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Extension of M-convexity and L-convexity to Polyhedral Convex Functions * (Extended Abstract)

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Abstract

The concepts of M-convex and L-convex functions were proposed by Murota in 1996 as two mutually conjugate classes of discrete functions over integer lattice points. M/L-convex functions are deeply connected with the well-solvability in nonlinear combinatorial optimization with integer variables. In this paper, we extend the concept of M-convexity and L-convexity to polyhedral convex functions, aiming at clarifying the well-behaved structure in well-solved nonlinear combinatorial optimization problems in real variables. The extended M/L-convexity often appear in nonlinear combinatorial optimization problems with piecewise-linear convex cost. We investigate the structure of polyhedral M-convex and L-convex functions from the dual viewpoint of analysis and combinatorics, and provide some properties and characterizations. It is also shown that polyhedral M/L-convex functions have nice conjugacy relationship.

1 Introduction

In the area of combinatorial optimization, there exist many "well-solved" problems, i.e., the problems which have nice combinatorial structure and which can be solved efficiently (see, [2, 12]). Many researchers have been trying to identify the well-behaved structure in combinatorial optimization problems.

The concept of matroid, introduced by Whitney [28], plays an important role in the field of combinatorial optimization (see [27, 29]). Matroidal structure is closely related to the well-solvability of combinatorial optimization problems such as those on graphs and matroids, and can be found in fairly large number of efficiently solvable problems. Matroidal structure yields the tractability of problems in the following way:

- Global optimality is equivalent to local optimality, which implies the success of the so-called greedy algorithm for the problem of optimizing a linear function over a single matroid.
- A nice duality theorem, Edmonds' intersection theorem [6], guarantees the existence of a certificate for the optimality in the matroid intersection problem in terms of dual variables.

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In 1970, Edmonds introduced the concept of polymatroid by extending that of matroid to sets of real vectors ([6], see also [27]). A polymatroid $P \subseteq \mathbf{R}_+^V$ is a polyhedron given as

$$P = \{x \in \mathbf{R}_+^V \mid \sum_{w \in V} x(w) \leq \rho(X) \ (\forall X \subseteq V)\}$$

by a submodular set function $\rho : 2^V \to \mathbf{R}$ with certain additional conditions, where $\mathbf{R}_+$ denotes the set of nonnegative reals, and ρ is called submodular if

$$\rho(X) + \rho(Y) \geq \rho(X \cap Y) + \rho(X \cup Y) \qquad (\forall X, Y \subseteq V). \tag{1}$$

Polymatroids share nice combinatorial properties of matroids: for example, the greedy algorithm for matroids still works for polymatroids, and a duality holds for the polymatroid intersection problem. Fujishige, emphasizing the essential role of submodularity of ρ, generalized the concept of polymatroid to that of submodular system [9].

In recent years, nonlinear combinatorial optimization problems are investigated more often due to theoretical interest and necessity in practical application. The nonlinear resource allocation problem and the convex cost submodular flow problem are examples of nonlinear combinatorial optimization problems. Both of the problems have nice combinatorial structures, which lead to efficient combinatorial algorithms. These results, however, do not completely fit in the framework of matroid, polymatroid, and submodular system.

The concepts of M-convex and L-convex functions, introduced by Murota [16, 17, 19], afford a nice framework for well-solved nonlinear combinatorial optimization problems. M-convex function is a natural extension of the concept of valuated matroid introduced by Dress–Wenzel [4, 5] (see also [14, 15]) as well as a quantitative generalization of the set of integral points in an integral base polyhedron [9]. L-convex function is an extension of submodular set function.

Let V be a finite set. A function $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ is called M-convex if it satisfies (M-EXC[**Z**]):

(M-EXC[Z]) $\forall x, y \in \mathrm{dom}_{\mathbf{Z}} f$, $\forall u \in \mathrm{supp}^+(x-y)$, $\exists v \in \mathrm{supp}^-(x-y)$ such that

$$f(x) + f(y) \geq f(x - \chi_u + \chi_v) + f(y + \chi_u - \chi_v),$$

where $\mathrm{dom}_{\mathbf{Z}} f = \{x \in \mathbf{Z}^V \mid -\infty < f(x) < +\infty\}$, $\mathrm{supp}^+(x-y) = \{w \in V \mid x(w) > y(w)\}$, $\mathrm{supp}^-(x-y) = \{w \in V \mid x(w) < y(w)\}$, and $\chi_w \in \{0,1\}^V$ is the characteristic vector of $w \in V$. A function $g : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ is called L-convex† if it satisfies (LF1[**Z**]) and (LF2[**Z**]):

(LF1[Z]) $\quad g(p) + g(q) \geq g(p \wedge q) + g(p \vee q) \qquad (\forall p, q \in \mathrm{dom}_{\mathbf{Z}} g)$,
(LF2[Z]) $\quad \exists r \in \mathbf{R}$ such that $g(p + \lambda \mathbf{1}) = g(p) + \lambda r \quad (\forall p \in \mathrm{dom}_{\mathbf{Z}} g,\ \lambda \in \mathbf{Z})$,

where $p \wedge q, p \vee q \ (\in \mathbf{R}^V)$ denote the vectors with $(p \wedge q)(v) = \min\{p(v), q(v)\}$, $(p \vee q)(v) = \max\{p(v), q(v)\}$ $(v \in V)$, and $\mathbf{1} \ (\in \mathbf{R}^V)$ is the vector with each component being equal to one. M/L-convex functions have nice properties:

- local optimality is equivalent to global optimality.
- M/L-convex functions can be extended to ordinary convex functions.
- M/L-convex functions are conjugate to each other.
- a (discrete) separation theorem and a Fenchel-type duality theorem hold for a pair of M-convex/M-concave (L-convex/L-concave) functions.

