数理解析研究所講究録
1116 巻 1999 年 165-180

BOUNDEDNESS OF OPERATORS ON BESOV SPACES ON A FRACTAL SET

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1. Introduction

Let D be a bounded domain in $\mathbf{R}^d$ $(d \geq 2)$ such that the boundary ∂D of D is a β-set satisfying $d-1 \leq \beta < d$. We say that a closed set F is a β-set if there exist a positive Radon measure μ on F and positive real numbers b_1, b_2, r_0 such that

$$b_1 r^\beta \leq \mu(B(x,r)\cap F) \leq b_2 r^\beta \tag{1.1}$$

for all $z \in F$ and all $r \leq r_0$, where $B(z,r)$ stands for the open ball in $\mathbf{R}^d$ with center z and radius r. Such a measure μ is called a β-measure.

We give examples.

1. If D is a bounded Lipschitz domain in $\mathbf{R}^d$, then ∂D is a $(d-1)$-set and the surface measure is a $(d-1)$-measure.

2. If ∂D consists of a finite number of self-similar sets, which satisfies the open set condition, and whose similarity dimensions are β, then ∂D is a β-set and the β-dimensional Hausdorff measure restricted to ∂D is a β-measure. The Von Koch snowflake is a typical example for $d=2$ and $\beta = \log 4/\log 3$.

We consider Besov spaces on a β-set ∂D. In general let F be a closed β-set in $\mathbf{R}^d$ and μ be a β-measure on F. Let $0 \leq \beta-(d-1) < \alpha \leq 1$. We define a Besov space $\Lambda^p_\alpha(F)$ by the Banach spaçe of all function $f \in L^p(\mu)$ such that

$$\iint \frac{|f(x)-f(z)|^p}{|x-z|^{\beta+p\alpha}} d\mu(x)d\mu(z) < \infty$$

with norm

$$\|f\|_{\alpha,p} = \left(\int |f(x)|^p d\mu(x)\right)^{1/p} + \left(\iint \frac{|f(x)-f(z)|^p}{|x-z|^{\beta+p\alpha}} d\mu(x)d\mu(z)\right)^{1/p}.$$

Hereafter we shall fix a β-measure μ on ∂D and suppose $\overline{D} \subset B(0,R/2)$ with $R \geq 1$. We may assume that (1.1) replaced F with ∂D holds for all $z \in \partial D$ and all $r \leq 3R$.

Further we denote by $\mathcal{V}(G)$ the Whitney decomposition of an open set G (cf. [S]) and simply set $\mathcal{V} = \mathcal{V}(\mathbf{R}^d \setminus \partial D)$.

According to Jonsson-Wallin, we constructed in [W3] an extension operator $\mathcal{E}$ having the following properties.

Proposition A *Assume that $\overline{D} \subset B(0, R/2)$. Then there exists a linear operator $\mathcal{E}$ from $L^p(\mu)$ to $L^p(\mathbf{R}^d)$ having the properties (i)-(vi):*
(i) $\mathcal{E}(f)$ *is a* C^∞*-function in* $\mathbf{R}^d \setminus \partial D$,
(ii) $\mathcal{E}(f) = f$ *on* ∂D,
(iii) supp $\mathcal{E}(f) \subset B(0, 2R)$,
(iv) $\mathcal{E}(1) = 1$ *on* $\overline{B(0,R)}$,
(v)

$$\int |\mathcal{E}(f)|^p dy \leq c \int |f|^p d\mu,$$

where c is a constant independent f,
(vi) *Let $Q \in \mathcal{V}$ be a cube with common side-length l. Then, for each $y \in Q \cap B(0, 2R)$,*

$$|\frac{\partial}{\partial y_i}\mathcal{E}(f)(y)| \leq c l^{-\beta-1} \int_{B(a,sl)} |f(z)| d\mu(z) \quad (i = 1, \cdots, d),$$

where a is a boundary point satisfying dist$(\partial D, Q) =$dist(a, Q) and $s = 6\sqrt{d}$, and c is a constant independent of l, y and f.

We note that dist(A, B) stands for the distance between a set A and B.
In the above Besov space our aim is to prove the boundedness of the operators K_1 and K_2, which are important to solve the Dirichlet problem for D and $\mathbf{R}^d \setminus \overline{D}$ by layer potential method.
The operators K_1 and K_2 are defined as follows: Define, for $f \in \Lambda^p_\alpha(\partial D)$ and $z \in \partial D$,

$$K_1 f(z) = \int_{\mathbf{R}^d \setminus \overline{D}} \langle \nabla_y \mathcal{E}(f)(y), \nabla_y N(z-y) \rangle dy$$

and if it is well-defined and $K_1 f(z) = 0$ otherwise, and

$$K_2 f(z) = -\int_D \langle \nabla_y \mathcal{E}(f)(y), \nabla_y N(z-y) \rangle dy$$

if it is well-defined and $K_2 f(z) = 0$ otherwise, where

$$N(x-y) = \begin{cases} \frac{1}{\omega_d (d-2)|x-y|^{d-2}} & \text{if } d \geq 3 \\ -\frac{3R}{2\pi} \log \frac{|x-y|}{3R} & \text{if } d = 2 \end{cases}$$

and ω_d stands for the surface area of the unit ball in $\mathbf{R}^d$.
But it is difficult to prove directly the boundedness of K_1 and K_2 on $\Lambda^p_\alpha(\partial D)$. So we introduce another Besov spaces $\mathcal{B}^+_{\alpha,p}$ and $\mathcal{B}^-_{\alpha,p}$, which are near spaces to $\Lambda^p_\alpha(\partial D)$. The

space $\mathcal{B}^+_{\alpha,p}$ (resp. $\mathcal{B}^-_{\alpha,p}$) is, for p, α satisfying $p > 1$ and $p - p\alpha - d + \beta > 0$, defined to be the Banach space of all $f \in L^p(\mu)$ satisfying

$$\int_D |\nabla \mathcal{E}(f)(y)|^p \delta(y)^{p-p\alpha-d+\beta} dy < \infty$$

