数理解析研究所講究録
1112 巻 1999 年 1-12

ARGUMENT ESTIMATES OF MEROMORPHICALLY MULTIVALENT FUNCTIONS

*NAK EUN CHO AND **SHIGEYOSHI OWA

ABSTRACT. The object of the present paper is to obtain some argument properties of meromorphically multivalent functions in the punctured open unit disk. We also derive the integral preserving properties in a sector.

1. Introduction

Let $\mathcal{U}=\{z\in\mathbb{C}:|z|<1\}$. For f and g which are analytic in $\mathcal{U}$, we say that f is subordinate to g, written $f\prec g$ or $f(z)\prec g(z)$, if there exists a Schwarz function w in $\mathcal{U}$ such that $f(z)=g(w(z))$.

Let Σ_p denote the class of all meromorphic functions of the form

$$f(z)=\frac{1}{z^p}+\frac{a_0}{z^{p-1}}+\ldots+a_{k+p-1}z^k+\ldots\ (p\in\mathbb{N}=\{1,2,\cdots\})$$

which are analytic in the annulus $\mathcal{D}=\{z:0<|z|<1\}$. We denote by $\Sigma_p^*(\beta)$ the subclass of Σ_p consisting of all functions which is meromorphically starlike of order β in $\mathcal{U}$.

The Hadamard product or convolution of two functions f and g in Σ_p will be denoted by $f*g$.

Let

$$D^{n+p-1}f(z)=\frac{1}{z^p(1-z)^{n+p}}*f(z)\ \ (z\in\mathcal{D}) \tag{1.1}$$

or, equivalently,

1991 Mathematics Subject Classification: 30C45.

Key words and phrases. meromorphically starlike, differential operator, argument estimates, integral operator.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

$$\begin{aligned} D^{n+p-1}f(z) =& \frac{1}{z^p}\left(\frac{z^{n+2p-1}f(z)}{(n+p-1)!}\right)^{(n+p-1)} \\ =& \frac{1}{z^p} + (n+p)a_0\frac{1}{z^{p-1}} + \frac{(n+p+1)(n+p)}{2!}a_1\frac{1}{z^{p-2}} + \ldots \\ & \ldots + \frac{(n+k+2p-1)\ldots(n+p)}{(k+p)!}a_{k+p-1}z^k + \ldots \quad (z \in \mathcal{D}), \end{aligned}$$

where n is any integer greater than $-p$.

For various interesting developments involving the operators D^{n+p-1} for functions belonging to Σ_p, the reader may be refereed to the recent works of author[1], Uralegaddi and Path[7], and others[8,9].

Let

$$\Sigma_p^*[n; A, B] = \left\{ f \in \Sigma_p : -\frac{z(D^{n+p-1}f(z))'(z)}{D^{n+p-1}f(z)} \prec p\frac{1+Az}{1+Bz},\ z \in \mathcal{U} \right\}, \tag{1.2}$$

where $-1 \leq B < A \leq 1$. In particular, we note that $\Sigma_p^*[-p+1; 1, -1]$ is the well known class of meromorphically p-valent starlike functions. From (1.2), we observe[6] that a function f is in $\Sigma_p^*[n; A, B]$ if and only if

$$\left| \frac{z(D^{n+p-1}f(z))'(z)}{D^{n+p-1}f(z)} + \frac{p(1-AB)}{1-B^2} \right| < \frac{p(A-B)}{1-B^2} \quad (-1 < B < A \leq 1\ ;\ z \in \mathcal{U}). \tag{1.3}$$

The object of the present paper is to give some argument estimates of meromorphically multivalent functions belonging to Σ_p and the integral preserving properties in connection with the differential operators D^{n+p-1} defined by (1.1).

2. Main results

To establish our main results, we need the following lemmas.

