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2 次元複素空間形内の定曲率極小曲面に付随したある常微分方程式系について

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これは1999 年 9 月16 日から 22 日にかけてルーマニアのブラショフ(Brasov)で行われる予定の第 4 回微分幾何学国際研究集会の講演記録集のための原稿である．それは数理解析研究所での講演（1999 年 6 月 25 日），東京大学で行われた第 46 回幾何学シンポジウムでの連続講演の後半部分（1999 年 8 月 3 日）の講演原稿をもとにして書かれている．

1 Introduction

This note is a report on part of an investigation undertaken by the author and Zhou [10] last year. The main motivation came from the constructions of interesting examples of constant mean curvature surfaces in complex space forms. In mathematics, "good" examples are very much important. For example, Wente's tori [14] and the surfaces found by Kapouleas [7], [8] and [9] are stimulating us to study the theory of constant mean curvature surfaces (that call cmc surfaces for short) in the Euclidean three space E^3.

The notion of the mean curvature of surfaces is old and familier, but untill 1985 the round sphere is the only compact cmc surface in E^3. This made the geometry for the mean curvature rather poor.

By a cmc surface in a complex space form, we mean the surface such that the length of the mean curvature vector is constant. In order to have the intriguing examples, we hope to study a restricted class of the surfaces, which are surfaces with parallel mean curvautre vector in a complex space form.

The surfaces with parallel mean curvaure vector in four dimensional real space forms are studied in Hoffman [6], Chen [1], and Yau [15]. These are locally cmc surfaces in a totally geodesic or totally umbilic hypersurface of the ambient manifold.

For the complex case, the situation is different. Minimal surfaces in the complex projective plane, CP^2, are extensively studied by Eschenburg, Guadalupe and Tribuzy [5]. When the mean curvature vectors of the surfaces are not zero, Chen [3] has found a non-trivial example of the surface with parallel mean curvature vector in a complex hyperbolic plane. Let $M^2(K)$ and $CH^2(4\rho)$ be a two dimensional Riemannian manifold of constant Gaussian curvature K and complex two dimensional complex hyperbolic space form of constant holomorphic sectional curvature $4\rho(\leq 0)$ respectively. It is the isometric immersion

$$X : M^2(-\frac{2}{3}) \longrightarrow CH^2(-4) \tag{1}$$

such that both the Kaehler angle, θ, and the length of the mean curvature vector, $|H|$, are constant given by

$$\theta = \cos^{-1}\frac{1}{3}, \quad |H| = \frac{2}{\sqrt{3}}.$$

Later on, Chen-Tazawa [4] obtained the explicit representation of the immersion using the bundle struture of $CH^2(-4)$.

Surfaces with parallel mean curvature vector in the complex Euclidean plane C^2 are locally classified [2] when the Kaehler angles are constant.

Recently, by the joint work with Zhou [10], the author has found new examples of the surfaces with parallel mean curvature vector in C^2 and $CH^2(4\rho)$ such that their Kaehler angles are not constant. It shall be remarked that the important examples only appear in the complex space forms of non-positive curvature. This contrasts with the usual submanifold theory in Kaehler manifolds (cf. [13]).

The following is our main result:

Theorem 1 (Kenmotsu-Zhou, 1998) *Let M^2 be a real two dimensional connected Riemannian manifold and $\overline{M}^2(4\rho)$ the complex two dimensional complex space form of constant holomorphic sectional curvature 4ρ. Let $X : M^2 \longrightarrow \overline{M}^2(4\rho)$ be an isometric immersion from M^2 into $\overline{M}^2(4\rho)$ with non-zero parallel mean curvature vector.*

(1) *When $\rho > 0$, the Kaehler angle of X is constant and the image is locally congruent to the Clifford torus.*

(2) *When $\rho \leq 0$, there exist the surfaces such that the Kaehler angles are not constant. Moreover we classified them.*

2 The overdetermined system of Ogata

Ogata [12] initiated the study of surfaces with non-zero parallel mean curvature vector in a complex space form $\overline{M}^2(4\rho)$. For the immersion $X : M^2 \longrightarrow \overline{M}^2(4\rho)$ with non-zero parallel mean curvature vector, he found the following overdetermined system:

$$\left\{\begin{array}{l} \dfrac{d\lambda}{du} = -2\lambda(u)^2(a(u)-b)\cot\theta(u), \quad \lambda(u) > 0, \quad u \in I \\ \dfrac{d\theta}{du} = 2\lambda(u)(a(u)+b) \\ \dfrac{da}{du} = 2\lambda(u)\left\{2a(u)(a(u)-b)\cot\theta(u) + \dfrac{3}{4}\rho\sin 2\theta(u)\right\} \\ \log\left\{\lambda(u)^4\left(a(u)^2 - \dfrac{\rho}{2}\left(3\cos^2\theta(u) - 1\right)\right)\right\} = k_1 u + k_2, \end{array}\right. \tag{2}$$

where I is an open interval, and $b,\ \rho,\ k_1,$ and $(-\infty \leq)\ k_2$ are real numbers.