† In the original definition [19], an L-convex function is assumed to be integer-valued.

The minimization of M/L-convex functions can be done in polynomial time [7, 25]. Application of M-convex functions can be found in system analysis through polynomial matrices [18, 20], and in mathematical economics [3].

M-convexity and L-convexity appear in various nonlinear combinatorial optimization problems with integer variables. Such nice combinatorial properties, however, are enjoyed not only by combinatorial optimization problems in integer variables but also by those in real variables. We dwell on this point by considering the minimum cost flow/tension problems.

Let $G = (V, A)$ be a directed graph with a specified vertex subset $T \subseteq V$. Suppose we are given a family of piecewise-linear convex functions $f_a : \mathbf{R} \to \mathbf{R} \cup \{+\infty\}$ $(a \in A)$, each of which represents the cost of flow on the arc a. A function $\xi : A \to \mathbf{R}$ is called a flow. The boundary $\partial\xi : V \to \mathbf{R}$ of a flow ξ is given by

$$\partial\xi(v) = \sum\{\xi(a) \mid a(\in A) \text{ leaves } v\} - \sum\{\xi(a) \mid a(\in A) \text{ enters } v\} \qquad (v \in V).$$

Then, the cost function $f : \mathbf{R}^T \to \mathbf{R} \cup \{\pm\infty\}$ of the minimum cost flow that realizes a supply/demand vector $x \in \mathbf{R}^T$ is defined by

$$f(x) = \inf\left\{\sum_{a\in A} f_a(\xi(a)) \;\middle|\; \xi \in \mathbf{R}^A,\ \partial\xi(w) = \left\{\begin{array}{ll} -x(w) & (w \in T) \\ 0 & (w \in V \setminus T) \end{array}\right\}\right\} \quad (x \in \mathbf{R}^T). \tag{2}$$

Suppose we are given another family of piecewise-linear convex functions $g_a : \mathbf{R} \to \mathbf{R} \cup \{+\infty\}$ $(a \in A)$, each of which represents the cost of tension on the arc a. Any function $p : V \to \mathbf{R}$ is called a potential. Given a potential p, its coboundary $\delta p : A \to \mathbf{R}$ is defined by

$$\delta p(a) = p(u) - p(v) \qquad (a = (u, v) \in A).$$

Then, the cost function $g : \mathbf{R}^T \to \mathbf{R} \cup \{\pm\infty\}$ of the minimum cost tension that realizes a potential vector $p' \in \mathbf{R}^T$ is written as

$$g(p') = \inf\{\sum_{a\in A} g_a(-\delta p(a)) \mid p \in \mathbf{R}^V,\ p(w) = p'(w)\ (w \in T)\} \qquad (p' \in \mathbf{R}^T). \tag{3}$$

It is well-known that the minimum cost flow/tension problems with piecewise-linear convex cost can be solved efficiently by various combinatorial algorithms (see [24]). It can be shown that both f and g are polyhedral convex functions, which is a direct extension of results in Iri [11] and Rockafellar [24] for the case of $|T| = 2$.

We consider here the cost functions $f_{\mathbf{Z}}$ and $g_{\mathbf{Z}}$ for the integer version of the minimum cost flow/tension problems:

$$\begin{aligned} f_{\mathbf{Z}}(x) &= \inf\left\{\sum_{a\in A} f_a(\xi(a)) \;\middle|\; \xi \in \mathbf{Z}^A,\ \partial\xi(w) = \left\{\begin{array}{ll} -x(w) & (w \in T) \\ 0 & (w \in V \setminus T) \end{array}\right\}\right\} \quad (x \in \mathbf{Z}^T), \\ g_{\mathbf{Z}}(p') &= \inf\{\sum_{a\in A} g_a(-\delta p(a)) \mid p \in \mathbf{Z}^V,\ p(w) = p'(w)\ (w \in T)\} \qquad (p' \in \mathbf{Z}^T). \end{aligned}$$

It is shown in [19, 21] that $f_{\mathbf{Z}}$ satisfies (M-EXC[**Z**]) and $g_{\mathbf{Z}}$ satisfies (LF1[**Z**]) and (LF2[**Z**]), i.e., $f_{\mathbf{Z}}$ are $g_{\mathbf{Z}}$ are M-convex and L-convex, respectively.

