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ON SOME ANGULAR ESTIMATES OF CLOSE-TO-CONVEX FUNCTIONS

AKIRA IKEDA AND MEGUMI SAIGO

ABSTRACT. The paper is devoted to generalizing the results by Libera [4], MacGregor [5], Pommerenke [6] and Ponnusamy and Karunakaran [7] relating to properties of close-to-convex functions.

1. Introduction

Let \( p \in \mathcal{N} = \{1, 2, 3, \ldots\} \) and \( \mathcal{A}(p) \) denote the class of functions

\[
f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k
\]

which are analytic in the unit disk \( \mathcal{U} = \{ z : |z| < 1 \} \). A function \( f(z) \in \mathcal{A}(p) \) is called \( p \)-valently starlike if

\[
\text{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} > 0 \quad \text{in} \quad \mathcal{U}.
\]

We denote by \( S^*(p) \) the subclass of \( \mathcal{A}(p) \) consisting of \( p \)-valently starlike functions. Further, a function in \( \mathcal{A}(p) \) is said to be \( p \)-valently convex if

\[
1 + \text{Re} \left\{ \frac{zf''(z)}{f'(z)} \right\} > 0 \quad \text{in} \quad \mathcal{U}.
\]

Let \( \mathcal{C}(p) \) denote the subclass of \( \mathcal{A}(p) \) of such \( p \)-valently convex functions in \( \mathcal{U} \). A function \( f(z) \in \mathcal{A}(p) \) is said to be \( p \)-valently close-to-convex if there is a function \( g(z) \in \mathcal{C}(p) \) such that

\[
\text{Re} \left\{ \frac{f'(z)}{g'(z)} \right\} > 0 \quad \text{in} \quad \mathcal{U}.
\]

We shall denote by \( \mathcal{K}(p) \) the class of \( p \)-valently close-to-convex functions. As is well know, we have the inclusions

\[
\mathcal{C}(p) \subset S^*(p) \subset \mathcal{K}(p).
\]

Now, we define the subordination. Let \( f(z) \) and \( g(z) \) be analytic in \( \mathcal{U} \), with \( f(0) = g(0) \). Suppose \( f(z) \) is univalent, and the range of \( \mathcal{U} \) by \( g(z) \) is contained in that of \( f(z) \). Then we say the function \( g(z) \) subordinates to \( f(z) \) and write \( g(z) \prec f(z) \).

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**Theorem A.** [3] Let $f(z) \in \mathcal{A}(p)$. Let $g(z) \in \mathcal{S}^*(p)$ satisfy

$$\text{Re} \left\{ \frac{f'(z)}{g'(z)} \right\} > 0 \quad \text{in } \mathcal{U},$$

then we have

$$\text{Re} \left\{ \frac{f(z)}{g(z)} \right\} > 0 \quad \text{in } \mathcal{U}.$$

Theorem A was proved by Sakaguchi [3], which is generalized by Libera [4], MacGregor [5], Pommerenke [6], and Ponnusamy and Karunakaran [7].

The generalization of MacGregor [5] is the following, which is quite similar to that of Libera [4]:

**Theorem B.** [5, Lemma 2] Suppose that functions $f(z)$ and $g(z)$ are analytic in $\mathcal{U}$ with $f(0) = g(0) = 0$, and $g(z)$ maps $\mathcal{U}$ onto a region which is starlike with respect to the origin. Let $0 \leq \gamma < 1$. If

$$\text{Re} \left\{ \frac{f'(z)}{g'(z)} \right\} > \gamma \quad \text{in } \mathcal{U},$$

then

$$\text{Re} \left\{ \frac{f(z)}{g(z)} \right\} > \gamma \quad \text{in } \mathcal{U}.$$

Likewise, if

$$\text{Re} \left\{ \frac{f'(z)}{g'(z)} \right\} < \gamma \quad \text{in } \mathcal{U},$$

then

$$\text{Re} \left\{ \frac{f(z)}{g(z)} \right\} < \gamma \quad \text{in } \mathcal{U}.$$

In [6], Pommerenke obtained the following theorem.