Lemma 2.1 [2]. *Let h be convex univalent in $\mathcal{U}$ with $h(0) = 1$ and* $\mathrm{Re}\,(\beta h(z) + \gamma) > 0 (\beta, \gamma \in \mathbb{C})$. If q is analytic in $\mathcal{U}$ with $q(0) = 1$, then

$$q(z) + \frac{zq'(z)}{\beta q(z) + \gamma} \prec h(z) \quad (z \in \mathcal{U})$$

implies

$$q(z) \prec h(z) \quad (z \in \mathcal{U}).$$

Lemma 2.2 [4]. *Let h be convex univalent in $\mathcal{U}$ and $\lambda(z)$ be analytic in $\mathcal{U}$ with* Re $\lambda(z) \geq 0$. *If q is analytic in $\mathcal{U}$ and $q(0) = h(0)$, then*

$$q(z) + \lambda(z)zq'(z) \prec h(z) \quad (z \in \mathcal{U})$$

implies

$$q(z) \prec h(z) \quad (z \in \mathcal{U}).$$

Lemma 2.3 [5]. *Let q be analytic in $\mathcal{U}$ with $q(0) = 1$ and $q(z) \neq 0$ in $\mathcal{U}$. Suppose that there exists a point $z_0 \in \mathcal{U}$ such that*

$$\left|\arg\, q(z)\right| < \frac{\pi}{2}\alpha \text{ for } |z| < |z_0| \tag{2.1}$$

and

$$\left|\arg\, q(z_0)\right| = \frac{\pi}{2}\alpha \quad (0 < \alpha \leq 1). \tag{2.2}$$

Then we have

$$\frac{z_0 q'(z_0)}{q(z_0)} = ik\alpha, \tag{2.3}$$

where

$$k \geq \frac{1}{2}\left(a + \frac{1}{a}\right) \quad \text{when} \quad \arg\, q(z_0) = \frac{\pi}{2}\alpha \tag{2.4}$$

$$k \leq -\frac{1}{2}\left(a + \frac{1}{a}\right) \quad \text{when} \quad \arg\, q(z_0) = -\frac{\pi}{2}\alpha \tag{2.5}$$

and

$$q(z_0)^{\frac{1}{\alpha}} = \pm ia \;\; (a > 0). \tag{2.6}$$

At first, with the help of Lemma 2.1, we obtain the following

Proposition 2.1. *Let h be convex univalent in $\mathcal{U}$ with $h(0) = 1$ and* Re h *be bounded in $\mathcal{U}$. If $f \in \Sigma_p$ satisfies the condition*

$$-\frac{z(D^{n+p}f(z))'}{pD^{n+p}f(z)} \prec h(z) \quad (z \in \mathcal{U}),$$

then

$$-\frac{z(D^{n+p-1}f(z))'}{pD^{n+p-1}f(z)} \prec h(z) \quad (z \in \mathcal{U})$$

for $\max_{z\in\mathcal{U}} \operatorname{Re} h(z) < \frac{n+2p}{p}$ *(provided* $D^{n+p-1}f(z) \neq 0$ *in* $\mathcal{U}$*).*

Proof. Let

$$q(z) = -\frac{z(D^{n+p-1}f(z))'}{pD^{n+p-1}f(z)}.$$

By using the equation

$$z(D^{n+p-1}f(z))' = (n+p)D^{n+p}f(z) - (n+2p)D^{n+p-1}f(z), \tag{2.7}$$

we get

$$q(z) - \frac{n+2p}{p} = -\frac{(n+p)D^{n+1}f(z)}{pD^{n+p-1}f(z)}. \tag{2.8}$$

Taking logarithemic derivatives in both sides of (2.8) and multiplying by z, we have

$$\frac{zq'(z)}{-pq(z)+n+2p} + q(z) = -\frac{z(D^{n+p}f(z))'}{pD^{n+p}f(z)} \prec h(z) \quad (z \in U).$$

From Lemma 2.1, it follows that $q(z) \prec h(z)$ for Re $(-h(z) + \frac{n+2p}{p}) > 0$ $(z \in \mathcal{U})$, which means

$$-\frac{z(D^{n+p-1}f(z))'}{pD^{n+p-1}f(z)} \prec h(z) \quad (z \in \mathcal{U})$$

for $\max_{z\in\mathcal{U}} \operatorname{Re} h(z) < \frac{n+2p}{p}$.