In the system (2), the variable u is one of the functions of the isothermal coordinates (u, v) on M^2, $\lambda(u)$ represents the Riemannian metric on M^2, $\theta(u)$ and $a(u)$ describe the Kaehler angle and the second fundamental forms of X respectively.

Conversely, given real numbers $\rho, b(>0), k_1$ and $(-\infty \leq)\ k_2$, any solutions $\lambda(u), \theta(u)$, and $a(u)$ of the system (2) define an isometric immersion X from an M^2 in the complex space form $\overline{M}^2(4\rho)$ with parallel mean curvature vector such that the length of the mean curvature vector is equal to b. In fact, M^2 is a domain of the product space $I \times R$, the first fundamental form of the immersion X is given by

$$ds^2 = \lambda(u)^2(du^2 + dv^2), \quad (u, v) \in M^2 \subset I \times R \tag{3}$$

and the second fundamental forms given, with respect to an orthonormal normal frame $\{e_3, e_4\}$,

$$\begin{aligned} h_{e_3} &= \begin{pmatrix} -a(u) - 2b - \Re c(u,v) & -\Im c(u,v) \\ -\Im c(u,v) & a(u) - 2b + \Re c(u,v) \end{pmatrix}, \\ h_{e_4} &= \begin{pmatrix} -\Im c(u,v) & -a(u) + \Re c(u,v) \\ -a(u) + \Re c(u,v) & \Im c(u,v) \end{pmatrix}, \end{aligned} \tag{4}$$

where we put, for a real number t,

$$c(u,v) = \sqrt{a(u)^2 - \frac{\rho}{2}\left(3\cos^2\theta(u) - 1\right)}\exp\sqrt{-1}\left(-\frac{k_1}{2}v + t\right).$$

Example Let ρ and b be any real numbers satisfying $b^2 + \rho/2 > 0$. Then,

$$\lambda(u) = \text{constant } (=\lambda), \quad \theta(u) = \frac{\pi}{2}, \text{ and } \quad a(u) = -b \tag{5}$$

satisfy the system (2) for $k_1 = 0$ and $k_2 = \log\{\lambda^4(b^2 + \rho/2)\}$.

This defines a totally real immersion from the two dimensional flat Riemannian manifold into $\overline{M}^2(4\rho)$ with parallel second fundamental forms. We know the explicit formulas for these immersions in Ogata [12] when $\rho > 0$, Chen [2] when $\rho = 0$, and Naitoh [11] when $\rho < 0$.

All solutions with $\theta(u) = constant$ are easily obtained:

Proposition 1 • *When $\rho \geq 0$, the solution of* (2) *with $\theta(u) = constant$ is only* (5).

- *When $\rho < 0$, the solutions of* (2) *with $\theta(u) = constant$ are* (1) *and* (5).

Proposition 1 classifies the slant immersions in $\overline{M}^2(4\rho)$ with parallel mean curvature vector.

3 The case of $k_1 = 0$

In the beginning, let us solve the system (2) when $k_1 = 0$. We may assume that $\theta(u)$ is not constant, hence so $a(u)$.

Assuming that ρ is positive, we find a contradiction. Therefore, we do not have new solutions of (2) in this case.

When ρ is zero , any solution of (2) such that $\theta(u)$ is not a constant function is represented by, using the function $a = a(u)$ as new variable,

$$\left\{ \begin{array}{l} \lambda^2(a) = \dfrac{c_1}{a}\,, \quad a > 0 \\ \sin^2\theta(a) = c_2\dfrac{(a-b)^2}{a}\,, \end{array} \right. \tag{6}$$

where c_1 and c_2 are positive numbers, and $a = a(u)$ satisfies the first order differential equation.

When ρ is negative, at first we have $\rho = -3b^2$. Next, let us consider the differential equation

$$\frac{d\theta}{du} = \sqrt{b}\sqrt{8 - 9\sin^2\theta(u)}\,. \tag{7}$$

Using the solution $\theta = \theta(u)$ of the equation (7), put

$$\left\{ \begin{array}{l} \lambda(u) = \dfrac{2}{\sqrt{b}\sqrt{8 - 9\sin^2\theta(u)}} \\ a(u) = b\left(1 - \frac{9}{4}\sin^2\theta(u)\right). \end{array} \right. \tag{8}$$

Then, $\lambda(u), \theta(u)$ and $a(u)$ defined by (7) and (8) are the general solutions of (2). It shall be remarked that the system (2) with $k_1 = 0$ has non-trivial solutions only when $\rho \leq 0$.

Representation of the surfaces. We have the explicit representation of the immersion given by (3), (4) and (6) as follows:

$$\left(F_1(u) \cdot e^{\sqrt{-1}\tau_1 v}, \quad F_2(u) \cdot e^{\sqrt{-1}\tau_2 v}\right) \in C^2 \tag{9}$$

where $F_1(u)$ and $F_2(u)$ are the complex valued functions written by $\lambda(u), \theta(u)$, and $a(u)$, and τ_1 and τ_2 are some real numbers.