These results indicate that the polyhedral convex functions f and g defined by (2) and (3) must have nice combinatorial properties like M-convexity and L-convexity, respectively. We can show that f satisfies the property (M-EXC):

(M-EXC) $\forall x, y \in \mathrm{dom}\, f$, $\forall u \in \mathrm{supp}^+(x-y)$, $\exists v \in \mathrm{supp}^-(x-y)$, $\exists \alpha_0 > 0$ such that

$$f(x) + f(y) \geq f(x - \alpha(\chi_u - \chi_v)) + f(y + \alpha(\chi_u - \chi_v)) \qquad (0 \leq \forall \alpha \leq \alpha_0),$$

which is a generalization of (M-EXC[**Z**]), and g satisfies (LF1) and (LF2):

(LF1) $g(p) + g(q) \geq g(p \wedge q) + g(p \vee q) \quad (\forall p, q \in \mathrm{dom}\, g)$,
(LF2) $\exists r \in \mathbf{R}$ such that $g(p + \lambda \mathbf{1}) = g(p) + \lambda r \quad (\forall p \in \mathrm{dom}\, g, \forall \lambda \in \mathbf{R})$,

which can be obtained by generalizing (LF1[**Z**]) and (LF2[**Z**]), where $\mathrm{dom}\, f = \{x \in \mathbf{R}^V \mid -\infty < f(x) < +\infty\}$, $\mathrm{dom}\, g = \{p \in \mathbf{R}^V \mid -\infty < g(p) < +\infty\}$.

The observation above indicates the possibility of extending the concepts of M-convexity and L-convexity to polyhedral convex functions. This can be done in the following way. For a polyhedral convex function $f : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$, we call f M-convex if $\mathrm{dom}\, f \neq \emptyset$ and f satisfies the property (M-EXC). Similarly, for a polyhedral convex function $g : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$ we call g L-convex if $\mathrm{dom}\, g \neq \emptyset$ and g satisfies (LF1) and (LF2).

The aim of this paper is to investigate the structures of polyhedral M-convex and L-convex functions from the dual viewpoint of analysis and combinatorics, and to provide a nice framework for well-solvable nonlinear combinatorial optimization problems in real variable. The organization of this paper is as follows. The details and proofs of theorems can be found in the full paper [22].

To investigate polyhedral M/L-convex functions, we need to consider the set version of M/L-convexity. A polyhedron $B \subseteq \mathbf{R}^V$ is called M-convex if it is not empty and satisfies (B-EXC):

(B-EXC) $\forall x, y \in B$, $\forall u \in \mathrm{supp}^+(x-y)$, $\exists v \in \mathrm{supp}^-(x-y)$, $\exists \alpha_0 > 0$ such that

$$x - \alpha(\chi_u - \chi_v) \in B, \; y + \alpha(\chi_u - \chi_v) \in B \qquad (0 \leq \forall \alpha \leq \alpha_0).$$

As is explained later in Theorem 2.1, an M-convex polyhedron is nothing but the base polyhedron of a submodular system [9]. Similarly, a polyhedron $D \subseteq \mathbf{R}^V$ is called L-convex if it is not empty and satisfies (LS1) and (LS2):

(LS1) $p \wedge q, \; p \vee q \in D \; (\forall p, \; q \in D)$, **(LS2)** $p \in D \Longrightarrow p + \lambda \mathbf{1} \in D \; (\forall \lambda \in \mathbf{R})$.

We show the polyhedral description of M/L-convex polyhedra in Section 2.

Section 3 shows fundamental properties of polyhedral M/L-convex functions. We give some properties on local structure of polyhedral M/L-convex functions such as directional derivatives, subdifferentials, minimizers, etc. In Section 3, we also investigate positively homogeneous polyhedral M/L-convex functions, which are important subclasses of polyhedral M/L-convex functions. It is shown that positively homogeneous polyhedral M/L-convex functions have one-to-one correspondences with certain set functions, and also with L/M-convex polyhedra.

For a function $f : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$, its conjugate function $f^\bullet : \mathbf{R}^V \to \mathbf{R} \cup \{\pm\infty\}$ is defined by

$$f^\bullet(p) = \sup_{x \in \mathbf{R}^V} \{\langle p, x \rangle - f(x)\} \qquad (p \in \mathbf{R}^V),$$

where $\langle p, x \rangle = \sum\{p(v)x(v) \mid v \in V\}$. It is shown in [19, 21] that there is a conjugacy relationship between M/L-convex functions over the integer lattice. In Section 4, we show that the conjugacy relationship also exists for polyhedral M/L-convex functions. Section 4 also provides various characterization of polyhedral M/L-convex functions by local structures such as directional derivative, the set of minimizers, and subdifferentials.

2 M-convex and L-convex Polyhedra

2.1 M-convex Polyhedra

We denote by $\mathcal{M}_0$ the family of M-convex polyhedra, i.e.,

$$\mathcal{M}_0 = \{B \subseteq \mathbf{R}^V \mid B: \text{ M-convex polyhedron}\}.$$

It is well-known as a folklore that what we call an "M-convex polyhedron" is nothing but the base polyhedron of a submodular system [9] (see also Theorem 2.1). We use the term "M-convex polyhedron" for denotational symmetry to "L-convex polyhedron."

