数理解析研究所講究録
1112 巻 1999 年 36-43

ON SOME ANGULAR ESTIMATES OF CLOSE-TO-CONVEX FUNCTIONS

AKIRA IKEDA AND MEGUMI SAIGO

ABSTRACT. The paper is devoted to generalizing the results by Libera [4], MacGregor [5], Pommerenke [6] and Ponnusamy and Karunakaran [7] relating to properties of close-to-convex functions.

1. Introduction

Let $p \in \mathcal{N} = \{1, 2, 3, \cdots\}$ and $\mathcal{A}(p)$ denote the class of functions

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k$$

which are analytic in the unit disk $\mathcal{U} = \{z : |z| < 1\}$. A function $f(z) \in \mathcal{A}(p)$ is called *p-valently starlike* if

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > 0 \qquad \text{in} \quad \mathcal{U}.$$

We denote by $\mathcal{S}^*(p)$ the subclass of $\mathcal{A}(p)$ consisting of p-valently starlike functions. Further, a function in $\mathcal{A}(p)$ is said to be *p-valently convex* if

$$1 + \mathrm{Re}\left\{\frac{zf''(z)}{f'(z)}\right\} > 0 \qquad \text{in} \quad \mathcal{U}.$$

Let $\mathcal{C}(p)$ denote the subclass of $\mathcal{A}(p)$ of such p-valently convex functions in $\mathcal{U}$. A function $f(z) \in \mathcal{A}(p)$ is said to be *p-valently close-to-convex* if there is a function $g(z) \in \mathcal{C}(p)$ such that

$$\mathrm{Re}\left\{\frac{f'(z)}{g'(z)}\right\} > 0 \qquad \text{in} \quad \mathcal{U}.$$

We shall denote by $\mathcal{K}(p)$ the class of p-valently close-to-convex functions. As is well know, we have the inclusions

$$\mathcal{C}(p) \subset \mathcal{S}^*(p) \subset \mathcal{K}(p).$$

Now, we define the subordination. Let $f(z)$ and $g(z)$ be analytic in $\mathcal{U}$, with $f(0) = g(0)$. Suppose $f(z)$ is univalent, and the range of $\mathcal{U}$ by $g(z)$ is contained in that of $f(z)$. Then we say the function $g(z)$ *subordinates* to $f(z)$ and write $g(z) \prec f(z)$.

1991 *Mathematics Subject Classification.* 30C45.

Theorem A. [3] *Let* $f(z) \in \mathcal{A}(p)$. *Let* $g(z) \in \mathcal{S}^*(p)$ *satisfy*

$$\mathrm{Re}\left\{\frac{f'(z)}{g'(z)}\right\} > 0 \qquad in \quad \mathcal{U},$$

then we have

$$\mathrm{Re}\left\{\frac{f(z)}{g(z)}\right\} > 0 \qquad in \quad \mathcal{U}.$$

Theorem A was proved by Sakaguchi [3], which is generalized by Libera [4], MacGregor [5], Pommerenke [6], and Ponnusamy and Karunakaran [7].

The generalization of MacGregor [5] is the following, which is quite similar to that of Libera [4]:

Theorem B. [5, Lemma 2] *Suppose that functions* $f(z)$ *and* $g(z)$ *are analytic in* $\mathcal{U}$ *with* $f(0) = g(0) = 0$, *and* $g(z)$ *maps* $\mathcal{U}$ *onto a region which is starlike with respect to the origin. Let* $0 \leq \gamma < 1$. *If*

$$\mathrm{Re}\left\{\frac{f'(z)}{g'(z)}\right\} > \gamma \qquad in \quad \mathcal{U},$$

then

$$\mathrm{Re}\left\{\frac{f(z)}{g(z)}\right\} > \gamma \qquad in \quad \mathcal{U}.$$

Likewise, if

$$\mathrm{Re}\left\{\frac{f'(z)}{g'(z)}\right\} < \gamma \qquad in \quad \mathcal{U},$$

then

$$\mathrm{Re}\left\{\frac{f(z)}{g(z)}\right\} < \gamma \qquad in \quad \mathcal{U}.$$

In [6], Pommerenke obtained the following theorem.

Theorem C. [6, Lemma 1] *Let* $f(z), g(z) \in \mathcal{A}(p)$. *For* $0 \leq \alpha \leq 1$,

$$\left|\arg\left\{\frac{f'(z)}{g'(z)}\right\}\right| \leq \frac{\pi}{2}\alpha \qquad in \quad \mathcal{U},$$

then

$$\left|\arg\left\{\frac{f(z_2) - f(z_1)}{g(z_2) - g(z_1)}\right\}\right| \leq \frac{\pi}{2}\alpha$$

for $z_1, z_2 \in \mathcal{U}$.