In the case of $CH^2(-3b^2)$, we get an explict representation of the coordinate functions of the surface by using the bundle structure of the complex hyperbolic space form. Let π be the projection of the Anti-de Sitter space time $H_1^5(-1)(\subset C_1^2)$ onto $CH^2(-4)$. Put

$$E_0(u,v) = \left(e^{\rho_0(u)v}, e^{\rho_1(u)v}, e^{\rho_2(u)v}\right) S(u) \ \in \ H_1^5(-1)$$

where $S(u)$ is a 3×3-matrix function, and $\rho_0(u), \rho_1(u)$ and $\rho_2(u)$ are eigenvalues of the following matrix:

$$\sqrt{-1}\lambda(u)\begin{pmatrix} 0 & -\sqrt{3}b\cos\frac{\theta(u)}{2} & \sqrt{3}b\sin\frac{\theta(u)}{2} \\ \sqrt{3}b\cos\frac{\theta(u)}{2} & (a(u)-b)\cot\frac{\theta(u)}{2} & b - c(u;t) \\ -\sqrt{3}b\sin\frac{\theta(u)}{2} & b-\bar{c}(u;t) & (a(u)-b)\tan\frac{\theta(u)}{2} \end{pmatrix}.$$

Then, the surface defined by (3), (4), (7) and (8) is integrated as $\pi \cdot E_0(u,v)$.

The surfaces which we found are isometric and have the same length of the mean curvature vector. This may correspond to the associated family of the cmc surfaces in R^3.

4 The case of $k_1 \neq 0$

The crucial point in Theorem 1 is to prove that even locally there is no solution of the system (2) when $k_1 \neq 0$ and $\theta(u)$ is not constant.

We will find a contradiction assuming that the system (2) with $k_1 \neq 0$ has a solution $\lambda(u), \theta(u)$ and $a(u)$ such that $\theta(u)$ is not constant. We may also assume that $\lambda(u)$ and $\theta(u)$ are not constant. By the change of the variable $x = \sin^2\theta$,

$\lambda(x)$ and $a(x)$ are non constant solutions of the following system:

$$\left\{\begin{array}{l} \dfrac{d\lambda}{dx}=-\dfrac{1}{2}\lambda(x)\dfrac{(a(x)-b)}{(a(x)+b)}\cdot\dfrac{1}{x}, \quad x>0 \\ \dfrac{da}{dx}=\dfrac{a(x)(a(x)-b)}{(a(x)+b)}\cdot\dfrac{1}{x}+\dfrac{3\rho}{4}\cdot\dfrac{1}{(a(x)+b)} \\ 2\rho\lambda(x)\sqrt{1-x}\cdot(4(a(x)-b)+9bx)=k_1\sqrt{x}\cdot\left(a(x)^2-\rho+\dfrac{3\rho}{2}x\right). \end{array}\right. \tag{10}$$

Let $I=(x_1,x_2)$ be the maximal interval of the existence of the solutions of the system (10). We have $x_1\geq 0$ and $\rho\neq 0$. In the analysis of (10), the hardest part is to prove $x_1=0$. Then, it is shown that *there exixts* $\lim_{x\to 0}a(x)$ *and the limit is equal to* 0 *or* b. We need the following uniqueness theorem:

Proposition 2 *For any real numbers* ρ, b, a_0 *and* a_0', *the differential equation*

$$\frac{da}{dx}=\frac{a(x)(a(x)-b)}{a(x)+b}\cdot\frac{1}{x}+\frac{3\rho}{4}\cdot\frac{1}{a(x)+b}, \quad 0<x<x_2,$$

has at most one solution under the initial conditions of

$$\lim_{x\to 0}a(x)=a_0, \quad \lim_{x\to 0}a'(x)=a_0'.$$

By using these results, we can prove that

Proposition 3 (main result) *For any real numbers* ρ *and* $k_1(\neq 0)$, *the system* (2) *has no solution such that* $\lambda(u)>0$ *and* $\theta(u)\neq$ *constant.*

We thus proved Theorem 1.

Remark By [1], [15], surfaces with parallel mean curvature vector in the real four dimensional Euclidean space E^4 are locally constant mean curvature surfaces in a hyperplane or a round sphere of E^4 and by [6], we know how to construct such surfaces. Theorem 1 gives us new insight for the immersion: Let $X:M\longrightarrow E^4$ be an isometric immersion with parallel mean curvature vector in E^4. We consider E^4 as the complex two plane C^2 by taking an almost complex structure on E^4. Then, Theorem 1 says that either it is totally real for the complex structure or locally congruent to a surface of the family cited above. In particular, such a surface must be rotational if and only if τ_1/τ_2 is rational.

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以上