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ON \((p, \omega)\)-PRECISE FUNCTIONS WHOSE DERIVATIVES ARE SINGULAR INTEGRALS

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1. Theorems. As kernels of potentials we shall be concerned only with Riesz and Bessel kernels in \(\mathbb{R}^d, d \geq 2\). We write \(k_a(x) = |x|^{a-d}\) for \(0 < a < d\) and \(U_a^f = k_a \ast f\) for a function \(f\) in case the potential is well-defined.

Let \(\alpha = (\alpha_1, \ldots, \alpha_d)\) with integers \(\alpha_i \geq 0\), and set \(|\alpha| = \alpha_1 + \cdots + \alpha_d\). We shall call \(\alpha\) an index of order \(|\alpha|\). For a function \(f\) we write \(D^\alpha f = D_2^\alpha f\) for \(\partial^{\alpha_1}f/\partial x_1^{\alpha_1} \cdots \partial x_d^{\alpha_d}\) when this has a meaning. In case \(\alpha = (0, \ldots, 0)\) we write \(\alpha = 0\) and let \(D^0 f\) mean \(f\).

Let \(d \geq 2, 1 < p < \infty, \Gamma\) be a family of locally rectifiable curves in \(\mathbb{R}^d\) and \(\omega\) a weight. We take the definition of extremal length \(\lambda_p(\Gamma; \omega)\) for granted. A function \(f\) is called \((p, \omega)\)-precise in an open set \(G\) if the extremal length of the family of locally rectifiable curves in \(G\) along each of which \(f\) is not absolutely continuous is infinite and \(\int_G |\text{grad } f|^p \omega dx\) is finite.

We announce

Theorem 1. Let \(d \geq 2, 1 < p < \infty, \omega\) be a weight in \(\mathbb{R}^d\) satisfying Muckenhoupt's \(A_p\) condition, \(f \in L^{p, \omega}(\mathbb{R}^d)\) and \(\alpha\) be an index of order \(|\alpha| \geq 0\). Then writing \(K^{i, \alpha}\) for \(\partial D^\alpha k_{1+|\alpha|}/\partial x_i\), we see that

\[
\lim_{r \to 0} \int_{|x-y|>r} K^{i, \alpha}(x-y)f(y)dy
\]

exists in \(L^{p, \omega}(\mathbb{R}^d)\). We denote this limit by \(T^{i, \alpha}f\). If, in addition, \(\int_{\mathbb{R}^d} (1+|x|)^{1-d}|f(x)|dx < \infty\), then \(D^\alpha k_{1+|\alpha|} \ast f\) is \((p, \omega)\)-precise, the relation

\[
\frac{\partial D^\alpha k_{1+|\alpha|} \ast f}{\partial x_i} = T^{i, \alpha}f
\]

holds a.e. in \(\mathbb{R}^d\) for each \(i\), and

\[
\|\text{grad}(D^\alpha k_{1+|\alpha|} \ast f)\|_{p, \omega} \leq \text{const. } \|f\|_{p, \omega}
\]
Next for $a > 0$ we consider the Bessel kernel
\[
G_a(r) = \frac{1}{(4\pi)^{a/2} \Gamma(a/2)} \int_0^\infty e^{-x|x|^2/\pi} e^{-t/2} t^{(a-1)/2} \frac{dt}{t}.
\]

**Theorem 2.** Let $d, p, \omega, f, \alpha$ be the same as in Theorem 1. Writing $K_{i,\alpha}$ for $\partial D^\alpha G_{1+|\alpha|}/\partial x_i$, we see that
\[
\lim_{r \to 0} \int_{|x-y| > r} K_{i,\alpha}(x-y) f(y) dy
\]
exists in $L^{p,\omega} (\mathbb{R}^d)$. Denote this limit by $T_{i,\alpha} f$. Then $D^\alpha G_{1+|\alpha|} * f$ is $(p, \omega)$-precise, the relation
\[
\frac{\partial D^\alpha G_{1+|\alpha|}}{\partial x_i} f = T_{i,\alpha} f + a_i f
\]
holds with some constant $a_i$ a.e. in $\mathbb{R}^d$ for each $i$, and
\[
\| \text{grad}(D^\alpha G_{1+|\alpha|} * f) \|_{p,\omega} \leq \text{const.} \| f \|_{p,\omega}
\]
is valid.