$$(\text{resp. } \int_{\mathbf{R}^d \setminus \overline{D}} |\nabla \mathcal{E}(f)(y)|^p \delta(y)^{p-p\alpha-d+\beta} dy < \infty),$$

with norm

$$\|f\|_{\mathcal{B}^+_{\alpha,p}} := \left(\int |f|^p d\mu\right)^{1/p} + \left(\int_D |\nabla \mathcal{E}(f)(y)|^p \delta(y)^{p-p\alpha-d+\beta} dy\right)^{1/p}$$

$$\left(\text{resp. } \|f\|_{\mathcal{B}^-_{\alpha,p}} := \left(\int |f|^p d\mu\right)^{1/p} + \left(\int_{\mathbf{R}^d \setminus \overline{D}} |\nabla \mathcal{E}(f)(y)|^p \delta(y)^{p-p\alpha-d+\beta} dy\right)^{1/p}\right),$$

where $\delta(y)$ stands for the distance of y from ∂D.

Hereafter we assume that $p > 1$ and $1 - (d - \beta) < \alpha < 1 - \frac{d-\beta}{p}$ and denote by G the set D or $\mathbf{R}^d \setminus \overline{D}$.

To study the relations of $\mathcal{B}^-_{\alpha,p}$ or $\mathcal{B}^+_{\alpha,p}$ and $\Lambda^p_\alpha(\partial D)$, we introduce the following maximal function on $\partial D \times \partial D$. To do so, define

$$F_0 = \{y \in \mathbf{R}^d; \delta(y) \le \frac{R}{10}\}$$

and fix a real number b satisfying $1 < b \le 11/10$. We define, for $h \in L^p(\mu \times \mu)$ and $y \in G \cap F_0$,

$$\begin{aligned} &M(\mu \times \mu)h(y) \\ &= \sup\{\frac{1}{\mu(B(y,r) \cap \partial D)^2} \int_{B(y,r)\cap\partial D} \int_{B(y,r)\cap\partial D} |h(x,z)| d\mu(x) d\mu(z); \\ &\quad b\delta(y) \le r \le \frac{R}{4}\}. \end{aligned}$$

Denote by ν_0 the positive measure on G defined by

$$\nu_0(E) = \int_{E\cap G\cap F_0} \delta(y)^{2\beta-d} dy \tag{1.2}$$

for a Borel set E.

We shall obtain the following lemma in §2.

Lemma 1.1 (i) *Let $t > 0$, $h \in L^1(\mu \times \mu)$ and set*

$$E_t = \{y \in G \cap F_0; M(\mu \times \mu)h(y) > t\}.$$

Then

$$\nu_0(E_t) \le ct^{-1} \iint |h(x,z)| d\mu(x) d\mu(z),$$

where c is a constant independent of f and t.

(ii) *Let $p > 1$ and $h \in L^p(\mu \times \mu)$. Then*

$$\int M(\mu \times \mu) h(y)^p d\nu_0(y) \le c \iint |h(x,z)|^p d\mu(x) d\mu(z).$$

The above lemma will be applied to prove the following theorem in §3.

Theorem 1 *Let $p > 1$ and $0 \le 1 - (d - \beta) < \alpha < 1 - (d - \beta)/p$. Further let $f \in \Lambda_\alpha^p(\partial D)$. Then*

$$\int_{\mathbf{R}^d} |\nabla \mathcal{E}(f)(y)|^p \delta(y)^{p - p\alpha - d + \beta} dy \le c \|f\|_{p,\alpha}^p,$$

where c is a constant independent of f.

We shall also introduce another maximal function. To do so, we define two measures. Fix a real number b satisfying $1 < b \le 11/10$ and let $\lambda \in \mathbf{R}$ satisfying $d - \beta + \lambda > 0$. The measure $\tau_{\lambda,G}^+$ (resp. $\tau_{\lambda,G}^-$) is defined by

$$\tau_{\lambda,G}^+(E) = \int_{E \cap G \cap F_0} \delta(y)^\lambda dy \quad (\text{resp. } \tau_{\lambda,G}^-(E) = \int_{E \cap (F_0 \setminus \overline{G})} \delta(y)^\lambda dy)$$

for a Borel measurable set E. We use τ_λ^+ (resp. τ_λ^-) instead of $\tau_{\lambda,G}^+$ (resp. $\tau_{\lambda,G}^-$) if there is no confusion.

Let $u \in L^1(\tau_\lambda^-)$. The maximal function $M(\tau_\lambda^-)u$ is defined by

$$M(\tau_\lambda^-)u(y) = \sup\{\frac{1}{\tau_\lambda^-(B(y,r))} \int_{B(y,r)} |u(x)| d\tau_\lambda^-(x); b\delta(y) \le r \le \frac{R}{4}\}$$

for $y \in G \cap F_0$.

We say that G satisfies the condition (b) if there exist a constant c and $r_1 > 0$ such that

$$|B(z,r) \cap G| \ge cr^d \tag{1.3}$$

for each $z \in \partial D$ and each $r \le r_1$, where $|A|$ stands for the d-dimensional volume of a set A. We note that, if G satisfies the condition (b), we may assume that (1.3) holds for $r \le 3R$.

We shall prove the following lemma in §2.

Lemma 1.2 *Let* $d-\beta+\lambda>0$ *and assume that* $\mathbf{R}^d\setminus\overline{G}$ *satisfies the condition (b).*
(i) *Let* $t>0$ *and* $u\in L^1(\tau_\lambda^-)$, *and set*

$$E_t=\{y\in G\cap F_0; M(\tau_\lambda^-)u(y)>t\}.$$

Then

$$\tau_\lambda^+(E_t)\le\frac{c}{s}\int|u|d\tau_\lambda^-(x),$$

where c *is a constant independent of* u *and* s.
(ii) *Let* $p>1$. *Then*

$$\int M(\tau_\lambda^-)u(y)^p d\tau_\lambda^+(y)\le c\int|u(x)|^p d\tau_\lambda^-(x)$$

for every $u\in L^p(\tau_\lambda^-)$.