We shall show that an M-convex polyhedron is described by a submodular set function. We denote the class of (normalized) submodular set functions by

$$\mathcal{S} \quad = \quad \{\rho : 2^V \to \mathbf{R} \cup \{+\infty\} \mid \rho : \text{submodular}, \ \rho(\emptyset) = 0, \ \rho(V) < +\infty\}.$$

For any nonempty $B \subseteq \mathbf{R}^V$, we define $\rho_B : 2^V \to \mathbf{R} \cup \{+\infty\}$ by

$$\rho_B(X) = \sup_{x \in B} x(X) \quad (X \subseteq V).$$

For a set function $\rho : 2^V \to \mathbf{R} \cup \{+\infty\}$, we define $\mathrm{B}(\rho) \subseteq \mathbf{R}^V$ by

$$\mathrm{B}(\rho) = \{x \in \mathbf{R}^V \mid x(X) \leq \rho(X) \ (X \subseteq V), \ x(V) = \rho(V)\}.$$

The following fact has been known to experts (cf. [1], [6], [27, Chapter 18]), but the precise statement cannot be found in the literature.

Theorem 2.1 (i) *For $B \in \mathcal{M}_0$, we have $\rho_B \in \mathcal{S}$ and $\mathrm{B}(\rho_B) = B$.*
(ii) *For $\rho \in \mathcal{S}$, we have $\mathrm{B}(\rho) \in \mathcal{M}_0$ and $\rho_{\mathrm{B}(\rho)} = \rho$.*
(iii) *The mappings $B \mapsto \rho_B$ ($B \in \mathcal{M}_0$) and $\rho \mapsto \mathrm{B}(\rho)$ ($\rho \in \mathcal{S}$) provide one-to-one correspondences between $\mathcal{M}_0$ and $\mathcal{S}$, and are the inverse of each other.*

2.2 L-convex Polyhedra

We denote by $\mathcal{L}_0$ the family of L-convex polyhedra, i.e.,

$$\mathcal{L}_0 = \{D \subseteq \mathbf{R}^V \mid D: \text{ L-convex polyhedron}\}.$$

We show the system of inequalities which describes the polyhedral structure of L-convex polyhedra. A function $\gamma : V \times V \to \mathbf{R} \cup \{+\infty\}$ with $\gamma(v, v) = 0$ ($\forall v \in V$) is called a *distance function*. For a distance function γ we define the set $\mathrm{D}(\gamma) \subseteq \mathbf{R}^V$ by

$$\mathrm{D}(\gamma) = \{p \in \mathbf{R}^V \mid p(v) - p(u) \leq \gamma(u, v) \ (u, v \in V)\}.$$

Given a nonempty set $D \subseteq \mathbf{R}^V$, the function $\gamma_D : V \times V \to \mathbf{R} \cup \{+\infty\}$ is defined by

$$\gamma_D(u, v) = \sup_{p \in D} \{p(v) - p(u)\}.$$

Note that γ_D is a distance function, and $D \subseteq \mathrm{D}(\gamma_D)$ holds in general.

We consider the *triangle inequality*

$$\gamma(v_1, v_2) + \gamma(v_2, v_3) \geq \gamma(v_1, v_3) \quad (\forall v_1, v_2, v_3 \in V) \tag{4}$$

for distance functions, and let $\mathcal{T}$ be the family of distance functions with triangle inequality, i.e.,

$$\mathcal{T} = \{\gamma \mid \gamma : V \times V \to \mathbf{R} \cup \{+\infty\},\ \gamma(v, v) = 0\ (v \in V),\ \gamma \text{ satisfies (4)}\}.$$

Theorem 2.2 (i) *For $D \in \mathcal{L}_0$, we have $\gamma_D \in \mathcal{T}$ and $\mathrm{D}(\gamma_D) = D$.*
(ii) *For $\gamma \in \mathcal{T}$, we have $\mathrm{D}(\gamma) \in \mathcal{L}_0$ and $\gamma_{\mathrm{D}(\gamma)} = \gamma$.*
(iii) *The mappings $D \mapsto \gamma_D$ ($D \in \mathcal{L}_0$) and $\gamma \mapsto \mathrm{D}(\gamma)$ ($\gamma \in \mathcal{T}$) provide a one-to-one correspondence between $\mathcal{L}_0$ and $\mathcal{T}$, and are the inverse of each other.*

3 Polyhedral M-convex and L-convex Functions

3.1 Polyhedral M-convex Functions

We denote

$$\begin{aligned} \mathcal{M} &= \{f \mid f : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\},\ \text{polyhedral M-convex}\}, \\ {}_0\mathcal{M} &= \{f \mid f : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\},\ \text{positively homogeneous polyhedral M-convex}\}. \end{aligned}$$

It may be obvious from the definition that polyhedral M-convex functions are quantitative extension of M-convex polyhedra.