2. **Proof.** We shall prove only Theorem 1 in this paper. We begin with

**Lemma 1.** Let $d, p, \omega$ and $f$ be as in Theorem 1. Let $\Phi_n$ be an integrable function in $\mathbb{R}^d$ such that $|\Phi_n(x)| \leq c_1 n^d$ for $|x| < 1/n$ and $|\Phi_n(x)| \leq c_2 n^{-1} |x|^{-d-1}$ for $|x| > 1/n$, where $c_1$ and $c_2$ are constants. Set $a_n = \int_{\mathbb{R}^d} \Phi_n(x) dx$ and $h_n = f * \Phi_n$. Then $\|h_n - a_n f\|_{p,\omega} \to 0$ as $n \to \infty$.

**Proof.** By a famous Muckenhoupt theorem [Mu, p.222, Theorem 9] $\|Mf\|_{p,\omega} \leq \text{const.} \| f \|_{p,\omega}$, where $Mf$ is the Hardy-Littlewood maximal function. It follows that $Mf(x) < \infty$ a.e. in $\mathbb{R}^d$. Let us see that $f$ is locally integrable in $\mathbb{R}^d$. In fact, for any compact set $K$ in $\mathbb{R}^d$ we have
\[
\int_K |f| dx \leq \left( \int_K |f|^p \omega dx \right)^{1/p} \left( \int_K \omega^{1/(1-p)} dx \right)^{1/p'} \leq \| f \|_{p,\omega} \left( \int_K \omega^{1/(1-p)} dx \right)^{1/p'} < \infty
\]
because $\omega \in A_p$ shows that the last integral is finite.

Now set
\[
k(r) = \int_{|y| < r} |f(x-y) - f(x)| dy.
\]
Then $r^{-d}k(r) \leq c_3(Mf(x) + |f(x)|) < \infty$ for a.e. $x \in \mathbb{R}^d$. Let $x$ be such a point. Moreover, since $f$ is locally integrable, $r^{-d}k(r) \to 0$ as $r \to 0$ also for a.e. $x$ by a well-known result in integration theory; see, for instance, [R, p.168, Theorem 8.8]. We suppose $x$ has such a property too. Given $\epsilon > 0$ choose $r_0 > 0$ so that $r^{-d}k(r) < \epsilon$ if $0 < r < r_0$. In order to show $\lim_{n \to \infty} (h_n - a_nf) = 0$, it is sufficient to consider $n$ such that $1/n < r_0$. We write

$$h_n(x) - a_n f(x) = \left( \int_{|y|<1/n} + \int_{1/n<|y|<r_0} + \int_{|y|>r_0} \right) (f(x-y) - f(x))\Phi_n(y)dy$$

$$= I_1(x) + I_2(x) + I_3(x).$$

Since $|\Phi_n(y)| \leq c_1/n^{-d}$ if $|y| < 1/n$,

$$|I_1(x)| \leq \frac{c_1}{n^{-d}} \int_{|y|<1/n} |f(x-y) - f(x)|dy \to 0$$

as $n \to \infty$ at our $x$. We note that

$$|I_2(x)| \leq \frac{c_2}{n} \int_{1/n}^{r_0} \frac{1}{r^{d+1}} dk(r) \leq \frac{c_2}{n} \left( \frac{k(r_0)}{r_0^{d+1}} + \epsilon \int_{1/r_0}^{r_0} \frac{1}{r^{2}} dr \right) \leq \frac{c_4}{n} (1 + \epsilon n) \leq 2c_4\epsilon$$

if $n$ is large. To avoid a repetition of similar computations we shall give a preliminary evaluation before evaluating $I_3(x)$. If $1/n \leq \alpha < \infty$, then

$$\int_{\alpha}^{\infty} |(f(x-y) - f(x))\Phi_n(y)|dy \leq \frac{c_2}{n} \int_{\alpha}^{\infty} \frac{1}{r^{d+1}} dk(r)$$

$$= \frac{c_2}{n} \left( \frac{k(r)}{r^{d+1}} \right)_{\alpha}^{\infty} + (d+1) \int_{\alpha}^{\infty} \frac{k(r)}{r^{d+2}} dr$$