This lemma will be useful to prove the following theorem in §4.

Theorem 2 *Assume that* D *is a bounded domain in* $\mathbf{R}^d$ *such that* $\mathbf{R}^d\setminus\overline{D}$ *is also connected and* ∂D *is a* β*-set* $(d-1\le\beta<d)$. *Let* $p>1$ *and* $1-(d-\beta)<\alpha<1-(d-\beta)/p$.
(i) *If* $\mathbf{R}^d\setminus\overline{D}$ *satisfies the condition (b), then* K_1 *is a bounded operator from* $\mathcal{B}_{\alpha,p}^-$ *to* $\mathcal{B}_{\alpha,p}^+$.
(ii) *If* D *satisfies the condition (b), then* K_2 *is a bounded operator from* $\mathcal{B}_{\alpha,p}^+$ *to* $\mathcal{B}_{\alpha,p}^-$.

2. Maximal functions

We begin with estimates for two measures $\mu\times\mu$ and ν_0 defined by (1.2).

Lemma 2.1 *Fix* b *satisfying* $1<b\le 11/10$. *Then*

$$\nu_0(B(y,r)\cap G)\le c_1r^{2\beta}\le c_2\int_{B(y,r)\cap\partial D}\int_{B(y,r)\cap\partial D}d\mu(x)d\mu(z)$$

for every $y\in G\cap F_0$ *and every* r *satisfying* $b\delta(y)\le r\le(3/2)R$.

Proof. In [W1, Lemma 2.2] we saw that

$$\int_{B(z,\rho)}\delta(y)^k dy\le c_1\rho^{k+d} \tag{2.1}$$

for every $z\in\partial D$ and every $\rho\le 3R$ if $\beta-d<k$.

Let $y\in G\cap F_0$ and $b\delta(y)\le r\le(3/2)R$. Pick a point $z_y\in\partial D$ satisfying $\delta(y)=|y-z_y|$. Noting that $B(y,r)\subset B(z_y,2r)$ and using (2.1), we have

$$\int_{B(y,r)\cap G}\delta(x)^{2\beta-d}dx\le\int_{B(z_y,2r)}\delta(x)^{2\beta-d}\le c_2r^{2\beta},$$

which shows the first inequality.

Since $B(z_y, \frac{(b-1)}{b}r) \subset B(y,r)$ and ∂D is a β-set, we also get the second inequality. □

Let us prove Lemma 1.1.

Proof of Lemma 1.1. Let $h \in L^1(\mu \times \mu)$ and $t > 0$. Put

$$E_t = \{y \in G \cap F_0; M(\mu \times \mu)f(y) > t\}.$$

For each $y \in E_t$, there exists a ball $B(y,r)$ with $b\delta(y) \le r \le R/4$ such that

$$\int_{B(y,r)\cap\partial D}\int_{B(y,r)\cap\partial D} |h(x,z)| > t\int_{B(y,r)\cap\partial D} d\mu(x)\int_{B(y,r)\cap\partial D} d\mu(z). \tag{2.2}$$

Therefore we can find a countable covering $\{B(y_i,r_i)\}$ of E_t such that $B(y,r) = B(y_i,r_i)$ satisfies (2.2).

With the aid of Vitali's covering lemma we can choose a subfamily $\{B(w_j,\rho_j)\}$ of $\{B(y_i,r_i)\}$ such that $\{B(w_j,\rho_j)\}$ are mutually disjoint and $\{B(w_j,5\rho_j)\}$ covers E_t. Then, by Lemma 2.1 and (2.2),

$$\begin{aligned}\int_{E_t} \delta(y)^{2\beta-d}dy &\le \sum_j \int_{B(w_j,5\rho_j)\cap G} \delta(y)^{2\beta-d}dy \\ &\le c_1\sum_j (5\rho_j)^{2\beta} \le c_2\sum_j \int_{B(w_j,\rho_j)\cap\partial D} d\mu(x)\int_{B(w_j,\rho_j)\cap\partial D} d\mu(z) \\ &\le \frac{c_2}{t}\sum_j \int_{B(w_j,\rho_j)\cap\partial D}\int_{B(w_j,\rho_j)\cap\partial D} |h(x,z)|d\mu(x)d\mu(z).\end{aligned}$$

Noting that $\{B(w_j,\rho_j)\}$ are mutually disjoint,

$$\nu_0(E_t) \le \frac{c_2}{t}\iint |h(x,z)|d\mu(x)d\mu(z),$$

which shows (i).

The inequality (ii) deduces from (i) by the usual method. □

When G satisfies the condition (b), the following lemma is fundamental.

Lemma 2.2 *Assume that G satisfies condition (b). Let $0 < \epsilon \le 3R$, $0 < r \le 3R$, $z \in \partial D$ and put*

$$E_\epsilon = \{x \in G; \delta(x) < \epsilon\}.$$

Then

$$c_1\epsilon^{d-\beta}r^{\beta} \le \int_{E_\epsilon\cap B(z,r)} dx \le c_2\epsilon^{d-\beta}r^{\beta}, \tag{2.3}$$

where c_1 and c_2 are constants independent of ϵ, r and z.

Proof. In [W4, Lemma 2.1] we proved a lemma corresponding to this one under more strong condition. But the method used in the proof of Lemma 2.1 in [W4] is available under our weaker assumption without any change. $\square$

Lemma 2.3 *Suppose $\mathbf{R}^d \setminus \overline{G}$ satisfies the condition (b). Let $d - \beta + \lambda > 0$ and $1 < b \leq 11/10$. Further let $x_0 \in G \cap F_0$ and $b\delta(x_0) \leq r \leq (3/2)R$. Then*

$$\int_{B(x_0,r)\cap G} \delta(y)^\lambda \leq c_1 r^{\lambda+d} \leq c_2 \int_{B(x_0,r)\cap(F_0\setminus\overline{G})} \delta(y)^\lambda dy, \tag{2.4}$$

where c_1 and c_2 are constants independent of x_0 and r.

