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Kyoto University
Hele-Shaw flows moving boundary problem whose initial domain has a corner with right angle

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1. HELE-SHAW FLOWS

We discuss a flow which is produced by injection of fluid into the narrow gap between two parallel planes. We call it a Hele-Shaw flow.

A mathematical description of the flow is the following: Let $\Omega(0)$ be a bounded connected open set in the plane and let $p_0$ be a point in $\Omega(0)$. We define $\Omega(0)$ and $p_0$ as the projection of the averaged initial blob of fluid and the injection point of fluid into one of the two parallel planes, respectively. The Hele-Shaw flow $\{\Omega(t)\}_{t>0}$ is the monotone increasing family of bounded connected open sets $\Omega(t)$ such that

$$-\frac{1}{2\pi} \frac{\partial G(x,p_0,\Omega(t))}{\partial n_x} = v_{n_x}$$

for every $t \geq 0$ and every point $x$ on the boundary $\partial\Omega(t)$ of $\Omega(t)$, where $G(x,p_0,\Omega(t))$ denotes the Green function (of the Dirichlet problem for the Laplace operator) for $\Omega(t)$ with pole at $p_0$, $\partial/\partial n_x$ denotes the outer normal derivative at $x \in \partial\Omega(t)$ and $v_{n_x}$ denotes the velocity of $\partial\Omega(t)$ at $x$ in the direction of outer normal. Here we have assumed that $\partial\Omega(t)$ is smooth for every $t \geq 0$ and the function $t = t(x)$ which is defined by $x \in \partial\Omega(t)$ is also smooth. Thus, the problem of the Hele-Shaw flows with a free boundary is to find $\{\Omega(t)\}_{t>0}$ which satisfies the equation above for given $\Omega(0)$ and $p_0$.

It is very hard to discuss the problem as formulated above, because we do not know a priori the smoothness of $\partial\Omega(t)$ and $t(x)$ even if
the boundary $\partial\Omega(0)$ of the initial domain $\Omega(0)$ is sufficiently smooth. Therefore, we need another formulation of the problem. If we assume that $\partial\Omega(t)$ and $t(x)$ are sufficiently smooth, then we can easily prove that, for every $t > 0$, $\Omega(t)$ satisfies

$$\int_{\Omega(0)} s(x) dx + ts(p_0) \leq \int_{\Omega(t)} S(X) d_X$$

for every integrable and subharmonic function $s$ in $\Omega(t)$. That is to say, the Hele-Shaw flow is a family $\{\Omega(t)\}_{t>0}$ of quadrature domains $\Omega(t)$ of $\lambda|\Omega(0)| + t\delta_{p_0}$, where $\lambda$ denotes the two-dimensional Lebesgue measure and $\delta_{p_0}$ denotes the unit one-point measure at $p_0$. In this formulation, we do not need the smoothness of $\partial\Omega(t)$ and $t(x)$. The existence and uniqueness of the solution are known. For more detailed discussions, see e.g. Gustafsson and Sakai [2] and Sakai [6].

We take a point $x_0$ on $\partial\Omega(0)$ and discuss the shape of $\Omega(t)$ around $x_0$ for small $t > 0$. If $x_0 \in \partial\Omega(t)$ for some $t > 0$, then $x_0 \in \partial\Omega(s)$ for every $s$ satisfying $0 < s < t$. We call such a point $x_0$ a stationary point. If $x_0$ is not a stationary point, then $x_0 \in \Omega(t)$ for every $t > 0$, in other words, $x_0$ is contained in $\Omega(t)$ right immediately after the initial time.

To give a more concrete discussion, we treat a corner with interior angle $\varphi$. Assume that $(\partial\Omega(0)) \cap B$ is a continuous simple arc passing through $x_0$ for a small disk $B$ with center at $x_0$. Assume further that $B \setminus (\partial\Omega(0))$ consists of two connected components and $\Omega(0) \cap B$ is one of them. We express $(\partial\Omega(0)) \cap B$ as the union of two continuous simple arcs $\Gamma_1(0)$ and $\Gamma_2(0)$; $(\partial\Omega(0)) \cap B = \Gamma_1(0) \cup \Gamma_2(0)$ and $\Gamma_1(0) \cap \Gamma_2(0) = \{x_0\}$, and assume further that both $\Gamma_1(0)$ and $\Gamma_2(0)$ are of class $C^1$ and regular up to the endpoint $x_0$. Then the intersection of $\Omega(0)$ and the circle with center at $x_0$ and with small radius is a
circular arc. We say that $x_0$ is a corner with interior angle $\varphi$ if the ratio of the length of the circular arc to the radius tends to $\varphi$ as the radius tends to 0. It follows that $0 \leq \varphi \leq 2\pi$. If $\varphi = \pi$, we interpret $x_0$ as a smooth boundary point of $\Omega(0)$. If $\varphi = \pi/2$, we say that $x_0$ is a corner with right angle.