(1)

$$\leq \frac{c_2c_3}{n} \lim_{r \to \infty} \frac{Mf(x) + |f(x)|}{r} + (d+1)\frac{c_2c_3}{n} \frac{Mf(x) + |f(x)|}{r} \int_{\alpha}^{\infty} \frac{1}{r^{2}} dr$$

$$= (d+1)\frac{c_2c_3}{n} \frac{Mf(x) + |f(x)|}{\alpha n}. $$

By this evaluation we see that $|I_3(x)| \leq (d+1)c_2c_3(Mf(x) + |f(x)|)/(r_0n) \to 0$ as $n \to \infty$. Accordingly $\lim\sup_{n \to \infty} |h_n(x) - a_n f(x)| \leq \text{const.} \epsilon$ so that $\lim_{n \to \infty} (h_n(x) - a_n f(x)) = 0$.

Next we shall show that $|h_n - a_n f|$ is dominated by a function, which is independent of $n$ and belongs to $L^{p,\omega}(\mathbb{R}^d)$. We write

$$h_n(x) - a_n f(x) = \left( \int_{|y|<1/n} + \int_{1/n<|y|} \right) (f(x-y) - f(x))\Phi_n(y)dy = I_1(x) + I_2(x).$$
We observe that $|I_1(x)| \leq c_3(Mf(x) + |f(x)|)$, and that $|I_2'(x)|$ is dominated by $(d+1)c_2c_3(Mf(x) + |f(x)|)$ by (1). We recall that $Mf + |f|$ belongs to $L^{p,\omega}(\mathbb{R}^d)$ in virtue of the Muckenhoupt theorem. Consequently we can apply Lebesgue's dominated convergence theorem and obtain

$$\lim_{n \to \infty} ||hn - anf||_{p,\omega} = ||\lim_{n \to \infty}(hn - anf)||_{p,\omega} = 0.$$ 

Thus the lemma is completely proved.

We shall give three more lemmas. Their proofs require a number of properties of $(p,\omega)$-precise functions and so they are omitted. We only refer to the corresponding results in [O].

We shall call a set $E$ in $\mathbb{R}^d$ $(p,\omega)$-exc. if $\lambda_p(\Gamma;\omega) = \infty$ for the family $\Gamma$ of curves terminating at the points of $E$. This can be characterized as a kind of set of capacity zero. We shall say that a property holds $(p,\omega)$-a.e. if the exceptional set is a $(p,\omega)$-exc. set.

**Lemma 2.** [O, Theorems 4.4.4 and 4.4.5]. Let $\omega \in A_p$. Let $f_1, f_2, \ldots$ be $(p,\omega)$-precise in $\mathbb{R}^d$ and assume

$$\lim_{n,m \to \infty} ||\mathrm{grad} f_n - \mathrm{grad} f_m||_{p,\omega} = 0.$$

Then there exist a $(p,\omega)$-precise function $f$ in $\mathbb{R}^d$, a subsequence $\{n_j\}$ and a sequence $\{c_j\}$ of constants such that $||\mathrm{grad} f_n - \mathrm{grad} f||_{p,\omega} \to 0$ and $f_{n_j} - c_j \to f$ $(p,\omega)$-a.e. in $\mathbb{R}^d$.

**Lemma 3.** [O, Corollary to Theorem 4.4.6]. If a sequence $\{g_n\}$ of $(p,\omega)$-precise functions converges pointwise to a function $g$ $(p,\omega)$-a.e. in $\mathbb{R}^d$ and $\{\mathrm{grad} g_n\}$ is a Cauchy sequence in $L^{p,\omega}(\mathbb{R}^d)$, then $g$ is $(p,\omega)$-precise and $||\mathrm{grad} g_n - \mathrm{grad} g||_{p,\omega} \to 0$ as $n \to \infty$.