Proof. By (2.1) we get

$$\int_{B(x_0,r)\cap G} \delta(x)^\lambda dx \leq \int_{B(x_0',2r)\cap G} \delta(x)^\lambda dx \leq c_1 r^{\lambda+d},$$

where x_0' is a point of ∂D satisfying $\delta(x) = |x_0 - x_0'|$, which gives the first inequality of (2.4).

We next prove the second inequality of (2.4). First we assume that $\lambda > 0$. Let $x_0 \in G \cap F_0$, $b\delta(x_0) \leq r \leq (3/2)R$ and put

$$E_j = \{y \in F_0 \setminus \overline{G}; \delta(y)^\lambda < 2^{-j}\}.$$

Then $y \in E_j$ implies $\delta(y) < 2^{-j/\lambda}$. Noting that $r(1 - 1/b)) \leq r - \delta(x_0)$, we get

$$\begin{aligned} I \equiv \int_{B(x_0,r)\cap(F_0\setminus\overline{G})} \delta(y)^\lambda dy &\geq \int_{B(x_0',r(1-1/b))\cap(F_0\setminus\overline{G})} \delta(y)^\lambda dy \\ &\geq c_2 \sum_{j=j_0}^{\infty} 2^j \int_{B(x_0',r(1-1/b))\cap E_j} dy, \end{aligned}$$

where j_0 is the integer satisfying

$$\left(2^{-1/\lambda}\right)^{j_0-1} > r(1-1/b) \geq \left(2^{-1/\lambda}\right)^{j_0}.$$

Noting that $2^{-j/\lambda} \leq r(1 - 1/b) < r \leq (3/2)R$ for every $j \geq j_0$, we get, by Lemma 2.2,

$$\begin{aligned} I &\geq c_3 \sum_{j=j_0}^{\infty} 2^{-j} r^\beta (1-1/b)^\beta (2^{-j/\lambda})^{d-\beta} \\ &\geq c_4 \sum_{j=j_0}^{\infty} r^\beta 2^{-(1+(d-\beta)/\lambda)j} \geq c_5 2^{-(1+(d-\beta)/\lambda)j_0}. \end{aligned}$$

Noting that $d-\beta+\lambda>0$ and

$$2^{-(1+(d-\beta)/\lambda)j_0} = \left(2^{-j_0/\lambda}\right)^{\lambda+d-\beta} \geq c_6\,(r(1-1/b))^{d-\beta-\lambda} = c_7 r^{\lambda+d-\beta},$$

we get

$$I \geq c_8 r^{\beta} r^{d-\beta+\lambda} = c_8 r_0^{d+\lambda}.$$

This gives the second inequality of (2.4) in case $\lambda > 0$.

In case $\lambda < 0$, we put

$$E_j = \{y \in F_0 \setminus \overline{G}; \delta(y)^{\lambda} > 2^j\}$$

and can prove the second inequality of (2.4) by the above method.

Finally, assume that $\lambda = 0$. Since $\mathbf{R}^d \setminus \overline{G}$ satisfies the condition (b), we have

$$\int_{B(x_0,r)\cap(F_0\setminus\overline{G})} \delta(y)^{\lambda} dy \geq \int_{B(x_0',r(1-1/b))\cap(F_0\setminus\overline{G})} dy \geq c_9 r^d.$$

Thus we also see that the second inequality of (2.4) holds. □

Let us prove Lemma 1.2 by using the above lemma.

Proof of Lemma 1.2. Since the assertion (ii) deduces from (i) by the usual method, we shall prove only (i). Let $y \in E_t$. Then there exists a ball $B(y,r)$ such that $b\delta(y) \leq r \leq R/4$ and

$$\int_{B(y,r)} |u(x)| d\tau_{\lambda}^{-}(x) > t \int_{B(y,r)} d\tau_{\lambda}^{-}(x). \tag{2.5}$$

Hence we choose $\{y_j\} \subset E_t$ such that

$$E_s \subset \cup B(y_j, r_j), \qquad b\delta(y_j) \leq r_j \leq \frac{R}{4}$$

and $B(y,r) = B(y_j, r_j)$ satisfies (2.5).

Using Vilali's covering lemma, we select a subfamily $\{B(w_k,\rho_k)\}$ of $\{B(y_j,r_j)\}$ such that $\{B(w_k,\rho_k)\}$ are mutually disjoint and

$$E_t \subset \cup_k B(w_k, 5\rho_k).$$

Then, by Lemma 2.3 and (2.5),

$$\begin{aligned} \tau_{\lambda}^{+}(E_t) &\leq \sum_k \tau_{\lambda}^{+}(B(w_k,5\rho_k)) \leq c_1\,(5\rho_k)^{\lambda+d} \\ &\leq c_2 \sum_k \int_{B(w_k,\rho_k)} d\tau_{\lambda}^{-} \leq \frac{c_2}{t} \sum_k \int_{B(w_k,\rho_k)} |u(x)| d\tau_{\lambda}^{-}(x). \end{aligned}$$

Noting that $\{B(w_k,\rho_k)\}$ are mutually disjoint, we have the inequality of (i). □

3. Proof of Theorem 1

In this section we shall prove Theorem 1 by using Lemma 1.1.

Proof of Theorem 1. Let $\{Q_j\}$ be the Whitney decomposition of $\mathbf{R}^d \setminus \partial D$ in Proposition A. Denote by l_j and a_j the common side-length of Q_j and a boundary point satisfying $\operatorname{dist}(\partial D, Q_j) = \operatorname{dist}(a_j, O_j)$, respectively. Put

$$b_j = \frac{1}{\mu(B(a_j,\eta l_j))}\int_{B(a_j,\eta l_j)} f(w)d\mu(w),$$

where η is a fixed positive real number satisfying $0 < \eta < 1/4$ and used in the definition $\mathcal{E}(f)$.