**Theorem C.** [6, Lemma 1] Let $f(z), g(z) \in \mathcal{A}(p)$. For $0 \leq \alpha \leq 1$,

$$\left| \arg \left\{ \frac{f'(z)}{g'(z)} \right\} \right| \leq \frac{\pi}{2} \alpha \quad \text{in } \mathcal{U},$$

then

$$\left| \arg \left\{ \frac{f(z_2) - f(z_1)}{g(z_2) - g(z_1)} \right\} \right| \leq \frac{\pi}{2} \alpha$$

for $z_1, z_2 \in \mathcal{U}$.

In [7], Ponnusamy and Karunakaran lead the next theorem.
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Theorem D. [7, Corollary 2] Let \( p \geq 1, k \geq 1, \beta < 1 \) and \( 0 \leq \delta < 1/p \). If \( f(z), g(z) \in \mathcal{A}(p) \) and \( g(z) \) satisfies
\[
\text{Re} \left\{ \frac{g(z)}{zg'(z)} \right\} > \delta,
\]
then
\[
\text{Re} \left\{ \frac{f'(z)}{g'(z)} \right\} > \beta
\]
implies
\[
\text{Re} \left\{ \frac{f(z)}{g(z)} \right\} > \frac{2\beta + k\delta}{2 + k\delta}.
\]

Theorem D may be regarded as a generalization of the results of Theorems A and B.

In 1995, Nunokawa obtained the next two theorems.

Theorem E. [8, Theorem 1] Let \( f(z) \in \mathcal{A}(p), g(z) \in \mathcal{S}^*(p) \), \( 0 < \alpha \leq 1 \) and \( \beta \) be a real number. Suppose that
\[
\left| \arg \left\{ \frac{f'(z)}{g'(z)} - \beta \right\} \right| < \frac{\pi}{2} \alpha \quad \text{in} \quad \mathcal{U},
\]
then we have
\[
\left| \arg \left\{ \frac{f(z)}{g(z)} - \beta \right\} \right| < \frac{\pi}{2} \alpha \quad \text{in} \quad \mathcal{U}.
\]

Theorem F. [8, Theorem 2] Let \( f(z) \in \mathcal{A}(p), g(z) \in \mathcal{S}^*(p) \), where \( 0 < \alpha \leq 1 \) and \( \beta > 1 \). Suppose that
\[
\left| \arg \left\{ \beta - \frac{f'(z)}{g'(z)} \right\} \right| < \frac{\pi}{2} \alpha \quad \text{in} \quad \mathcal{U},
\]
then we have
\[
\left| \arg \left\{ \beta - \frac{f(z)}{g(z)} \right\} \right| < \frac{\pi}{2} \alpha \quad \text{in} \quad \mathcal{U}
\]
or
\[
\pi - \frac{\pi}{2} \alpha < \arg \left\{ \frac{f(z)}{g(z)} - \beta \right\} < \pi + \frac{\pi}{2} \alpha \quad \text{in} \quad \mathcal{U}.
\]

Remark 1. Theorem E is a generalization of Theorem A, the first half of Theorem B and Theorem C, while Theorem F is a generalization of the second half of Theorem B.
2. Preliminaries

In this paper, we need the following lemmas.

**Lemma 1.** [10] Let $p(z)$ be analytic in $\mathcal{U}$ with $p(0) = 1$ and $p(z) \neq 0$ in $\mathcal{U}$. Let $\beta > 0$ and suppose that there exists a point $z_0 \in \mathcal{U}$ such that

$$|\arg\{p(z)\}| < \frac{\pi}{2}\beta$$

for $|z| < |z_0|$

and

$$|\arg\{p(z_0)\}| = \frac{\pi}{2}\beta.$$

Then we have

$$\frac{z_0p'(z_0)}{p(z_0)} = ik\beta,$$

where

$$k \geq 1 \quad \text{when} \quad \arg\{p(z_0)\} = \frac{\pi}{2}\beta,$$

$$k \leq -1 \quad \text{when} \quad \arg\{p(z_0)\} = -\frac{\pi}{2}\beta$$

and

$$p(z_0)^{1/\beta} = \pm ia, \quad a > 0.$$

**Lemma 2.** Let $\alpha$ be a positive real number and let $p(z)$ be analytic in $\mathcal{U}$ with $p(0) = 1$ and $p(z) \neq 0$ in $\mathcal{U}$. Let $-1 \leq \delta < \lambda \leq 1$ and suppose that