**Lemma 4.** [O, Theorem 4.2.5]. Let $f, g$ be $(p,\omega)$-precise functions in $\mathbb{R}^d$ which are equal $(p,\omega)$-a.e. in $\mathbb{R}^d$. Then $\mathrm{grad} f = \mathrm{grad} g$ a.e. in $\mathbb{R}^d$.

**Proof of Theorem 1.** The first assertion of the theorem is a consequence of Theorem III due to Coifman and Fefferman [CF, Studia Math., 1974]. Assume $\int_{\mathbb{R}^d}(1 + |x|)^{1-d}|f(x)|dx < \infty$. This is a necessary and sufficient condition for $U^{|f|}_1$ to be finite a.e. in $\mathbb{R}^d$. 


Set \( \varphi_n = D^\alpha k_{1+|\alpha|,1/n} \) and \( g_n = \varphi_n * f \), where \( k_{a,b}(x) = (|x|^2 + b^2)^{(a-d)/2} \) in general.

We can observe easily
\[
\left| \frac{\partial}{\partial x_i} \varphi_n(x-y) f(y) \right| \leq \begin{cases} \text{const. } n^d |f(y)| & \text{if } |y| < 2(|x|+1), \\ \text{const. } \frac{|f(y)|}{|y|^d} & \text{if } |y| \geq 2(|x|+1). \end{cases}
\]

We infer that \( \partial g_n / \partial x_i = (\partial \varphi_n / \partial x_i) * f \) and it is continuous in \( \mathbb{R}^d \). Therefore \( g_n \) is absolutely continuous along all locally rectifiable curves in \( \mathbb{R}^d \). Setting \( K_{1/n}^{i,\alpha}(x) = K^{i,\alpha}_1 \chi_{|x|>1/n}(x) \), we know that \( K_{1/n}^{i,\alpha} * f \to T^{i,\alpha} f \) in \( L^{p,\omega} (\mathbb{R}^d) \) as \( n \to \infty \). We can write
\[
\frac{\partial g_n}{\partial x_i}(x) - K_{1/n}^{i,\alpha} * f(x) = \Phi_n * f,
\]
where
\[
\Phi_n(x) = \begin{cases} \frac{\partial}{\partial x_i} D^\alpha \left( |x|^2 + \frac{1}{n^2} \right)^{(1+|\alpha|-d)/2} & \text{on } |x| < 1/n, \\ \frac{\partial}{\partial x_i} D^\alpha \left( |x|^2 + \frac{1}{n^2} \right)^{(1+|\alpha|-d)/2} - \frac{\partial}{\partial x_i} D^\alpha |x|^{1+|\alpha|-d} & \text{on } |x| \geq 1/n. \end{cases}
\]

By the mean value theorem we have \( |\Phi_n(x)| \leq \text{const. } 1/(n^2|x|^{d+1}) \) on \( |x| \geq 1/n \). Hence \( \int_{\mathbb{R}^d} |\Phi_n(x)| dx < \infty \). As in [Mi, p.219, Lemma 4.1] we see that \( \int_{\mathbb{R}^d} \Phi_n(x) dx \) vanishes.

In view of Lemma 1 and the equality \( \int_{\mathbb{R}^d} \Phi_n(x) dx = 0 \), we obtain
\[
(1) \quad \lim_{n \to \infty} \left\| \frac{\partial g_n}{\partial x_i} - K_{1/n}^{i,\alpha} * f \right\|_{p,\omega} = \lim_{n \to \infty} \left\| \int_{\mathbb{R}^d} f(x-y) \Phi_n(y) dy \right\|_{p,\omega} = 0.
\]