With the aid of Proposition A we have, for each $y \in Q_j$

$$\begin{aligned}
&|\nabla\mathcal{E}(f-b_j)(y)| \\
&\le c_1\frac{1}{l_j^{\beta+1}l_j^{\beta}}\int_{B(a_j,sl_j)} d\mu(z)\int_{B(a_j,\eta l_j)} |f(z)-f(w)|d\mu(w) \\
&\le c_2 l_j^{\beta/p+\alpha-2\beta-1}\int_{B(a_j,sl_j)} d\mu(z)\int_{B(a_j,\eta l_j)} \frac{|f(z)-f(w)|}{|z-w|^{\beta/p+\alpha}}d\mu(w),
\end{aligned}$$

where $s = 6\sqrt{d}$. On the other hand let $y \in Q_j$ and x_j be a point in Q_j satisfying $|x_j - a_j| = \operatorname{dist}(a_j, Q_j)$. If $z \in B(a_j, sl_j) \cap \partial D$, then

$$\begin{aligned}
|y-z| &\le |y-x_j| + |x_j-a_j| + |a_j-z| \\
&\le \sqrt{d}l_j + 4\sqrt{d}l_j + sl_j = 11\sqrt{d}l_j.
\end{aligned}$$

Putting $s' = 11\sqrt{d}$, we have

$$\begin{aligned}
&|\nabla\mathcal{E}(f-b_j)(y)|\delta(y)^{1-\alpha-\beta/p} \\
&\le c_3\frac{1}{l_j^{2\beta}}\int_{B(y,s'l_j)\cap\partial D} d\mu(z)\int_{B(y,s'l_j)\cap\partial D} |h(z,w)|d\mu(w),
\end{aligned} \tag{3.1}$$

where $h(z,w) = \frac{|f(z)-f(w)|}{|z-w|^{\beta/p+\alpha}}$.

Put $s'' = R/(s'20\sqrt{d})$. First, let $l_j \le s''$ and $x \in Q_j$. Then

$$s'\delta(x) \le s'5\sqrt{d}l_j \le \frac{R}{4}.$$

Noting that

$$\mu(B(y,s'l_j)\cap\partial D) \le \mu(B(a_j,2s'l_j)\cap\partial D) \le c_4 l_j^{\beta},$$

we have, by Lemma 1.2 and (3.1),

$$|\nabla \mathcal{E}(f-b_j)(y)|\delta(y)^{1-\alpha-\beta/p} \le c_5 M(\mu\times\mu)h(y).$$

By virtue of Proposition A, (iv) we obtain

$$\begin{aligned}
&\sum_{l_j\le s''}\int_{Q_j}|\nabla\mathcal{E}(f)(y)|^p\delta(y)^{p-p\alpha-d+\beta}\\
&\le \sum_{l_j\le s''}\int_{Q_j}|\nabla\mathcal{E}(f-b_j)(y)|^p\delta(y)^{p-p\alpha-\beta}\delta(y)^{2\beta-d}dy\\
&\le c_6\sum_{l_j\le s''}\int_{Q_j}M(\mu\times\mu)h(y)^p d\nu_0(y)\le c_7\iint h(z,w)^p d\mu(z)\mu(w).
\end{aligned}$$

We next assume that $l_j\ge s''$. Then, by $y\in Q_j$, Proposition A, (vi) implies

$$|\nabla\mathcal{E}(f)(y)|\le c_8 l_j^{-\beta-1}\int_{B(a_j,sl_j)}|f(z)|d\mu(z)\le c_9\,(s'')^{-\beta/p-1}\,\|f\|_p.$$

Noting that supp $\mathcal{E}(f)\subset B(0,2R)$, we have

$$\begin{aligned}
&\sum_{l_j\ge s''}\int_{Q_j}|\nabla\mathcal{E}(f)(y)|^p\delta(y)^{p-p\alpha-d+\beta}dy\\
&\le c_{10}\,(s'')^{-\beta-p}\,\|f\|_p^p\int_{B(0,2R)}(2R)^{p-p\alpha-d+\beta}dy\le c_{11}\|f\|_p^p.
\end{aligned}$$

Thus we have

$$\int_{\mathbf{R}^d}|\nabla\mathcal{E}(f)(y)|^p\delta(y)^{p-p\alpha-d+\beta}dy\le c_{12}\left(\iint\frac{|f(z,w)|^p}{|z-w|^{\beta+p\alpha}}d\mu(z)d\mu(w)+\|f\|_p^p\right),$$

which completes the proof.

□

4. Proof of Theorem 2

In this section we prove Theorem 2. The proof of this theorem is essentially same as that of Theorem in [W4]. But we need improve on it to be available in the case $\beta=d-1$.

Proof of Theorem 2. (i) We first show that

$$\left(\int|K_1 f(z)|^p d\mu(z)\right)^{1/p}\le c_1\|f\|_{B^-_{\alpha,p}}. \tag{4.1}$$

Set $q = p/(p-1)$. Choosing $\epsilon_1 > 0$ satisfying $\epsilon_1 < \alpha$, we have, for $z \in \partial D$,

$$|K_1 f(z)| \leq c_2 \left(\int_{\mathbf{R}^d \setminus \overline{D}} |\nabla \mathcal{E}(f)(y)|^p \delta(y)^{p(1-\alpha-(d-\beta)/p)} |z-y|^{-\beta+\epsilon_1 p} dy \right)^{1/p}$$
$$\times \left(\int_{\mathbf{R}^d \setminus \overline{D}} \delta(y)^{-q(1-\alpha-(d-\beta)/p)} |z-y|^{q(1-d+\beta/p-\epsilon_1)} dy \right)^{1/q}.$$

Noting that $-q(1-\alpha-(d-\beta)/p)+d-\beta > 0$ and $-q(1-\alpha-(d-\beta)/p)+q(1-d+\beta/p-\epsilon_1) = q(\alpha-\epsilon_1) > 0$ and using Lemma 2.3 in [W1], we get

$$|K_1 f(z)| \leq c_3 \left(\int_{\mathbf{R}^d \setminus \overline{D}} |\nabla \mathcal{E}(f)(y)|^p \delta(y)^{p(1-\alpha-(d-\beta)/p)} |z-y|^{-\beta+\epsilon_1 p} dy \right)^{1/p}.$$

Hence

$$\begin{aligned}
&\int |K_1 f(z)|^p d\mu(z) \\
&\leq c_4 \int_{\mathbf{R}^d \setminus \overline{D}} |\nabla \mathcal{E}(f)(y)|^p \delta(y)^{p(1-\alpha-(d-\beta)/p)} dy \int |z-y|^{-\beta+\epsilon_1 p} d\mu(z) \\
&\leq c_5 \|f\|^p_{\mathcal{B}^-_{\alpha,p}}.
\end{aligned}$$

This shows (4.1).