If $x_0$ is a corner with interior angle $\varphi$, we can give a more accurate discussion than whether it is a stationary point or not. We introduce the following notion.

The corner $x_0$ is called a laminar-flow stationary corner with interior angle $\varphi$, if there is a small disk $B_0$ with center at $x_0$ and small $t_0 > 0$ such that $(\partial \Omega(t)) \cap B_0$ is a continuous simple arc for every $t$ with $0 < t < t_0$ and $(\partial \Omega(t)) \cap B_0$ can be expressed as the union of two continuous simple arcs $\Gamma_1(t)$ and $\Gamma_2(t)$; $(\partial \Omega(t)) \cap B_0 = \Gamma_1(t) \cup \Gamma_2(t)$ and $\Gamma_1(t) \cap \Gamma_2(t) = \{x_0\}$, and both $\Gamma_1(t)$ and $\Gamma_2(t)$ are of class $C^1$ and regular up to the endpoint $x_0$, and real-analytic except for $x_0$. Furthermore $x_0$ is a corner of $\partial \Omega(t)$ with interior angle $\varphi$, and $\varphi$ does not depend on $t$ satisfying $0 < t < t_0$. It follows that $(\partial \Omega(s) \cap B_0) \setminus \{x_0\} \subset \Omega(t) \cap B_0$ for every $s$ with $0 \leq s < t$.

The corner $x_0$ is called a laminar-flow point, if there is a small disk $B_0$ with center at $x_0$ and small $t_0 > 0$ such that $(\partial \Omega(t)) \cap B_0$ is a regular real-analytic simple arc for every $t$ with $0 < t < t_0$. In this case, $(\partial \Omega(s) \cap B_0) \subset \Omega(t) \cap B_0$ for every $s$ with $0 \leq s < t$.

We have already announced the following theorems:

**Theorem A.** Let $x_0 \in \partial \Omega(0)$ be a corner with interior angle $\varphi$.

1. If $0 \leq \varphi < \pi/2$, then $x_0$ is a laminar-flow stationary corner with interior angle $\varphi$.

2. If $\varphi = \pi/2$, then $x_0$ is a laminar-flow stationary corner with
right angle or a laminar-flow point.

(3) If $\pi/2 < \varphi < 2\pi$, then $x_0$ is a laminar-flow point.

**Theorem B.** Let $x_0 \in \partial\Omega(0)$ be a corner with right angle.

(1) There is an example of corner $x_0$ which is a laminar-flow stationary corner with right angle.

(2) If $\Gamma_1(0)$ and $\Gamma_2(0)$ are of class $C^{1,\alpha}$ or $x_0$ is a Lyapunov-Dini corner with right angle, then $x_0$ is a laminar-flow point.

In this paper, we give a more detailed discussion and give a sufficient condition for a corner with right angle to be a laminar-flow stationary corner with right angle and also give a sufficient condition to be a laminar-flow point. Each of them is not a necessary and sufficient condition, but very close to a necessary and sufficient condition.

2. GENERAL ARGUMENTS

We have already interpreted $\Omega(t)$ as the quadrature domain of $\lambda|\Omega(0) + t\delta_{p_0}$. For the sake of simplicity, we write $\Omega(0)$ for $\lambda|\Omega(0)$, that is to say, $\Omega(t)$ is a quadrature domain of $\Omega(0) + t\delta_{p_0}$. Now we introduce the restricted quadrature domain and measure of $D + \mu$, where $D$ is a bounded domain and $\mu$ is a finite positive measure supported in $D$. Let $R$ be a domain, which may not be bounded, with smooth boundary. We call this domain a restriction domain. For the sake of simplicity, we assume that $\text{supp}\mu \subset D \cap R$ and $D \cap R$ is connected.