Proposition 2.2. *Let h be convex univalent in $\mathcal{U}$ with $h(0) = 1$ and* Re h *be bounded in $\mathcal{U}$. Let F be the integral operator defined by*

$$F(z) = \frac{c}{z^{c+p}} \int_0^z t^{c+p-1} f(t)dt \quad (c > 0). \tag{2.9}$$

If $f \in \Sigma_p$ satisfies the condition

$$-\frac{z(D^{n+p-1}f(z))'}{pD^{n+p-1}f(z)} \prec h(z) \quad (z \in \mathcal{U}),$$

then

$$-\frac{z(D^{n+p-1}F(z))'}{pD^{n+p-1}F(z)} \prec h(z) \quad (z \in \mathcal{U})$$

for $\max_{z\in\mathcal{U}} \operatorname{Re} h(z) < \frac{c+p}{p}$ *(provided* $D^{n+p-1}F(z) \neq 0$ *in* $\mathcal{U}$*).*

Proof. From (2.9), we have

$$z(D^{n+p-1}F(z))' = cD^{n+p-1}f(z) - (c+p)D^{n+p-1}F(z). \tag{2.10}$$

Let

$$p(z) = -\frac{z(D^{n+p-1}F(z))'}{pD^{n+p-1}F(z)}.$$

Then, by using (2.10), we get

$$q(z) - (c+p) = -c\frac{D^{n+p-1}f(z)}{D^{n+p-1}F(z)}. \tag{2.11}$$

Taking logarithemic derivatives in both sides of (2.11) and multiplying by z, we have

$$\frac{zq'(z)}{-pq(z)+(c+p)} + q(z) = -\frac{z(D^{n+p-1}f(z))'}{pD^{n+p-1}f(z)} \prec h(z) \quad (z \in \mathcal{U}).$$

Therefore, by Lemma 2.1, we have

$$-\frac{z(D^{n+p-1}F(z))'}{pD^{n+p-1}F(z)} \prec h(z) \quad (z \in \mathcal{U})$$

for $\max_{z\in\mathcal{U}}\mathrm{Re}\, h(z) < \frac{c+p}{p}$ (provided $D^{n+p-1}F(z) \neq 0$ in $\mathcal{U}$).

Remark. Taking $p=1$ and $h(z) = \frac{1+z}{1-z}$ in Proposition 2.1 and Proposition 2.2, we have the results obtained by Ganigi and Uralegaddi[3].

Applying Lemma 2.2, Lemma 2.3 and Proposition 2.1, we now derive

Theorem 2.1. *Let $f \in \Sigma_p$. Choose an integer n such that*

$$n \geq \frac{p(1+A)}{1+B} - 2p,$$

where $-1 < B < A \leq 1$ and $p \in \mathbb{N}$. If

$$\left|\arg\left(-\frac{z(D^{n+p}f(z))'}{D^{n+p}g(z)} - \gamma\right)\right| < \frac{\pi}{2}\delta \ (0 \leq \gamma < p\ ;\ 0 < \delta \leq 1)$$

for some $g \in \Sigma_p^*[n+1; A, B]$, then

$$\left|\arg\left(-\frac{z(D^{n+p-1}f(z))'}{D^{n+p-1}g(z)} - \gamma\right)\right| < \frac{\pi}{2}\alpha,$$

where α $(0 < \alpha \leq 1)$ is the solution of the equation

$$\delta = \alpha + \frac{2}{\pi}\tan^{-1}\left(\frac{\alpha\sin\frac{\pi}{2}(1-t(A,B))}{\frac{(n+2p)(1-B)+A-1}{1-B}+\alpha\cos\frac{\pi}{2}(1-t(A,B))}\right) \tag{2.12}$$

when

$$t(A,B) = \frac{2}{\pi}\sin^{-1}\left(\frac{p(A-B)}{(n+2p)(1-B^2)-p(1-AB)}\right). \tag{2.13}$$