We next prove that there exists $t_0 > 0$ and $c_6 > 0$ such that

$$\text{(4.2.)} \qquad \left(\int_{D \cap \{\delta(x) \leq t_0 R\}} |\nabla \mathcal{E}(K_1 f)(x)|^p \delta(x)^{p-p\alpha-d+\beta} dx \right)^{1/p} \leq c_6 \|f\|_{\mathcal{B}^-_{\alpha,p}}$$

for every $f \in \mathcal{B}^-_{\alpha,p}$.

To do so, let $Q \in \mathcal{V}$, $Q \subset D$ and a be a boundary point satisfying $\mathrm{dist}(\partial D, Q) = \mathrm{dist}(a, Q)$. Further denote by x_0 and l the center and the common side length of Q, respectively. We set

$$\Phi f(x_0) = \int_{\mathbf{R}^d \setminus \overline{D}} \langle \nabla \mathcal{E}(f)(y), \nabla_y N(x_0 - y) \rangle dy.$$

Let $x \in Q$. We write, by Proposition A,

$$\begin{aligned}
I(x) &\equiv \left| \frac{\partial \mathcal{E}(K_1 f - \Phi f(x_0))}{\partial x_i}(x) \right| \\
&\leq c_7 \delta(x)^{-1-\beta} \int_{B(a,sl)} d\mu(z) \\
&\quad \int_{B(0,2R) \setminus \overline{D}} |\nabla_y \mathcal{E}(f)(y)| |\nabla_y N(z-y) - \nabla_y N(x_0 - y)| dy,
\end{aligned}$$

where $s = 6\sqrt{d}$. Note that a cube $Q \in \mathcal{V}$ with the common side length l has the following property.

$$l\sqrt{d} \le \operatorname{dist}(Q, \partial D) \le 4l\sqrt{d}.$$

Since $l\sqrt{d} \le \delta(x)$, we write

$$\begin{aligned} I(x) &\le c_8\delta(x)^{-1-\beta}\int_{B(a,6\delta(x))} d\mu(z)\int_{B(z,\delta(x))\cap(\mathbf{R}^d\setminus\overline{D})} \frac{|\nabla_y\mathcal{E}(f)(y)|}{|z-y|^{d-1}}dy \\ &+ c_8\delta(x)^{-1-\beta}\int_{B(a,6\delta(x))} d\mu(z)\int_{B(x_0,\delta(x))\cap(\mathbf{R}^d\setminus\overline{D})} \frac{|\nabla_y\mathcal{E}(f)(y)|}{|x_0-y|^{d-1}}dy \\ &+ c_8\delta(x)^{-\beta}\int_{B(a,6\delta(x))} d\mu(z)\int_{\{|z-y|>\delta(x)\}\cap(\mathbf{R}^d\setminus\overline{D})} \frac{|\nabla_y\mathcal{E}(f)(y)|}{|z-y|^{d}}dy \\ &+ c_8\delta(x)^{-\beta}\int_{B(a,6\delta(x))} d\mu(z)\int_{\{|x_0-y|>\delta(x)\}\cap(\mathbf{R}^d\setminus\overline{D})} \frac{|\nabla_y\mathcal{E}(f)(y)|}{|x_0-y|^{d}}dy \\ &\equiv I_1(x)+I_2(x)+I_3(x)+I_4(x). \end{aligned}$$

We set $G = D$ and estimate $I_1(x)$. If $y \in \mathbf{R}^d \setminus \overline{D}$, $z \in B(a, 6\delta(x))$ and $|y-z| \le \delta(x)$, then

$$\begin{aligned} |x-y| &\le |x-a| + |a-z| + |z-y| \\ &\le 2\delta(x) + 6\delta(x) + \delta(x) \le 2^{k_0+1}\delta(x), \end{aligned}$$

where $k_0 = 3$. Since $d-\beta-1+\alpha > 0$, we pick $\epsilon > 0$ satisfying $d-\beta-1+\alpha-\epsilon > 0$, and put $t = q(1-\alpha-(d-\beta)/p+\epsilon/p)$ and $\lambda = -t-\epsilon$. Note that $d-\beta+\lambda = q(d-\beta-1+\alpha-\epsilon) > 0$. We set $F_1(y) = |\nabla_y\mathcal{E}(y)|\delta(y)^t$. Then

$$\begin{aligned} &I_1(x)\delta(x)^t \\ &\le c_9\delta(x)^{t-1-\beta}\int_{\{|x-y|\le 2^{k_0+1}\delta(x)\}\cap(\mathbf{R}^d\setminus\overline{D})} F_1(y)\delta(y)^\lambda dy \\ &\int_{|z-y|\le\delta(x)} |z-y|^{1-d+\epsilon}d\mu(z) \\ &\le c_{10}\delta(x)^{t-d+\epsilon}\int_{\{|x-y|\le 2^{k_0+1}\delta(x)\}\cap(\mathbf{R}^d\setminus\overline{D})} F_1(y)\delta(y)^\lambda dy. \end{aligned} \tag{4.3}$$