Theorem 3.1 (i) *For a function $f : \mathbf{R}^V \to \{0, +\infty\}$, we have $f \in \mathcal{M} \iff \mathrm{dom}\, f \in \mathcal{M}_0$.*
(ii) *For $f \in \mathcal{M}$, we have $\mathrm{dom}\, f \in \mathcal{M}_0$.*

We consider two slightly different exchange axioms, where the former is weaker and the latter is stronger than (M-EXC).

(M-EXC$_\mathbf{w}$) $\forall x, y \in \mathrm{dom}\, f$, $\forall u \in \mathrm{supp}^+(x - y)$, $\exists v \in \mathrm{supp}^-(x - y)$, $\exists \alpha > 0$ such that

$$f(x) + f(y) \geq f(x - \alpha(\chi_u - \chi_v)) + f(y + \alpha(\chi_u - \chi_v)).$$

(M-EXC$_\mathbf{s}$) $\forall x, y \in \mathrm{dom}\, f$, $\forall u \in \mathrm{supp}^+(x - y)$, $\exists v \in \mathrm{supp}^-(x - y)$ such that

$$f(x) + f(y) \geq f(x - \alpha(\chi_u - \chi_v)) + f(y + \alpha(\chi_u - \chi_v)) \quad (0 \leq \forall \alpha \leq \{x(u) - y(u)\}/2k),$$

where $k = |\mathrm{supp}^-(x - y)|$.

Theorem 3.2 *For a polyhedral convex function $f : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$ with $\mathrm{dom}\, f \neq \emptyset$,* (M-EXC) $\iff$ (M-EXC$_\mathrm{w}$) $\iff$ (M-EXC$_\mathrm{s}$).

Global optimality of a polyhedral M-convex function is characterized by local optimality. For a polyhedral convex function $f : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$ and $x \in \mathrm{dom}\, f$, define $f'(x; \cdot, \cdot) : V \times V \to \mathbf{R} \cup \{+\infty\}$ by

$$f'(x; v, u) = \lim_{\alpha \downarrow 0} \frac{f(x + \alpha(\chi_v - \chi_u)) - f(x)}{\alpha} \qquad (u, v \in V).$$

Theorem 3.3 *Let $f \in \mathcal{M}$ and $x \in \mathrm{dom}\, f$. Then, $f(x) \leq f(y)$ $(\forall y \in \mathbf{R}^V)$ $\iff$ $f'(x; v, u) \geq 0$ $(\forall u, v \in V)$.*

For a polyhedral convex function $f : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$ and $x \in \mathrm{dom}\, f$, the *subdifferential* $\partial f(x)$ of f at x is defined by

$$\partial f(x) = \{p \in \mathbf{R}^V \mid f(y) \geq f(x) + \langle p, y - x\rangle \ (\forall y \in \mathbf{R}^V)\}.$$

Directional derivative functions and subdifferentials of a polyhedral M-convex function have nice structures such as M/L-convexity, and they can be explicitly described by certain distance functions with triangle inequality (cf. Theorem 3.4 (i)).

For any distance function $\gamma : V \times V \to \mathbf{R} \cup \{+\infty\}$ (i.e., $\gamma(v, v) = 0$ for all $v \in V$), we define $f_\gamma : \mathbf{R}^V \to \mathbf{R} \cup \{\pm\infty\}$ by

$$f_\gamma(x) = \inf\{ \sum_{u,v \in V} \lambda_{uv}\gamma(u, v) \mid \sum_{u,v \in V} \lambda_{uv}(\chi_v - \chi_u) = x,\ \lambda_{uv} \geq 0\ (u, v \in V)\} \quad (x \in \mathbf{R}^V). \tag{5}$$

Theorem 3.4 *Let $f \in \mathcal{M}$ and $x \in \mathrm{dom}\, f$.*
(i) *The function $\gamma_x : V \times V \to \mathbf{R} \cup \{+\infty\}$ defined by*

$$\gamma_x(u, v) = f'(x; v, u) \qquad (u, v \in V)$$

satisfies $\gamma_x(v, v) = 0$ $(\forall v \in V)$ and the triangle inequality (4), i.e., $\gamma_x \in \mathcal{T}$.
(ii) *We have $f'(x; \cdot) = f_{\gamma_x}$ and $f'(x; \cdot) \in {}_0\mathcal{M}$.*

L-convexity appears in subdifferentials of a polyhedral M-convex function.

Theorem 3.5 *Let $f \in \mathcal{M}$ and $x \in \mathrm{dom}\, f$.*
(i) *$\partial f(x) \in \mathcal{L}_0$, and $\partial f(x)$ is represented as*

$$\partial f(x) = \mathrm{D}(\gamma_x) = \{p \in \mathbf{R}^V \mid p(v) - p(u) \leq f'(x; v, u)\ (u, v \in V)\}.$$

(ii) *For any $y \in \mathbf{R}^V$ we have $f(y) - f(x) \geq \sup_{p \in \partial f(x)} \langle p, y - x\rangle = f_{\gamma_x}(y - x)$.*

The next theorem shows that each face of the epigraph of a polyhedral M-convex function is an M-convex polyhedron when it is projected to $\mathbf{R}^V$. For any $p \in \mathbf{R}^V$, the function $f[p] : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$ is defined by $f[p](x) = f(x) + \langle p, x\rangle$ $(x \in \mathbf{R}^V)$.