Proof. Let

$$q(z) = -\frac{1}{p-\gamma}\left(\frac{z(D^{n+p-1}f(z))'}{D^{n+p-1}g(z)}+\gamma\right).$$

By (2.7), we have

$$\begin{aligned}&(p-\gamma)zq'(z)D^{n+p-1}g(z)+(1-\gamma)q(z)z(D^{n+p-1}g(z))' \\ &-(n+2p)z(D^{n+p-1}f(z))' = -(n+p)z(D^{n+p}f(z))' - \gamma z(D^{n+p-1}g(z))'(z).\end{aligned} \tag{2.14}$$

Dividing (2.14) by $D^{n+p-1}g(z)$ and simplifying, we get

$$q(z)+\frac{zq'(z)}{-r(z)+n+2p} = -\frac{1}{p-\gamma}\left(\frac{z(D^{n+p}f(z))'}{D^{n+p}g(z)}+\gamma\right), \tag{2.15}$$

where

$$r(z) = -\frac{z(D^{n+p-1}g(z))'}{D^{n+p-1}g(z)}.$$

Since $g \in \Sigma_p^*[n+1;A,B]$, from Proposition 2.1, we have

$$r(z) \prec p\frac{1+Az}{1+Bz}.$$

Using (1.3), we have

$$-r(z)+n+2p = \rho e^{i\frac{\pi}{2}\phi},$$

where

$$\begin{cases}\frac{(n+2p)(1+B)-(1+A)}{1+B} < \rho < \frac{(n+2p)(1-B)+A-1}{1-B} \\ -t(A,B) < \phi < t(A,B)\end{cases}$$

when $t(A,B)$ is given by (2.13). Let h be a function which maps $\mathcal{U}$ onto the angular domain $\{w : |\arg w| < \frac{\pi}{2}\delta\}$ with $h(0)=1$. Applying Lemma 2.2 for this h with $\lambda(z) = \frac{1}{-r(z)+n+2p}$, we see that $\operatorname{Re} q(z) > 0$ in $\mathcal{U}$ and hence $q(z) \neq 0$ in $\mathcal{U}$.

If there exists a point $z_0 \in \mathcal{U}$ such that the conditions (2.1) and (2.2) are satisfied, then(by Lemma 2.3) we obtain (2.3) under the restrictions (2.4), (2.5) and (2.6).

At first, suppose that $q(z_0)^{\frac{1}{\alpha}} = ia$ $(a > 0)$. Then we obtain

$$\begin{aligned}
&\arg\left[-\frac{1}{p-\gamma}\left(\frac{z_0(D^{n+p}f(z_0)'}{D^{n+p}g(z_0)}+\gamma\right)\right] = \arg\left(q(z_0)+\frac{z_0q'(z_0)}{-r(z_0)+n+2p}\right)\\
&= \frac{\pi}{2}\alpha + \arg\left(1+i\alpha k(\rho e^{i\frac{\pi}{2}\phi})^{-1}\right)\\
&= \frac{\pi}{2}\alpha + \tan^{-1}\left(\frac{\eta k\sin\frac{\pi}{2}(1-\phi)}{\rho+\alpha k\cos\frac{\pi}{2}(1-\phi)}\right)\\
&\geq \frac{\pi}{2}\alpha + \tan^{-1}\left(\frac{\alpha\sin\frac{\pi}{2}(1-t(A,B))}{\frac{(n+2p)(1-B)+A-1}{1-B}+\alpha\cos\frac{\pi}{2}(1-t(A,B))}\right)\\
&= \frac{\pi}{2}\delta,
\end{aligned}$$

where δ and $t(A,B)$ are given by (2.12) and (2.13), respectively. This is a contradiction to the assumption of our theorem.