We set $b = 11/10$ in the definition of $M(\tau_\lambda^-)$. Further set $t_0 = \dfrac{1}{20}2^{-k_0-1}$ and

$$D_1 = \{x \in D; \delta(x) \le t_0R\}.$$

Suppose $x \in Q$ and $Q\cap D_1 \ne \emptyset$ and $x_1 \in Q\cap D_1$. Then $\delta(x) \le 5\sqrt{d}l \le 5\delta(x_1) \le 5t_0R$. Hence $2^{k_0+1}\delta(x) \le R/4$ and $2^{k_0+1} > \frac{11}{10}$. Noting that

$$\int_{B(x,r)\cap(F_0\setminus\overline{D})} \delta(y)^\lambda dy \le c_{11}r^{d+\lambda},$$

we have, by (4.3),

$$I_1(x) \le c_{12} M(\tau_\lambda^-) F_1(x).$$

We next estimate $I_2(x)$. To do so, let $Q \cap D_1 \neq \emptyset$ and $x \in Q$. Then the inequalities

$$\delta(x_0) \ge \delta(x) - |x - x_0| \ge \frac{\sqrt{d}}{2} l \quad \text{and} \quad \delta(x) \le 5\sqrt{d} l$$

imply $\delta(x_0) \ge \frac{\delta(x)}{10}$. Hence

$$\begin{aligned} I_2(x)\delta(x)^t &\le c_{13}\delta(x)^{t-1-\beta}\delta(x)^{\beta}\delta(x)^{1-d+\epsilon} \int_{\{|x_0-y|\le\delta(x)\}\cap(\mathbf{R}^d\setminus\overline{D})} F_1(y)\delta(y)^{\lambda} dy \\ &\le c_{14}\delta(x)^{-\lambda-d} \int_{\{|x-y|\le 2\delta(x)\}\cap(\mathbf{R}^d\setminus\overline{D})} F(y)\delta(y)^{\lambda} dy. \end{aligned}$$

Noting that $2\delta(x) \le 2^{k_0+1}\delta(x) \le R/4$, we also get

$$I_2(x) \le c_{15} M(\tau_\lambda^-) F_1(x).$$

Since $pt + \lambda = p - p\alpha - d + \beta$, we have, by Lemma 1.2,

$$\begin{aligned} (4.4) \qquad & \sum_{Q\cap D_1\neq\emptyset} \sum_{j=1}^{2} \int_Q I_j(x)^p \delta(x)^{p-p\alpha-d+\beta} dx \\ &= \sum_{Q\cap D_1\neq\emptyset} \sum_{j=1}^{2} \int_Q I_j(x)^p \delta(x)^{pt} d\tau_\lambda^+(x) \\ &\le c_{16} \sum_{Q\cap D_1\neq\emptyset} \int_Q M(\tau_\lambda^-) F_1(x)^p d\tau_\lambda^+(x) \le c_{17} \int_{F_0\setminus\overline{D}} F_1(y)^p d\tau_\lambda^-(y) \\ &\le c_{17} \int_{\mathbf{R}^d\setminus\overline{D}} |\nabla_y \mathcal{E}(f)(y)|^p \delta(y)^{p-p\alpha-d+\beta} dy \end{aligned}$$

We next consider $I_3(x)$. Let $x \in Q$ and $Q \cap D_1 \neq \emptyset$ and $x_1 \in Q \cap D_1$, and put $u = -q(1 - \alpha - (d-\beta)/p)$ and $F_2(y) = |\nabla_y \mathcal{E}(f)(y)|\delta(y)^{-u}$. We write

$$\begin{aligned} & I_3(x)\delta(x)^{-u} \\ &\le c_{18} \sum_{k=1}^{m} \delta(x)^{-u-\beta} \int_{B(a,6\delta(x))} d\mu(z) \int_{2^{k-1}\delta(x)<|z-y|\le 2^k\delta(x)} F_2(y)\delta(y)^u \frac{1}{|z-y|^d} dy \\ &+ c_{18}\delta(x)^{-u-\beta} \int_{B(a,6\delta(x))} d\mu(z) \int_{|z-y|>2^m\delta(x)} F_2(y)\delta(y)^u \frac{1}{|z-y|^d} dy \\ &\equiv I_{31}(x) + I_{32}(x), \end{aligned}$$

where m is the greatest integer satisfying $2^{k_0+m}\delta(x) \leq R/4$.

If $1 \leq k \leq m$ and $|z-y| \leq 2^k\delta(x)$, then $|x-y| \leq 2^{k_0+k}\delta(x) \leq R/4$ and $2^{k_0+k} \geq 2^{k_0+1} \geq 2$. Using Lemma 1.2 and noting that $u < 0$, we have

$$\begin{aligned}
I_{31}(x) &\leq c_{19}\sum_{k=1}^{m}\delta(x)^{-u}2^{-(k-1)d}\delta(x)^{-d}\int_{|x-y|\leq 2^{k_0+k}\delta(x)} F_2(y)\delta(y)^u dy \\
&\leq c_{20}\sum_{k=1}^{m}(2^u)^k\left(2^{k_0+k}\delta(x)\right)^{-u-d}\int_{|x-y|\leq 2^{k_0+k}\delta(x)} F_2(y)\delta(y)^u dy \\
&\leq c_{21}\left(\sum_{k=1}^{m}(2^u)^k\right)M(\tau_u^-)F_2(x) \leq c_{22}M(\tau_u^-)F_2(x).
\end{aligned}$$

We next estimate $I_{32}(x)$. Since

$$\begin{aligned}
I_{32} &\leq c_{23}\delta(x)^{-u-\beta}\left(2^m\delta(x)\right)^{-d}\delta(x)^{\beta}\int_{B(0,2R)\setminus\overline{D}}|\nabla\mathcal{E}(f)(y)|dy \\
&= c_{23}\delta(x)^{-u-d}\left(2^m\right)^{-d}\int_{B(0,2R)\setminus\overline{D}}|\nabla\mathcal{E}(f)(y)|dy
\end{aligned}$$

and $R/4 < 2^{k_0+m+1}\delta(x)$, we get

$$I_{32}(x) \leq c_{24}\delta(x)^{-u}\int_{B(0,2R)\setminus\overline{D}}|\nabla\mathcal{E}(f)(y)|dy.$$