(1) $$\left|\arg\left\{p(z) + \frac{g(z)}{g'(z)}p'(z)\right\}\right| < \frac{\pi}{2}\alpha$$

in $\mathcal{U}$

or

$$p(z) + \frac{g(z)}{g'(z)}p'(z) \prec \left(\frac{1+z}{1-z}\right)^\alpha$$

in $\mathcal{U}$,

where $g(z)$ belongs to $S^*(p)$ and satisfies

(2) $$\frac{g(z)}{zg'(z)} \lesssim \frac{1+\lambda z}{p 1 + \delta z}.$$

Then for $\beta > 0$ being determined by

(3) $$\alpha = \beta + \frac{2}{\pi} \tan^{-1} \left\{ \frac{(1 - \lambda) \{(\lambda - \delta)\beta + p(1 - \lambda)(1 - \delta^2)\}}{p(1 - \delta)(\lambda - \delta)} \right\},$$

we have

$$|\arg\{p(z)\}| < \frac{\pi}{2}\beta$$

in $\mathcal{U}$. 
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Proof. Suppose that there exists a point \( z_0 \in \mathcal{U} \) such that

\[
|\arg\{p(z)\}| < \frac{\pi}{2} \beta \quad \text{for} \quad |z| < |z_0|
\]

and

\[
|\arg\{p(z_0)\}| = \frac{\pi}{2} \beta.
\]

Then, from Lemma 1, we have

\[
\frac{z_0 p'(z_0)}{p(z_0)} = ik\beta,
\]

where

\[
k \geq 1 \quad \text{when} \quad \arg\{p(z_0)\} = \frac{\pi}{2} \beta,
\]

\[
k \leq -1 \quad \text{when} \quad \arg\{p(z_0)\} = -\frac{\pi}{2} \beta
\]

and

\[
p(z_0)^{1/\beta} = \pm ia, \quad a > 0.
\]

Then it follows that

\[
\arg\left\{p(z_0) + \frac{g(z_0)}{g'(z_0)} p'(z_0)\right\} = \arg\{p(z_0)\} \left[ 1 + \frac{z_0 p'(z_0)}{p(z_0)} \frac{g(z_0)}{z_0 g'(z_0)} \right]
\]

\[
= \arg\{p(z_0)\} \left[ 1 + ik\beta \frac{g(z_0)}{z_0 g'(z_0)} \right]
\]

\[
= \arg\{p(z_0)\} (A + iB).
\]

Here real constants \( A \) and \( B \) can be estimated by virtue of the assumption (2) such as

\[
A \leq 1 + \frac{1}{p} \frac{\lambda - \delta}{1 - \delta^2} k\beta,
\]

(4)

\[
B \geq \frac{1}{p} \frac{1 - \lambda}{1 - \delta} k\beta.
\]

Note that the right hand side of (4) is positive.
When \( \arg \{ p(z_0) \} = \pi \beta / 2 \), we have
\[
\arg \left\{ p(z_0) + \frac{g(z_0)}{g'(z_0)} p'(z_0) \right\} = \arg \{ p(z_0) \} (A + iB)
\geq \frac{\pi}{2} \beta + \tan^{-1} \left\{ \frac{1 - \lambda}{p(1 - \delta)} \left( \frac{\beta}{k} + \frac{p(1 - \lambda)(1 - \delta^2)}{1 - \delta} \right) \right\}
= \frac{\pi}{2} \beta + \tan^{-1} \left\{ \frac{1 - \lambda}{p(1 - \delta)} \left( \frac{(\lambda - \delta) \beta + p(1 - \lambda)(1 - \delta^2)}{1 - \delta} \right) \right\}
\geq \frac{\pi}{2} \beta + \tan^{-1} \left( \frac{(1 - \lambda)(\lambda - \delta) \beta + p(1 - \lambda)(1 - \delta^2)}{p(1 - \delta)(\lambda - \delta)} \right)
= \frac{\pi}{2} \beta + \frac{2}{\pi} \tan^{-1} \left( \frac{(1 - \lambda)(\lambda - \delta) \beta + p(1 - \lambda)(1 - \delta^2)}{p(1 - \delta)(\lambda - \delta)} \right)
= \frac{\pi}{2} \alpha.
\]