Theorem 3.6 *For $f \in \mathcal{M}$ and $p \in \mathbf{R}^V$, we have* $\arg\min f[-p] \in \mathcal{M}_0$ *if* $\inf f[-p] > -\infty$.

The class of polyhedral M-convex functions is closed under various fundamental operations.

Theorem 3.7 *Let $f, f_1, f_2 \in \mathcal{M}$.*
(1) *For $a \in \mathbf{R}^V$, the functions $f(a - x)$ and $f(a + x)$ are polyhedral M-convex in x.*
(2) *For any $U \subseteq V$, the function $f_U : \mathbf{R}^U \to \mathbf{R} \cup \{+\infty\}$ defined by*

$$f_U(y) = f(y, \mathbf{0}_{V \setminus U}) \qquad (y \in \mathbf{R}^U)$$

is polyhedral M-convex if $\mathrm{dom}\, f_U \neq \emptyset$.
(3) *For a family of piecewise-linear convex functions* $\varphi_v : \mathbf{R} \to \mathbf{R} \cup \{+\infty\}$ $(v \in V)$, *the function* $\tilde{f} : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$ *defined by*

$$\tilde{f}(x) = f(x) + \sum_{v \in V} \varphi_v(x(v)) \quad (x \in \mathbf{R}^V)$$

is polyhedral M-convex if $\mathrm{dom}\, \tilde{f} \neq \emptyset$. *In particular, the function* $f[-p] : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$ *is polyhedral M-convex for any* $p \in \mathbf{R}^V$.
(4) *For any* $a : V \to \mathbf{R} \cup \{-\infty\}$ *and* $b : V \to \mathbf{R} \cup \{+\infty\}$ *with* $a \leq b$, *the restriction* $f_{[a,b]}$ *of* f *given by*

$$f_{[a,b]}(x) = \begin{cases} f(x) & (x \in [a,b]), \\ +\infty & (x \notin [a,b]) \end{cases}$$

is polyhedral M-convex if $\mathrm{dom}\, f \cap [a,b] \neq \emptyset$.

We show the relationship of positively homogeneous polyhedral M-convex functions to distance functions with triangle inequalities, and also to L-convex polyhedra.

For a positively homogeneous polyhedral convex function $f : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$ with $\mathbf{0} \in \mathrm{dom}\, f$, define $\gamma_f : V \times V \to \mathbf{R} \cup \{+\infty\}$ by

$$\gamma_f(u,v) = f'(\mathbf{0}; v, u) \ (= f(\chi_v - \chi_u)) \qquad (u, v \in V).$$

Recall the definition of f_γ in (5).

Theorem 3.8 (i) *For* $f \in {}_0\mathcal{M}$, *we have* $\gamma_f \in \mathcal{T}$ *and* $f_{\gamma_f} = f$.
(ii) *For* $\gamma \in \mathcal{T}$, *we have* $f_\gamma \in {}_0\mathcal{M}$ *and* $\gamma_{f_\gamma} = \gamma$.
(iii) *The mappings* $f \mapsto \gamma_f$ $(f \in {}_0\mathcal{M})$ *and* $\gamma \mapsto f_\gamma$ $(\gamma \in \mathcal{T})$ *provide a one-to-one correspondence between* ${}_0\mathcal{M}$ *and* $\mathcal{T}$, *and are the inverse of each other.*

For any $S \subseteq \mathbf{R}^V$ with $S \neq \emptyset$, the *support function* $\delta_S^* : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$ of S is defined by

$$\delta_S^*(p) = \sup_{x \in S} \langle p, x \rangle \qquad (p \in \mathbf{R}^V).$$

For any positively homogeneous function $f : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$, we define the set $S_f \subseteq \mathbf{R}^V$ by

$$S_f = \{x \in \mathbf{R}^V \mid \langle p, x \rangle \leq f(p) \ (\forall p \in \mathbf{R}^V)\}.$$

Theorem 3.9 (i) *For* $f \in {}_0\mathcal{M}$, *we have* $S_f \in \mathcal{L}_0$ *and* $\delta_{S_f}^* = f$.
(ii) *For* $D \in \mathcal{L}_0$, *we have* $\delta_D^* \in {}_0\mathcal{M}$ *and* $S_{\delta_D^*} = D$.
(iii) *The mappings* $f \mapsto S_f$ $(f \in {}_0\mathcal{M})$ *and* $D \mapsto \delta_D^*$ $(D \in \mathcal{L}_0)$ *provide a one-to-one correspondence between* ${}_0\mathcal{M}$ *and* $\mathcal{L}_0$, *are the inverse of each other.*

3.2 Polyhedral L-convex Functions

We denote

$$\begin{aligned} \mathcal{L} &= \{g \mid g : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}, \text{ polyhedral L-convex}\}, \\ {}_0\mathcal{L} &= \{g \mid g : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}, \text{ positively homogeneous polyhedral L-convex}\}. \end{aligned}$$

As is obvious from the definition, polyhedral L-convex functions are quantitative generalization of L-convex polyhedra.

