \frac{\alpha}{2} \arg \left\{ \frac{g(tre^{i\theta})}{g(re^{i\theta})} \right\}.
\]

From (4) and (5), \(s\) lies in convex sector

\[\left\{ s : |\arg s| \leq \frac{\pi}{2\alpha}\right\} \]

and the same is true of its integral mean of (3), (see e.g. [5, Lemma 1]). Therefore we have

\[\left| \arg \left\{ \frac{f(z)}{zf'(z)} \right\} \right| < \frac{\pi}{2\alpha} \text{ in } \mathcal{E},\]

or

\[\left| \arg \left\{ \frac{zf'(z)}{f(z)} \right\} \right| < \frac{\pi}{2\alpha} \text{ in } \mathcal{E}.\]

This shows that

\[\text{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} > 0 \text{ in } \mathcal{E},\]

which completes the proof of our main theorem.

**Remark.** This result is sharp for the case \(\alpha \rightarrow 0\) and \(\alpha = 1\).

(a) For the case \(\alpha \rightarrow 0\), let us put \(f(z) = z\), then \(f(z)\) is a convex function of order \(1 - \frac\alpha2 \rightarrow 1\) and \(f(z)\) is a strongly starlike function of order \(\alpha \rightarrow 0\).

(b) For the case \(\alpha = 1\), let us put

\[(6) \quad 1 + \frac{zf''(z)}{f'(z)} = \frac{1}{1-z}.
\]

Then we have

\[1 + \text{Re} \left\{ \frac{zf''(z)}{f'(z)} \right\} > \frac{1}{2} \text{ in } \mathcal{E},\]

and therefore \(f(z)\) is a convex function of order \(1/2\). From (5), we easily have

\[f'(z) = \frac{1}{1-z} \quad \text{and} \quad f(z) = \log \left\{ \frac{1}{1-z} \right\}.
\]

Putting \(|z| = 1, z = e^{i\theta}, 0 \leq \theta < 2\pi\), then it follows that

\[\frac{z}{1-z} = -\frac{1}{2} + i\frac{\cos \frac{\theta}{2}}{2 \sin \frac{\theta}{2}}\]
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and

\[
\log \left\{ \frac{1}{1 - z} \right\} = \log \left\{ \frac{1}{2 + i \frac{\cos \frac{\theta}{2}}{2 \sin \frac{\theta}{2}}} \right\} + i \arg \left\{ \frac{1}{2 + i \frac{\cos \frac{\theta}{2}}{2 \sin \frac{\theta}{2}}} \right\}.
\]

\[
\lim_{\theta \to +0} \arg \left\{ \frac{zf'(z)}{f(z)} \right\} = \lim_{\theta \to +0} \arg \left\{ \frac{1}{1 - z} \right\} \left( \log \frac{1}{1 - z} \right)
\]

\[
= \lim_{\theta \to +0} \arg \left\{ \frac{1}{2 + i \frac{\cos \frac{\theta}{2}}{2 \sin \frac{\theta}{2}}} \right\}
\]

\[
- \lim_{\theta \to +0} \arg \left\{ \log \left\{ \frac{1}{2 + i \frac{\cos \frac{\theta}{2}}{2 \sin \frac{\theta}{2}}} \right\} + i \arg \left( \frac{1}{2 + i \frac{\cos \frac{\theta}{2}}{2 \sin \frac{\theta}{2}}} \right) \right\} = \frac{\pi}{2}.
\]

The above shows that the main theorem is sharp for the case \(\alpha \to 0\) and \(\alpha = 1\).

Applying the same method as the above and [2], we can obtain the following result.

**Theorem C.** If \(f(z) \in \mathcal{A}(p)\) and satisfies

\[
p - \frac{\alpha}{2} < 1 + \Re \left\{ \frac{zf''(z)}{f'(z)} \right\}
\]

in \(\mathcal{E}\)

where \(0 < \alpha \leq 1\), then \(f(z) \in STS(p, \alpha)\).

**REFERENCES**

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