We recall that \( K_{1/n}^{i,\alpha} * f \to T^{i,\alpha} f \) in \( L^{p,\omega} (\mathbb{R}^d) \) as \( n \to \infty \). So naturally each \( \|K_{1/n}^{i,\alpha} * f\|_{p,\omega} \) is finite. Hence (1) gives \( \|\partial g_n / \partial x_i\|_{p,\omega} < \infty \) for each \( n \). The absolute continuity along all locally rectifiable curves being known, it follows that \( g_n \) is \((p,\omega)\)-precise. From (1) and the fact \( \lim_{n \to \infty} \|K_{1/n}^{i,\alpha} * f - T^{i,\alpha} f\|_{p,\omega} = 0 \) we infer that \( \{\partial g_n / \partial x_i\} \) form a Cauchy sequence in \( L^{p,\omega}(\mathbb{R}^d) \). Using Lemma 2 we find a \((p,\omega)\)-precise function \( g_0 \), a subsequence \( \{n_j\} \) and a sequence \( \{c_j\} \) of constants such that \( \|\text{grad}(g_n - g_0)\|_{p,\omega} \to 0 \) and \( g_{n_j} - c_j \to g_0 \) \((p,\omega)\)-a.e. The assumption \( \int_{\mathbb{R}^d} (1+|x|)^{1-d} |f(x)| dx < \infty \) implies that
\[
|g_{n_j}(x)| = |D^\alpha k_{1+|\alpha|,1/n} * f(x)| \leq \text{const. } U_1^{1/1}(x) < \infty
\]
for a.e. \( x \). Hence we may assume that all \( c_j \) are zero so that \( g_{n_j} \to g_0 \) \((p,\omega)\)-a.e.
From (1) it follows that there exists a subsequence of \(\{n_j\}\), which will be denoted again by \(\{n_j\}\), such that

\[
\lim_{j \to \infty} \left( \frac{\partial g_{n_j}}{\partial x_i} - K^{i,\alpha}_{1/n_j} * f \right) = 0
\]
a.e. in \(\mathbb{R}^d\). The relations \(\lim_{n \to \infty} \|\text{grad}(g_n - g_0)\|_{p,\omega} = 0\) and \(\lim_{n \to \infty} \|K^{i,\alpha}_{1/n} * f - T^{i,\alpha}f\|_{p,\omega} = 0\) show that we may assume that \(\lim_{j \to \infty} \frac{\partial g_{n_j}}{\partial x_i} = \frac{\partial g_0}{\partial x_i}\) and \(\lim_{j \to \infty} K^{i,\alpha}_{1/n_j} * f = T^{i,\alpha}f\) a.e. in \(\mathbb{R}^d\) for each \(i\). Taking into account (2) we obtain the equality

\[
\frac{\partial g_0}{\partial x_i} = T^{i,\alpha}f
\]
a.e. in \(\mathbb{R}^d\) for each \(i\).

In the special case \(\alpha = 0\) and \(f \geq 0\) \(g_n\) increases to \(k_1 * f = U_1^f\) everywhere in \(\mathbb{R}^d\) and \(\{\text{grad} g_n\}\) form a Cauchy sequence. Lemma 3 shows that \(U_1^f\) is \((p, \omega)\)-precise. In the general case \(|D^{\alpha}k_{1+|\alpha|,1/n}| \leq \text{const.} k_1\) and \(U_1^f\) is finite a.e. in \(\mathbb{R}^d\). Hence applying Lebesgue's dominated convergence theorem, we have

\[
\lim_{n \to \infty} g_n(x) = \lim_{n \to \infty} D^{\alpha}k_{1+|\alpha|,1/n} * f(x) = D^{\alpha}k_{1+|\alpha|} * f(x)
\]
at every point \(x\) with finite \(U_1^f(x)\). Again by Lemma 3 we infer that \(D^{\alpha}k_{1+|\alpha|} * f\) is \((p, \omega)\)-precise.

Next, we recall that \(g_{n_j} \to g_0\) as \(j \to \infty\) \((p, \omega)\)-a.e. and obtain \(g_0 = D^{\alpha}k_{1+|\alpha|} * f\) \((p, \omega)\)-a.e. in \(\mathbb{R}^d\). Using Lemma 4 and (3) we derive

\[
\frac{\partial D^{\alpha}k_{1+|\alpha|} * f}{\partial x_i} = \frac{\partial g_0}{\partial x_i} = T^{i,\alpha}f
\]
a.e. in \(\mathbb{R}^d\) for each \(i\). Finally since \(\|T^{i,\alpha}f\|_{p,\omega} \leq \text{const.} \|f\|_{p,\omega}, \|\text{grad}(D^{\alpha}k_{1+|\alpha|} * f)\|_{p,\omega} \leq \text{const.} \|f\|_{p,\omega}\) as announced.

References