In [7], Ponnusamy and Karunakaran lead the next theorem.

Theorem D. [7, Corollary 2] *Let* $p \geq 1$, $k \geq 1$, $\beta < 1$ *and* $0 \leq \delta < 1/p$. *If* $f(z), g(z) \in \mathcal{A}(p)$ *and* $g(z)$ *satisfies*

$$\operatorname{Re}\left\{\frac{g(z)}{zg'(z)}\right\} > \delta,$$

then

$$\operatorname{Re}\left\{\frac{f'(z)}{g'(z)}\right\} > \beta$$

implies

$$\operatorname{Re}\left\{\frac{f(z)}{g(z)}\right\} > \frac{2\beta + k\delta}{2 + k\delta}.$$

Theorem D may be regarded as a generalization of the results of Theorems A and B.

In 1995, Nunokawa obtained the next two theorems.

Theorem E. [8, Theorem 1] *Let* $f(z) \in \mathcal{A}(p)$, $g(z) \in \mathcal{S}^*(p)$, $0 < \alpha \leq 1$ *and* β *be a real number. Suppose that*

$$\left|\arg\left\{\frac{f'(z)}{g'(z)} - \beta\right\}\right| < \frac{\pi}{2}\alpha \quad in \quad \mathcal{U},$$

then we have

$$\left|\arg\left\{\frac{f(z)}{g(z)} - \beta\right\}\right| < \frac{\pi}{2}\alpha \quad in \quad \mathcal{U}.$$

Theorem F. [8, Theorem 2] *Let* $f(z) \in \mathcal{A}(p)$, $g(z) \in \mathcal{S}^*(p)$, *where* $0 < \alpha \leq 1$ *and* $\beta > 1$. *Suppose that*

$$\left|\arg\left\{\beta - \frac{f'(z)}{g'(z)}\right\}\right| < \frac{\pi}{2}\alpha \quad in \quad \mathcal{U},$$

then we have

$$\left|\arg\left\{\beta - \frac{f(z)}{g(z)}\right\}\right| < \frac{\pi}{2}\alpha \quad in \quad \mathcal{U}$$

or

$$\pi - \frac{\pi}{2}\alpha < \arg\left\{\frac{f(z)}{g(z)} - \beta\right\} < \pi + \frac{\pi}{2}\alpha \quad in \quad \mathcal{U}.$$

Remark 1. Theorem E is a generalization of Theorem A, the first half of Theorem B and Theorem C, while Theorem F is a generalization of the second half of Theorem B.

2. Preliminaries

In this paper, we need the following lemmas.

Lemma 1. [10] *Let $p(z)$ be analytic in $\mathcal{U}$ with $p(0) = 1$ and $p(z) \neq 0$ in $\mathcal{U}$. Let $\beta > 0$ and suppose that there exists a point $z_0 \in \mathcal{U}$ such that*

$$|\arg\{p(z)\}| < \frac{\pi}{2}\beta \qquad for \quad |z| < |z_0|$$

and

$$|\arg\{p(z_0)\}| = \frac{\pi}{2}\beta.$$

Then we have

$$\frac{z_0 p'(z_0)}{p(z_0)} = ik\beta,$$

where

$$k \geq 1 \qquad when \quad \arg\{p(z_0)\} = \frac{\pi}{2}\beta,$$

$$k \leq -1 \qquad when \quad \arg\{p(z_0)\} = -\frac{\pi}{2}\beta$$

and

$$p(z_0)^{1/\beta} = \pm ia, \qquad a > 0.$$

Lemma 2. *Let α be a positive real number and let $p(z)$ be analytic in $\mathcal{U}$ with $p(0) = 1$ and $p(z) \neq 0$ in $\mathcal{U}$. Let $-1 \leq \delta < \lambda \leq 1$ and suppose that*

$$\left|\arg\left\{p(z) + \frac{g(z)}{g'(z)}p'(z)\right\}\right| < \frac{\pi}{2}\alpha \qquad in \quad \mathcal{U} \tag{1}$$

or

$$p(z) + \frac{g(z)}{g'(z)}p'(z) \prec \left(\frac{1+z}{1-z}\right)^{\alpha} \qquad in \quad \mathcal{U},$$

where $g(z)$ belongs to $\mathcal{S}^*(p)$ and satisfies

$$\frac{g(z)}{zg'(z)} \prec \frac{1}{p}\frac{1+\lambda z}{1+\delta z}. \tag{2}$$

Then for $\beta > 0$ being determined by

$$\alpha = \beta + \frac{2}{\pi}\tan^{-1}\left\{\frac{(1-\lambda)\{(\lambda-\delta)\beta + p(1-\lambda)(1-\delta^2)\}}{p(1-\delta)(\lambda-\delta)}\right\}, \tag{3}$$