Theorem 3.10 (i) *For a function* $g : \mathbf{R}^V \to \{0, +\infty\}$, $g \in \mathcal{L} \iff \mathrm{dom}\, g \in \mathcal{L}_0$.
(ii) *For* $g \in \mathcal{L}$, *we have* $\mathrm{dom}\, g \in \mathcal{L}_0$.

Global optimality of a polyhedral L-convex function is characterized by local optimality.

Theorem 3.11 *Let* $g \in \mathcal{L}$ *and* $p \in \mathrm{dom}\, g$. *Then,* $g(p) \le g(q)$ $(\forall q \in \mathbf{R}^V)$ *if and only if* $g'(p; \chi_X) \ge 0$ $(\forall X \subseteq V)$ *and* $g'(p; \chi_V) = r = 0$, *where* r *is in* (LF2).

Given a set function $\rho : 2^V \to \mathbf{R} \cup \{+\infty\}$, we define $g_\rho : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$ by

$$g_\rho(p) = \sum_{j=1}^{k-1} (p_j - p_{j+1}) \rho(V_j) + p_k \rho(V_k), \tag{6}$$

where $p_1 > p_2 > \cdots > p_k$ are distinct values in $\{p(v)\}_{v \in V}$, and $V_j = \{v \in V \mid p(v) \ge p_j\}$ $(j = 1, \cdots, k)$. The function g_ρ is called the *Lovász extension* of ρ.

Theorem 3.12 *If* $\rho \in \mathcal{S}$, *then* $g_\rho(p) = \sup\{\langle p, x \rangle \mid x \in \mathrm{B}(\rho)\}$ $(\forall p \in \mathbf{R}^V)$.

Theorem 3.13 (Lovász [13]) *Let* $\rho : 2^V \to \mathbf{R} \cup \{+\infty\}$ *be a function such that* $\rho(\emptyset) = 0$ *and* $\rho(V) < +\infty$. *Then,* $\rho \in \mathcal{S} \iff g_\rho$ *is convex.*

Directional derivative functions and subdifferentials of a polyhedral L-convex function have nice structures such as M/L-convexity, and they can be explicitly described by certain submodular functions (cf. Theorem 3.14 (i)).

Theorem 3.14 *Let* $g \in \mathcal{L}$ *and* $p \in \mathrm{dom}\, g$.
(i) *The function* $\rho_p : 2^V \to \mathbf{R} \cup \{+\infty\}$ *defined by*

$$\rho_p(X) = g'(p; \chi_X) \qquad (X \subseteq V)$$

satisfies $\rho_p(\emptyset) = 0$, $-\infty < \rho_p(V) < +\infty$, *and the submodular inequality (1), i.e.,* $\rho_p \in \mathcal{S}$.
(ii) *We have* $g'(p; \cdot) = g_{\rho_p}$ *and* $g'(p; \cdot) \in {}_0\mathcal{L}$.

M-convexity appears in subdifferentials of a polyhedral L-convex function.

Theorem 3.15 *Let* $g \in \mathcal{L}$ *and* $p \in \mathrm{dom}\, g$.
(i) $\partial g(p) \in \mathcal{M}_0$, *and* $\partial g(p)$ *is represented as*

$$\partial g(p) = \mathrm{B}(\rho_p) = \{x \in \mathbf{R}^V \mid x(X) \le g'(p; \chi_X) \ (\forall X \subseteq V),\ x(V) = g'(p; \chi_V)\}.$$

(ii) *For any* $q \in \mathbf{R}^V$ *we have* $g(p+q) - g(p) \ge \sup\{\langle q, x \rangle \mid x \in \partial g(p)\} = g_{\rho_p}(q)$.

The next theorem shows that each face of the epigraph of a polyhedral L-convex function is an L-convex polyhedron when it is projected to $\mathbf{R}^V$.

Theorem 3.16 *For* $g \in \mathcal{L}$ *and* $x \in \mathbf{R}^V$, *we have* $\arg\min g[-x] \in \mathcal{L}_0$ *if* $\inf g[-x] > -\infty$.

The class of polyhedral L-convex functions are closed under various fundamental operations.

Theorem 3.17 *Let $g, g_1, g_2 \in \mathcal{L}$.*
(1) *For $x \in \mathbf{R}^V$, the function $g[-x] : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$ is polyhedral L-convex.*
(2) *For $a \in \mathbf{R}^V$ and $\beta \in \mathbf{R}$, the function $g(a + \beta p)$ is polyhedral L-convex in p.*
(3) *For any $U \subseteq V$, the function $g^U : \mathbf{R}^U \to \mathbf{R} \cup \{\pm\infty\}$ given by*

$$f^U(y) = \inf_{z \in \mathbf{R}^{V \setminus U}} f(y, z) \qquad (y \in \mathbf{R}^U)$$