On the other hand, when \( \arg \{ p(z_0) \} = -\pi \beta / 2 \), we have
\[
\arg \left\{ p(z_0) + \frac{g(z_0)}{g'(z_0)} p'(z_0) \right\} = \arg \{ p(z_0) \} (A + iB)
\leq -\frac{\pi}{2} \left( \beta + \frac{2}{\pi} \tan^{-1} \left( \frac{(1 - \lambda)(\lambda - \delta) \beta + p(1 - \lambda)(1 - \delta^2)}{p(1 - \delta)(\lambda - \delta)} \right) \right)
= -\frac{\pi}{2} \alpha.
\]

These contradict (1), which completes the proof of Lemma 2.

**Remark 2.** Note that when \( \lambda = 1 \), \( \beta = \alpha \) from the equation (1).

**Remark 3.** The existence of \( \beta \) satisfying (3) for any positive \( \alpha \) can be certificate easily.

### 3. Main results

**Theorem 1.** Let \( \gamma \) be a real number and \( 0 < \alpha \leq 1 \). Let \( f(z) \in A(p) \), \( g(z) \in S^*(p) \) and
\[
\frac{g(z)}{zg'(z)} < \frac{1 + \lambda z}{p(1 + \delta z)}
\]
for \(-1 \leq \delta \leq 1 \) and suppose that
\[
\arg \left\{ \frac{f'(z)}{g'(z)} - \gamma \right\} < \frac{\pi}{2} \alpha \quad \text{in} \quad U.
\]
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Then for $\beta > 0$ being determined by (3) we have

$$|\arg \left\{ \frac{f(z)}{g(z)} - \gamma \right\}| < \frac{\pi}{2} \beta \quad \text{in } \mathcal{U}.$$  

Proof. Let us put

$$p(z) = \frac{1}{1-\gamma} \left\{ \frac{f(z)}{g(z)} - \gamma \right\}.$$  

Then we have

$$p(z) + \frac{g(z)}{g'(z)} p'(z) = \frac{1}{1-\gamma} \left\{ \frac{f'(z)}{g'(z)} - \gamma \right\}.$$  

Applying Lemma 2 for this $p(z)$, we obtain the required result.

Remark 4. Theorem 1 is a revision of Theorem E in view of Remark 2.

Theorem 2. Let $\gamma > 1$ and $0 < \alpha \leq 1$. Let $f(z) \in \mathcal{A}(p), g(z) \in S^{*}(p)$. For $-1 \leq \delta < \lambda \leq 1$ we assume

$$\frac{g(z)}{zg'(z)} < \frac{1 + \lambda z}{p(1 + \delta z)}$$

and suppose that

$$|\arg \left\{ \gamma - \frac{f'(z)}{g'(z)} \right\}| < \frac{\pi}{2} \alpha \quad \text{in } \mathcal{U}.$$  

Then for $\beta > 0$ being determined by (3) we have

$$|\arg \left\{ \gamma - \frac{f(z)}{g(z)} \right\}| < \frac{\pi}{2} \beta \quad \text{in } \mathcal{U}$$

or

$$\pi - \frac{\pi}{2} \beta < \arg \left\{ \frac{f(z)}{g(z)} - \gamma \right\} < \pi + \frac{\pi}{2} \beta \quad \text{in } \mathcal{U}.$$  

Proof. Let us put

$$p(z) = \frac{1}{\gamma - 1} \left\{ \gamma - \frac{f(z)}{g(z)} \right\}.$$  

Then we have

$$p(z) + \frac{g(z)}{g'(z)} p'(z) = \frac{1}{\gamma - 1} \left\{ \gamma - \frac{f'(z)}{g'(z)} \right\},$$

which yields the result of the present theorem.

Remark 5. Theorem 2 is better than Theorem F, as we noted in Remark 3.

Remark 6. In case of $\lambda = 1, \alpha = \beta = 1$ and $\gamma = 0$, Theorem 1 is equivalent to Theorem A.
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