We call $(\Omega_R, \nu_R)$ the restricted quadrature domain and measure in $R$ of $D \cap R + \mu$ if
(i) \( \Omega_R \) is a bounded domain containing \( D \cap R \);
(ii) \( \nu_R \) is a finite positive measure on \((\partial\Omega_R) \setminus (R \cap \partial\Omega_R)\);
(iii) 
\[
\int_{D \cap R} s(x)dx + \int sd\mu \leq \int_{\Omega_R} s(x)dx + \int sd\nu_R
\]
for every integrable and subharmonic function \( s \) on \( \overline{\Omega_R \setminus (R \cap \partial\Omega_R)} \).

Here we interpret \( \nu_R \) as 0 if \((\partial\Omega_R) \setminus (R \cap \partial\Omega_R)\) is empty and we say that \( s \) is subharmonic on \( \overline{\Omega_R \setminus (R \cap \partial\Omega_R)} \) if \( s \) is subharmonic in some open set containing \( \overline{\Omega_R \setminus (R \cap \partial\Omega_R)} \). If \( \mu > 0 \), then there exists a smallest \( \Omega_R \). We always treat the case that \((\Omega_R, \nu_R)\) is determined uniquely. For the properties of the restricted quadrature domain and measure \((\Omega_R, \nu_R)\), see Gustafsson and Sakai [2, Sect.2] and Sakai [6, Chap.1, Sect.4]. Simple facts which we use afterward are

\[
D \cap R \subset \Omega_R \subset \Omega \cap R,
\]
where \( \Omega \) denotes the quadrature domain of \( D + \mu \) and

\[
\beta(\mu, D \cap R) |\partial R \leq \nu_R \leq \beta(\mu, \Omega_R) |\partial R,
\]
where \( \beta(\mu, D \cap R) \) denotes the balayage measure of \( \mu \) onto the boundary of \( D \cap R \).

Let \( x_0 \) be a corner with right angle and let \( R_a = \{y \in \mathbb{R}^2 : |y - x_0| > a\} \) be a restriction domain. Let \((\Omega_a(t), \nu_a(t))\) be the restricted quadrature domain and measure in \( R_a \) of \( \Omega(0) \cap R_a + t\delta_{p_0} \). Then we obtain the following proposition:
Proposition 1. $x_0$ is a laminar-flow stationary corner with right angle if and only if

$$\lim_{a \to 0} \inf \frac{||\nu_{a}(t)||}{a^2} = 0$$

for some $t > 0$.

Replacing $D$ with $\Omega(0)$, $R$ with $R_a$, $\mu$ with $t\delta_{p_0}$ and $\nu_R$ with $\nu_{a}(t)$ in the first inequality before Proposition 1, we obtain

$$\beta(t\delta_{p_0}, \Omega(0) \cap R_a)|\partial R_a \leq \nu_{a}(t).$$

Since

$$\beta(t\delta_{p_0}, \Omega(0) \cap R_a) = t\beta(\delta_{p_0}, \Omega(0) \cap R_a),$$

we obtain the following corollary:

Corollary 2. If

$$\lim_{a \to 0} \inf \frac{||\beta(\delta_{p_0}, \Omega(0) \cap R_a)|\partial R_a||}{a^2} > 0,$$

then $x_0$ is a laminar-flow point.

3. CONCRETE RESULTS

From now on, we discuss very concrete cases. We assume that $x_0 = 0$, $p_0 = (1,0) \in \Omega(0)$ and

$$\Omega(0) \cap \{(r, \theta) : 0 < r < 1\} = \{(r, \theta) : 0 < r < 1, -\frac{\pi}{4} + \delta_2(r) < \theta < \frac{\pi}{4} + \delta_1(r)\},$$

where $\delta_j$ is a function on the interval $[0,1]$ such that

(i) $\delta_j$ is continuous on $[0,1]$ and of class $C^1$ on $]0,1]$;

(ii) $\delta_j(0) = 0$ and $|\delta_j(r)| < \frac{\pi}{8}$ on $[0,1]$;
(iii) \( \lim_{r \to 0} r \delta_j'(r) = 0. \)

We need the last condition, because it holds if and only if \( \Gamma_j(0) \) is of class \( C^1 \) up to the origin. We set \( \delta(r) = \delta_1(r) - \delta_2(r) \). It follows that

\[
\left( \frac{\pi}{4} + \delta_1(r) \right) - \left( -\frac{\pi}{4} + \delta_2(r) \right) = \frac{\pi}{2} + \delta(r) \longrightarrow \frac{\pi}{2} \quad (r \to 0).
\]

Hence the origin is a corner with right angle.