we have

$$|\arg\{p(z)\}| < \frac{\pi}{2}\beta \qquad in \quad \mathcal{U}.$$

Proof. Suppose that there exists a point $z_0 \in \mathcal{U}$ such that

$$|\arg\{p(z)\}| < \frac{\pi}{2}\beta \qquad \text{for} \quad |z| < |z_0|$$

and

$$|\arg\{p(z_0)\}| = \frac{\pi}{2}\beta.$$

Then, from Lemma 1, we have

$$\frac{z_0 p'(z_0)}{p(z_0)} = ik\beta,$$

where

$$k \geq 1 \qquad \text{when} \quad \arg\{p(z_0)\} = \frac{\pi}{2}\beta,$$

$$k \leq -1 \qquad \text{when} \quad \arg\{p(z_0)\} = -\frac{\pi}{2}\beta$$

and

$$p(z_0)^{1/\beta} = \pm ia, \qquad a > 0.$$

Then it follows that

$$\begin{aligned}\arg\left\{p(z_0) + \frac{g(z_0)}{g'(z_0)}p'(z_0)\right\} &= \arg\{p(z_0)\}\left[1 + \frac{z_0 p'(z_0)}{p(z_0)}\frac{g(z_0)}{z_0 g'(z_0)}\right] \\ &= \arg\{p(z_0)\}\left[1 + ik\beta\frac{g(z_0)}{z_0 g'(z_0)}\right] \\ &= \arg\{p(z_0)\}(A + iB).\end{aligned}$$

Here real constants A and B can be estimated by virtue of the assumption (2) such as

$$A \leq 1 + \frac{1}{p}\frac{\lambda - \delta}{1 - \delta^2}k\beta,$$

(4)
$$B \geq \frac{1}{p}\frac{1 - \lambda}{1 - \delta}k\beta.$$

Note that the right hand side of (4) is positive.

When $\arg\{p(z_0)\} = \pi\beta/2$, we have

$$
\begin{aligned}
\arg\left\{p(z_0) + \frac{g(z_0)}{g'(z_0)}p'(z_0)\right\} &= \arg\{p(z_0)\}(A+iB) \\
&\geq \frac{\pi}{2}\beta + \tan^{-1}\left\{\frac{\frac{1}{p}\frac{1-\lambda}{1-\delta}k\beta}{1+\frac{1}{p}\frac{\lambda-\delta}{1-\delta^2}k\beta}\right\} \\
&= \frac{\pi}{2}\beta + \tan^{-1}\left\{\frac{(1-\lambda)\{(\lambda-\delta)k\beta + p(1-\lambda)(1-\delta^2)\}}{p(1-\delta)(\lambda-\delta)}\right\} \\
&\geq \frac{\pi}{2}\beta + \tan^{-1}\left\{\frac{(1-\lambda)\{(\lambda-\delta)\beta + p(1-\lambda)(1-\delta^2)\}}{p(1-\delta)(\lambda-\delta)}\right\} \\
&= \frac{\pi}{2}\left[\beta + \frac{2}{\pi}\tan^{-1}\left\{\frac{(1-\lambda)\{(\lambda-\delta)\beta + p(1-\lambda)(1-\delta^2)\}}{p(1-\delta)(\lambda-\delta)}\right\}\right] \\
&= \frac{\pi}{2}\alpha.
\end{aligned}
$$

On the other hand, when $\arg\{p(z_0)\} = -\pi\beta/2$, we have

$$
\begin{aligned}
\arg\left\{p(z_0) + \frac{g(z_0)}{g'(z_0)}p'(z_0)\right\} &= \arg\{p(z_0)\}(A+iB) \\
&\leq -\frac{\pi}{2}\left[\beta + \frac{2}{\pi}\tan^{-1}\left\{\frac{(1-\lambda)\{(\lambda-\delta)\beta + p(1-\lambda)(1-\delta^2)\}}{p(1-\delta)(\lambda-\delta)}\right\}\right] \\
&= -\frac{\pi}{2}\alpha.
\end{aligned}
$$

These contradict (1), which completes the proof of Lemma 2.

Remark 2. Note that when $\lambda = 1$, $\beta = \alpha$ from the equation (1).

Remark 3. The existence of β satisfying (3) for any positive α can be certificated easily.