Next, suppose that $p(z_0)^{\frac{1}{\alpha}} = -ia$ $(a > 0)$. Applying the same method as the above, we have

$$\begin{aligned}
&\arg\left[-\frac{1}{p-\gamma}\left(\frac{z_0(D^{n+p}f(z_0))'}{D^{n+p}g(z_0)}+\gamma\right)\right]\\
&\leq -\frac{\pi}{2}\alpha - \tan^{-1}\left(\frac{\alpha\sin\frac{\pi}{2}(1-t(A,B))}{\frac{(n+2p)(1-B)+A-1}{1-B}+\alpha\cos\frac{\pi}{2}(1-t(A,B))}\right)\\
&= -\frac{\pi}{2}\delta,
\end{aligned}$$

where δ and $t(A,B)$ are given by (2.12) and (2.13), respectively, which contradicts the assumption. Therefore we complete the proof of our theorem.

Letting $A=1,\ B=0$ and $\delta=1$ in Theorem 2.1, we have

Corollary 2.1. *Let $f \in \Sigma$. If*

$$-\operatorname{Re}\left\{\frac{z(D^{n+p}f(z))'}{D^{n+p}g(z)}\right\} > \gamma \quad (0 \leq \gamma < p)$$

for some $g \in \Sigma_p$ satisfying the condition

$$\left|\frac{z(D^{n+p}g(z))'}{D^{n+p}g(z)} + p\right| < p,$$

then

$$-\text{Re}\left\{\frac{z(D^{n+p-1}f(z))'}{D^{n+p-1}g(z)}\right\} > \gamma.$$

Taking $A = 1,\ B = 0$ and $g(z) = \frac{1}{z^p}$ in Theorem 2.1, we have

Corollary 2.2. *Let $f \in \Sigma_p$. If*

$$\left|\arg\ \left[-z^{p+1}(D^{n+p}f(z))' - \gamma\right]\right| < \frac{\pi}{2}\delta\ \ (0 \leq \gamma < p;\ \ 0 < \delta \leq 1),$$

then

$$\left|\arg\ \left[-z^{p+1}(D^{n+p-1}f(z))' - \gamma\right]\right| < \frac{\pi}{2}\delta.$$

Making $n = 0,\ p = 1$ and $\delta = 1$ in Corollary 2.2, we have

Corollary 2.3. *Let $f \in \Sigma_1$. If*

$$-\text{Re}\ \left\{z^2(zf''(z) + 3f'(z))\right\} > \gamma\ \ (0 \leq \gamma < 1),$$

then

$$-\text{Re}\ \left\{z^2 f'(z)\right\} > \gamma.$$

By the same techniques as in the proof of Theorem 2.1, we obtain

Theorem 2.2. *Let $f \in \Sigma$. Choose an integer n such that*

$$n \geq \frac{p(1+A)}{1+B} - 2p,$$

where $-1 < B < A \leq 1$ and $p \in \mathbb{N}$. If

$$\left|\arg\ \left(\frac{z(D^{n+p}f(z))'}{(D^{n+p}g(z))} + \gamma\right)\right| < \frac{\pi}{2}\delta\ \ (\gamma > p, 0 < \delta \leq 1)$$

for some $g \in \Sigma_p^*[n+1; A, B]$, *then*

$$\left|\arg\left(\frac{z(D^{n+p-1}f(z))'}{D^{n+p-1}g(z)}+\gamma\right)\right| < \frac{\pi}{2}\alpha,$$

where α $(0 < \alpha \leq 1)$ *is the solution of the equation given by* (2.12).