Similarly we can estimate

$$I_4(x) \leq c_{25}\left(M(\tau_u^-)F_2(x) + \delta(x)^{-u}\int_{B(0,2R)\setminus\overline{D}}|\nabla\mathcal{E}(f)(y)|dy\right).$$

Noting that $-pu+u = p-p\alpha-d+\beta$, we get

$$\begin{aligned}
&\sum_{Q\cap D_1\neq\emptyset}\sum_{j=3}^{4}\int_Q I_j(x)^p\delta(x)^{p-p\alpha-d+\beta}dx \\
&= \sum_{Q\cap D_1\neq\emptyset}\sum_{j=3}^{4}\int_Q I_j(x)^p\delta(x)^{-pu}d\tau_u^+(x) \\
&\leq c_{26}\sum_{Q\cap D_1\neq\emptyset}\int_Q M(\tau_u^-)F_2(x)^p d\tau_u^+(x) \\
&+ c_{26}\sum_{Q\cap D_1\neq\emptyset}\int_Q \delta(x)^{-pu}d\tau_u^-(x)\Big(\int_{B(0,2R)\setminus\overline{D}}|\nabla\mathcal{E}(f)(y)|dy\Big)^p \\
&\equiv J_1 + J_2
\end{aligned}$$

Lemma 1.2 yields

$$J_1 \le c_{27} \int_{F_0 \setminus \overline{D}} F_2(y)^p d\tau_u^-(y) \le c_{28} \int_{B(0,2R)\setminus \overline{D}} |\nabla \mathcal{E}(f)(y)|^p \delta(x)^{p-p\alpha-d+\beta} dy.$$

We next estimete J_2. Noting that $-(p-1)u > 0$ and $d-\beta+u = q(\alpha-1+d-\beta) > 0$, we get

$$\begin{aligned} J_2 &\le c_{29} \int_D \delta(x)^{-(p-1)u} dx \left(\int_{B(0,2R)\setminus \overline{D}} |\nabla \mathcal{E}(f)(y)| dy \right)^p \\ &\le c_{30} \left(\int_{B(0,2R)\setminus \overline{D}} |\nabla \mathcal{E}(f)(y)|^p \delta(x)^{p-p\alpha-d+\beta} dy \right) \left(\int_{B(0,2R)\setminus \overline{D}} \delta(y)^u dy \right)^{p/q} \\ &\le c_{31} \|f\|^p_{\mathcal{B}^-_{\alpha,p}}. \end{aligned}$$

Thus we see that

$$\sum_{Q\cap D_1 \ne \emptyset} \sum_{j=3}^{4} \int_Q I_j(x)^p \delta(x)^{p-p\alpha-d+\beta} \le c_{32} \|f\|^p_{\mathcal{B}^-_{\alpha,p}} \tag{4.5}$$

From (4.4) and (4.5) we deduce

$$\begin{aligned} &\int_{D_1} |\nabla_x \mathcal{E}(K_1 f)(x)|^p \delta(x)^{p-p\alpha-d+\beta} dx \\ &\le \sum_{Q\in \mathcal{V}(D),\, Q\cap D_1 \ne \emptyset} \int_Q |\nabla_x \mathcal{E}(K_1 f - \Phi f(x_0))(x)|^p \delta(x)^{p-p\alpha-d+d} dx \\ &\le c_{33} \|f\|^p_{\mathcal{B}^-_{\alpha,p}}, \end{aligned}$$

which shows (4.2).

Finally we shall show that

$$\int_{D\cap\{\delta(x)\ge t_0 R\}} |\nabla_x \mathcal{E}(K_1 f)(x)|^p \delta(x)^{p-p\alpha-d+\beta} dx \le c_{34} \|f\|^p_{\mathcal{B}^-_{\alpha,p}}. \tag{4.6}$$

To do so, let k_1 be the greatest integer such that $Q \cap \{x \in D; \delta(x) \ge t_0 R\} \ne \emptyset$ for some k_1-cube Q. Let Q be a k-cube satisfying $k \le k_1$ and put $2^{-k} = l$. Then, by Proposition A,

$$\begin{aligned} &\int_Q |\nabla_x \mathcal{E}(K_1 f)(x)|^p \delta(x)^{p-p\alpha-d+\beta} dx \\ &\le c_{35} \int_Q \delta(x)^{-p(1+\beta)} \delta(x)^{p-p\alpha-d+\beta} dx \left(\int_{B(a,sl)} |K_1 f(z)| d\mu(z) \right)^p \\ &\le c_{36} l^{-p\alpha} \|K_1 f\|_p^p. \end{aligned}$$

By [W1, Lemma 3.3] the number of k-cube included in D is at most $c_{37}2^{k\beta}$. Therefore we have

$$\sum_{Q\in\mathcal{V}_k(D),\, Q\cap D_1\neq\emptyset}\int_Q |\nabla_x\mathcal{E}(K_1f)(x)|^p\delta(x)^{p-p\alpha-d+\beta}dx \le c_{38}l^{-p\alpha-\beta}\|K_1f\|_p^p,$$

where $\mathcal{V}_k(D)=\{Q\in\mathcal{V}(D); Q \text{ is a } k\text{-cube}\}$. This and (4.1) imply

$$\begin{aligned}&\int_{D\cap\{\delta(x)\ge t_0R\}}|\nabla_x\mathcal{E}(K_1f)(x)|^p\delta(x)^{p-p\alpha-d+\beta}dx\\ &\le c_{39}\sum_{k=-\infty}^{k_1}\left(2^{-k}\right)^{-p\alpha-\beta}\|K_1f\|_p^p\le c_{40}\|f\|^p_{\mathcal{B}^-_{\alpha,p}},\end{aligned}$$

which gives (4.6).

Thus we see that K_1 is a bounded operator from $\mathcal{B}^-_{\alpha,p}$ to $\mathcal{B}^+_{\alpha,p}$.

(ii) Setting $G=\mathbf{R}^d\setminus\overline{D}$, we can also prove by a similar method that K_2 is a bounded operator from $\mathcal{B}^+_{\alpha,p}$ to $\mathcal{B}^-_{\alpha,p}$.

□

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