Now, we apply estimates of harmonic measure which were given originally by Ahlfors [1] and improved by Warschawski [7] and others. By using our notation, we express them as follows:

\[
|| \beta(\delta_{p_0}, \Omega(0) \cap R_a) | \partial R_a || \leq C_1 \exp \left( -\pi \int_a^1 \frac{dr}{r \theta(r)} \right),
\]

where \( C_1 \) denotes an absolute constant and \( \theta(r) = \frac{\pi}{2} + \delta(r) \) and

\[
|| \beta(\delta_{p_0}, \Omega(0) \cap R_a) | \partial R_a || \geq C_2 \exp \left( -\pi \int_a^1 \frac{dr}{r \theta(r)} \right),
\]

where \( C_2 \) denotes a constant which depends on the total variations of \( \delta_1 \) and \( \delta_2 \).

Substituting \( \frac{\pi}{2} + \delta(r) \) for \( \theta(r) \), we obtain

\[
\pi \int_a^1 \frac{dr}{r \theta(r)} = -2 \log a - \frac{4}{\pi} \int_a^1 \frac{\delta(r)}{1 + \frac{2}{\pi} \delta(r)} \frac{dr}{r}.
\]

We set

\[
\Delta(r) = \frac{\frac{4}{\pi} \delta(r)}{1 + \frac{2}{\pi} \delta(r)}.
\]

We denote by \( V(I; \delta_j) \) the total variation on an interval \( I \) of \( \delta_j \) and set

\[
V(r) = V([r, 1]; \delta_1) + V([r, 1]; \delta_2).
\]

Then we obtain the following main theorem:
Theorem 3. Let the origin be a corner with right angle.

(1) If there is a positive constant $\epsilon$ such that

$$\int_0^1 \exp \left( \int_r^1 \Delta(s) \frac{ds}{s} + \epsilon V(r) \right) \frac{dr}{r} < +\infty,$$

then the origin is a laminar-flow stationary corner with right angle.

(2) If there is a positive constant $\epsilon$ such that

$$\int_0^1 \exp \left( \int_r^1 \Delta(s) \frac{ds}{s} - \epsilon V(r) \right) \frac{dr}{r} = +\infty,$$

then the origin is a laminar-flow point.

Example. Let

$$\delta(r) = \delta_1(r) - \delta_2(r) = \frac{A}{\log \left( \frac{1}{r} \right)}$$

for small $r$, where $A$ denotes a constant, and $\delta_1$ and $\delta_2$ are monotone functions satisfying (i) through (iii). Then $\int_0^1 \delta(r)^2 \frac{dr}{r} < +\infty$, and so

$$\int_0^1 \exp \left( \int_r^1 \Delta(s) \frac{ds}{s} \right) \frac{dr}{r} < +\infty$$

if and only if

$$\int_0^1 \exp \left( \frac{4}{\pi} \int_r^1 \delta(s) \frac{ds}{s} \right) \frac{dr}{r} < +\infty.$$

Since the last inequality holds if and only if

$$\int_0^{r_0} \left( \log \left( \frac{1}{r} \right) \right)^{\frac{4}{\pi} A} \frac{dr}{r} < +\infty.$$
for some $r_0 < 1$, the origin is a laminar-flow stationary corner with right angle if and only if $A < -\frac{\pi}{4}$.

The proof of Theorem 3 is complicated and long. We prove the first assertion by applying the Ahlfors distortion theorem which we have already mentioned before Theorem 3 as the first estimate of harmonic measure. Ahlfors [1] called it Die erste Hauptungleichung. In the paper he also discussed the opposite inequality, which he called Die zweite Hauptungleichung. This second inequality was improved extensively by Warschawski [7], Lelong-Ferrand [4], Jenkins and Oikawa [3] and Rodin and Warschawski [5]. We prove the second assertion by applying the second inequality formulated and proved by Warschawski.

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