Next, we prove

Theorem 2.3. *Let* $f \in \Sigma_p$ *and choose a positive number* c *such that*

$$c \geq \frac{1+A}{1+B} - p,$$

where $-1 < B < A \leq 1$ *and* $p \in \mathbb{N}$. *If*

$$\left|\arg\left(-\frac{z(D^{n+p-1}f(z))'}{D^{n+p-1}g(z)}-\gamma\right)\right| < \frac{\pi}{2}\delta \ \ (0 \leq \gamma < p\ ;\ 0 < \delta \leq 1)$$

for some $g \in \Sigma_p^*[n; A, B]$, *then*

$$\left|\arg\left(-\frac{z(D^{n+p-1}F(z))'}{D^{n+p-1}G(z)}-\gamma\right)\right| < \frac{\pi}{2}\alpha,$$

where F *is the integral operator given by* (2.9),

$$G(z) = \frac{c}{z^{c+p}}\int_0^z t^{c+p-1}g(t)dt, \ \ (c > 0), \tag{2.16}$$

and $\alpha(0 < \alpha \leq 1)$ *is the solution of the equation*

$$\delta = \alpha + \frac{2}{\pi}\tan^{-1}\left(\frac{\alpha\sin\frac{\pi}{2}(1-t(A,B,c))}{\frac{(c+p)(1-B)+A-1}{1-B}+\alpha\cos\frac{\pi}{2}(1-t(A,B,c))}\right) \tag{2.17}$$

when

$$t(A,B,c) = \frac{2}{\pi}\sin^{-1}\left(\frac{p(A-B)}{(c+p)(1-B^2)-p(1-AB)}\right).$$

Proof. Let

$$q(z) = -\frac{1}{p-\gamma}\left(\frac{z(D^nF(z))'}{D^nG(z)}+\gamma\right).$$

Since $g \in \Sigma_p^*[n; A, B]$, from Proposition 2.2, $g \in \Sigma_p^*[n; A, B]$.

Using (2.10), we have

$$(p-\gamma)q(z)D^{n+p-1}G(z)-(c+p)D^{n+p-1}F(z)=-cD^{n+p-1}f(z)-\gamma D^{n+p-1}G(z).$$

Then, by a simple calculation, we get

$$(p-\gamma)(zq'(z)+q(z)(-r(z)+c+p))+\gamma(-r(z)+c+p)=-\frac{cz(D^{n+p-1}f(z))'}{D^{n+p-1}G(z)},$$

where

$$r(z)=-\frac{z(D^{n+p-1}G(z))'}{D^{n+p-1}G(z)}.$$

Hence we have

$$q(z)+\frac{zq'(z)}{-r(z)+c+p}=-\frac{1}{p-\gamma}\left(\frac{z(D^{n+p-1}f(z))'}{D^{n+p-1}g(z)}+\gamma\right).$$

The remaining part of the proof is similar to that of Theorem 2.1 and so we omit it.

Letting $n=-p+1$, $A=1$, $B=0$ and $\delta=1$ in Theorem 2.3, we have

Corollary 2.4. *Let $c>0$ and $f\in\Sigma$. If*

$$-\mathrm{Re}\left\{\frac{zf'(z)}{g(z)}\right\}>\gamma\ \ (0\leq\gamma<p)$$

for some $g\in\Sigma_p$ satisfying the condition

$$\left|\frac{zg'(z)}{g(z)}+p\right|<p,$$

then

$$-\mathrm{Re}\left\{\frac{zF'(z)}{G(z)}\right\}>\gamma,$$

where F and G are given by (2.9) and (2.16), respectively.

Taking $n=0$, $B\to A$ and $g(z)=\frac{1}{z^p}$ in Theorem 2.3, we have

Corollary 2.5. *Let $c>0$ and $f\in\Sigma_p$. If*

$$|\arg(-z^{p+1}f'(z)-\gamma)|<\frac{\pi}{2}\delta,\ \ (0\le\gamma<p\ ;\ 0<\delta\le 1)$$

then

$$|\arg(-z^{p+1}F'(z)-\gamma)|<\frac{\pi}{2}\alpha,$$

where F is the integral operator given by (2.9) and α $(0<\alpha\le 1)$ is the solution of the equation