3. Main results

Theorem 1. *Let γ be a real number and $0 < \alpha \leq 1$. Let $f(z) \in \mathcal{A}(p)$, $g(z) \in \mathcal{S}^*(p)$ and*

$$\frac{g(z)}{zg'(z)} \prec \frac{1}{p}\frac{1+\lambda z}{1+\delta z}$$

for $-1 \leq \delta < \lambda \leq 1$ and suppose that

$$\left|\arg\left\{\frac{f'(z)}{g'(z)} - \gamma\right\}\right| < \frac{\pi}{2}\alpha \quad \text{in } \mathcal{U}. \tag{5}$$

Then for $\beta > 0$ *being determined by* (3) *we have*

$$\left|\arg\left\{\frac{f(z)}{g(z)} - \gamma\right\}\right| < \frac{\pi}{2}\beta \qquad in \quad \mathcal{U}.$$

Proof. Let us put

$$p(z) = \frac{1}{1-\gamma}\left\{\frac{f(z)}{g(z)} - \gamma\right\}.$$

Then we have

$$p(z) + \frac{g(z)}{g'(z)}p'(z) = \frac{1}{1-\gamma}\left\{\frac{f'(z)}{g'(z)} - \gamma\right\}.$$

Applying Lemma 2 for this $p(z)$, we obtain the required result.

Remark 4. Theorem 1 is a revision of Theorem E in view of Remark 2.

Theorem 2. *Let* $\gamma > 1$ *and* $0 < \alpha \leq 1$. *Let* $f(z) \in \mathcal{A}(p)$, $g(z) \in \mathcal{S}^*(p)$. *For* $-1 \leq \delta < \lambda \leq 1$ *we assume*

$$\frac{g(z)}{zg'(z)} \prec \frac{1}{p}\frac{1+\lambda z}{1+\delta z}$$

and suppose that

$$\left|\arg\left\{\gamma - \frac{f'(z)}{g'(z)}\right\}\right| < \frac{\pi}{2}\alpha \qquad in \quad \mathcal{U}.$$

Then for $\beta > 0$ *being determined by* (3) *we have*

$$\left|\arg\left\{\gamma - \frac{f(z)}{g(z)}\right\}\right| < \frac{\pi}{2}\beta \qquad in \quad \mathcal{U}$$

or

$$\pi - \frac{\pi}{2}\beta < \arg\left\{\frac{f(z)}{g(z)} - \gamma\right\} < \pi + \frac{\pi}{2}\beta \qquad in \quad \mathcal{U}.$$

Proof. Let us put

$$p(z) = \frac{1}{\gamma - 1}\left\{\gamma - \frac{f(z)}{g(z)}\right\}.$$

Then we have

$$p(z) + \frac{g(z)}{g'(z)}p'(z) = \frac{1}{\gamma-1}\left\{\gamma - \frac{f'(z)}{g'(z)}\right\},$$

which yields the result of the present theorem.

Remark 5. Theorem 2 is better than Theorem F, as we noted in Remark 3.

Remark 6. In case of $\lambda = 1$, $\alpha = \beta = 1$ and $\gamma = 0$, Theorem 1 is equivalent to Theorem A.

REFERENCES

1. W. Kaplan, *Close-to-convex schlicht functions*, Michigan Math. J. **1** (1952), 169–185.
2. P. L. Duren, *Univalent Functions*, Springer-Verlag, New York, Berlin, Heidelberg (1983).
3. K. Sakaguchi, *On a certain univalent mapping*, J. Math. Soc. Japan **11** (1959), 72–75.
4. R. J. Libera, *Some classes of regular univalent functions*, Proc. Amer. Math. Soc. **16** (1965), 755–758.
5. T. H. MacGregor, *A subordination for convex functions of order* α, J. London Math. Soc. (2) **9** (1975), 530–536.
6. Ch. Pommerenke, *On close-to-convex analytic functions*, Trans. Amer. Math. Soc. **114** (1965), 176–186.
7. S. Ponnusamy and V. Karunakaran, *Differential subordination and conformal mappings*, Complex Variables **11** (1989), 79–86.
8. M. Nunokawa, *On some angular estimates of analytic functions*, Math. Japon. **41** (1995), 447–452.
9. M. Nunokawa, *On properties of non-Carathéodory functions*, Proc. Japan Acad. **68** (1992), 152–153.
10. M. Nunokawa, *On the order of strongly starlikeness of strongly convex functions*, Proc. Japan Acad. **69** (1993), 234–237.

AKIRA IKEDA:
MEGUMI SAIGO:
DEPARTMENT OF APPLIED MATHEMATICS, FUKUOKA UNIVERSITY,
8-19-1 NANAKUMA, JONAN-KU, FUKUOKA, 814-0180, JAPAN
E-mail address: aikeda@sf.sm.fukuoka-u.ac.jp, msaigo@fukuoka-u.ac.jp