is polyhedral L-convex if $g^U > -\infty$.
(4) *For a family of piecewise-linear convex function $\psi_v : \mathbf{R} \to \mathbf{R} \cup \{+\infty\}$ $(v \in V)$, the function $\tilde{g} : \mathbf{R}^V \to \mathbf{R} \cup \{\pm\infty\}$ defined by*

$$\tilde{g}(p) = \inf_{q \in \mathbf{R}^V} \{g(q) + \sum_{v \in V} \psi_v(p(v) - q(v))\} \quad (p \in \mathbf{R}^V)$$

is polyhedral L-convex if $\tilde{g} > -\infty$ and $\mathrm{dom}\,\tilde{g} \neq \emptyset$.
(5) *$g_1 + g_2 \in \mathcal{L}$ if $\mathrm{dom}\,g_1 \cap \mathrm{dom}\,g_2 \neq \emptyset$.*

We show the relationship of positively homogeneous polyhedral L-convex functions with submodular functions, and with M-convex polyhedra.

For a positively homogeneous polyhedral convex function $g : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$ with $\mathbf{0} \in \mathrm{dom}\,g$, define a set function $\rho_g : 2^V \to \mathbf{R} \cup \{+\infty\}$ by

$$\rho_g(X) = g'(\mathbf{0}; \chi_X) \ (= g(\chi_X)) \qquad (X \subseteq V).$$

Recall the definition of g_ρ in (6).

Theorem 3.18 (i) *For $g \in {}_0\mathcal{L}$, we have $\rho_g \in \mathcal{S}$ and $g_{\rho_g} = g$.*
(ii) *For $\rho \in \mathcal{S}$, we have $g_\rho \in {}_0\mathcal{L}$ and $\rho_{g_\rho} = \rho$.*
(iii) *The mappings $g \mapsto \rho_g$ $(g \in {}_0\mathcal{L})$ and $\rho \mapsto g_\rho$ $(\rho \in \mathcal{S})$ provide a one-to-one correspondence between ${}_0\mathcal{L}$ and $\mathcal{S}$, and are the inverse of each other.*

Theorem 3.19 (i) *For $g \in {}_0\mathcal{L}$, we have $S_g \in \mathcal{M}_0$ and $\delta^*_{S_g} = g$.*
(ii) *For $B \in \mathcal{M}_0$, we have $\delta^*_B \in {}_0\mathcal{L}$ and $S_{\delta^*_B} = B$.*
(iii) *The mappings $g \mapsto S_g$ $(g \in {}_0\mathcal{L})$ and $B \mapsto \delta^*_B$ $(B \in \mathcal{M}_0)$ provide a one-to-one correspondence between ${}_0\mathcal{L}$ and $\mathcal{M}_0$, and are the inverse of each other.*

From Theorem 3.18, we see that a polyhedral convex function is positively homogeneous polyhedral L-convex if and only if it is the Lovász extension of a submodular set function.

Corollary 3.20 ${}_0\mathcal{L} = \{g_\rho \mid \rho \in \mathcal{S}\}$.

4 Conjugacy and Characterizations

Polyhedral M-convex and L-convex functions are conjugate to each other.

Theorem 4.1 *For $f \in \mathcal{M}$ and $g \in \mathcal{L}$, we have $f^\bullet \in \mathcal{L}$ and $g^\bullet \in \mathcal{M}$. More specifically, the mappings $f \mapsto f^\bullet$ $(f \in \mathcal{M})$ and $g \mapsto g^\bullet$ $(g \in \mathcal{L})$ provide a one-to-one correspondence between $\mathcal{M}$ and $\mathcal{L}$, and are the inverse of each other.*

Polyhedral M/L-convex functions are characterized by local polyhedral structures such as directional derivative functions, subdifferentials, and the sets of minimizers.

Theorem 4.2 *Let* $f : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$ *be a polyhedral convex function with* $\mathrm{dom}\, f \neq \emptyset$. *Then,*

$$\text{(i) } f \in \mathcal{M} \iff \text{(ii) } f'(x;\cdot) \in {}_0\mathcal{M}\ (\forall x \in \mathrm{dom}\, f) \iff \text{(iii) } \partial f(x) \in \mathcal{L}_0\ (\forall x \in \mathrm{dom}\, f)$$
$$\iff \text{(iv) } \arg\min f[-p] \in \mathcal{M}_0\ (\forall p \in \mathbf{R}^V \textit{ with } \inf f[-p] > -\infty).$$

Theorem 4.3 *Let* $g : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$ *be a polyhedral convex function with* $\mathrm{dom}\, g \neq \emptyset$. *Then,*

$$\text{(i) } g \in \mathcal{L} \iff \text{(ii) } g'(p;\cdot) \in {}_0\mathcal{L}\ (\forall p \in \mathrm{dom}\, g) \iff \text{(iii) } \partial g(p) \in \mathcal{M}_0\ (\forall p \in \mathrm{dom}\, g)$$
$$\iff \text{(iv) } \arg\min g[-x] \in \mathcal{L}_0\ (\forall x \in \mathbf{R}^V \textit{ with } \inf g[-x] > -\infty).$$

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