$$\delta=\alpha+\frac{2}{\pi}\tan^{-1}\left(\frac{\alpha}{c+p-1}\right).$$

By using the same methods as in proving Theorem 2.3, we have

Theorem 2.4. *Let $f\in\Sigma_p$ and choose a positive number c such that*

$$c\ge\frac{1+A}{1+B}-p,$$

where $-1<B<A\le 1$ and $p\in\mathbb{N}$. If

$$\left|\arg\left(\frac{z(D^{n+p-1}f(z))'}{D^{n+p-1}g(z)}+\gamma\right)\right|<\frac{\pi}{2}\delta\ \ (\gamma>p\ ;\ 0<\delta\le 1)$$

for some $g\in\Sigma_p^*[n;A,B]$, then

$$\left|\arg\left(\frac{z(D^{n+p-1}F(z))'}{D^{n+p-1}G(z)}+\gamma\right)\right|<\frac{\pi}{2}\alpha,$$

where F and G are given by (2.9) and (2.16), respectively, and $\alpha(0<\alpha\le 1)$ is the solution of the equation given by (2.17)

Finally, we derive

Theorem 2.5. *Let $f\in\Sigma_p$. Choose an integer n such that*

$$n\ge\frac{p(1+A)}{1+B}-2p,$$

where $-1<B<A\le 1$ and $p\in\mathbb{N}$. If

$$\left|\arg\left(-\frac{z(D^{n+p-1}f(z))'}{D^{n+p-1}g(z)}-\gamma\right)\right|<\frac{\pi}{2}\delta\ \ (0\le\gamma<p\ ;\ 0<\delta\le 1)$$

for some $g\in\Sigma_p^*[n;A,B]$, then

$$\left|\arg\left(-\frac{z(D^{n+p}F(z))'}{D^{n+p}G(z)}-\gamma\right)\right|<\frac{\pi}{2}\delta,$$

where F and G are given by (2.9) and (2.16) with $c=n+p$, respectively.

Proof. From (2.7) and (2.8) with $c=n+p$, we have $D^{n+p-1}f(z)=D^{n+p}F(z)$ Therefore

$$\frac{z(D^{n+p-1}f(z))'}{D^{n+p-1}g(z)}=\frac{z(D^{n+p}F(z))'}{D^{n+p}G(z)}$$

and the result follows.

Acknowledgement. The author wishes to acknowledge the financial support of the Korea Research Foundation made in the program year of 1997.

References

1. Nak Eun Cho, *On certain subclasses of meromorphically multivalent convex functions*, J. Math. Anal. Appl., **193**(1995), 255-263.
2. P. Enigenberg,S. S. Miller, P. T. Mocanu and M. O. Reade, *On a Briot-Bouquet Differential subordination*, General Inequalities, **3**(Birkhauser Verlag-Basel), 339-348.
3. M. D. Ganigi and B. A. Uralegaddi, *New criteria for meromorphic univalent functions*, Bull. Math. Soc. Sci. Math. R. S. Roumanie(N. S.), **33**(1989), 9-13.
4. S. S. Miller and P. T. Mocanu,*Differential subordinations and univalent functions*, Michigan Math. J., **28**(1981), 157-171.
5. M. Nunokawa, *On the order of strongly starlikeness of strongly convex functions*, Proc. Japan Acad. Ser. A Math. Sci., **69**(1993), 234-237.
6. H. Silverman and E. M. Silvia, *Subclasses of starlike functions subordinate to convex functions*, Canad. J. Math., **37**(1985), 48-61.
7. B. A. Uralegaddi and H. S. Path, *A new criterion for p-valent meromorphic convex functions*, Math. Chronicle, **18**(1989), 75-77.
8. B. A. Uralegaddi and C. Somanatha, *A new criterion for meromorphic convex functions*, Tamkang J. Math., **19**(1988), 43-48.
9. B. A. Uralegaddi and C. Somanatha, *Certain subclasses of meromorphic convex functions*, Indian J. Math., **32**(1990), 49-57.

*Department of Applied Mathematics, Pukyong National University, Pusan 608-737, Korea

**Department of Mathematics, Kinki University, Higashi-Osaka, Osaka 577-8502, Japan