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Hele-Shaw flows moving boundary problem whose initial domain has a corner with right angle

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1. HELE-SHAW FLOWS

We discuss a flow which is produced by injection of fluid into the narrow gap between two parallel planes. We call it a Hele-Shaw flow.

A mathematical description of the flow is the following: Let $\Omega(0)$ be a bounded connected open set in the plane and let p_0 be a point in $\Omega(0)$. We define $\Omega(0)$ and p_0 as the projection of the averaged initial blob of fluid and the injection point of fluid into one of the two parallel planes, respectively. The Hele-Shaw flow $\{\Omega(t)\}_{t>0}$ is the monotone increasing family of bounded connected open sets $\Omega(t)$ such that

$$-\frac{1}{2\pi}\frac{\partial G(x,p_0,\Omega(t))}{\partial n_x}=v_{n_x}$$

for every $t\geq 0$ and every point x on the boundary $\partial\Omega(t)$ of $\Omega(t)$, where $G(x,p_0,\Omega(t))$ denotes the Green function (of the Dirichlet problem for the Laplace operator) for $\Omega(t)$ with pole at p_0, $\partial/\partial n_x$ denotes the outer normal derivative at $x\in\partial\Omega(t)$ and v_{n_x} denotes the velocity of $\partial\Omega(t)$ at x in the direction of outer normal. Here we have assumed that $\partial\Omega(t)$ is smooth for every $t\geq 0$ and the function $t=t(x)$ which is defined by $x\in\partial\Omega(t)$ is also smooth. Thus, the problem of the Hele-Shaw flows with a free boundary is to find $\{\Omega(t)\}_{t>0}$ which satisfies the equation above for given $\Omega(0)$ and p_0.

It is very hard to discuss the problem as formulated above, because we do not know *a priori* the smoothness of $\partial\Omega(t)$ and $t(x)$ even if the boundary $\partial\Omega(0)$ of the initial domain $\Omega(0)$ is sufficiently smooth. Therefore, we need another formulation of the problem. If we assume that $\partial\Omega(t)$ and $t(x)$ are sufficiently smooth, then we can easily prove that, for every $t > 0$, $\Omega(t)$ satisfies

$$\int_{\Omega(0)} s(x)dx + ts(p_0) \leq \int_{\Omega(t)} s(x)dx$$

for every integrable and subharmonic function s in $\Omega(t)$. That is to say, the Hele-Shaw flow is a family $\{\Omega(t)\}_{t>0}$ of quadrature domains $\Omega(t)$ of $\lambda|\Omega(0) + t\delta_{p_0}$, where λ denotes the two-dimensional Lebesgue measure and δ_{p_0} denotes the unit one-point measure at p_0. In this formulation, we do *not* need the smoothness of $\partial\Omega(t)$ and $t(x)$. The existence and uniqueness of the solution are known. For more detailed discussions, see e.g. Gustafsson and Sakai [2] and Sakai [6].

We take a point x_0 on $\partial\Omega(0)$ and discuss the shape of $\Omega(t)$ around x_0 for small $t > 0$. If $x_0 \in \partial\Omega(t)$ for some $t > 0$, then $x_0 \in \partial\Omega(s)$ for every s satisfying $0 < s < t$. We call such a point x_0 a stationary point. If x_0 is not a stationary point, then $x_0 \in \Omega(t)$ for every $t > 0$, In other words, x_0 is contained in $\Omega(t)$ right immediately after the initial time.

To give a more concrete discussion, we treat a corner with interior angle φ. Assume that $(\partial\Omega(0)) \cap B$ is a continuous simple arc passing through x_0 for a small disk B with center at x_0. Assume further that $B \setminus (\partial\Omega(0))$ consists of two connected components and $\Omega(0) \cap B$ is one of them. We express $(\partial\Omega(0)) \cap B$ as the union of two continuous simple arcs $\Gamma_1(0)$ and $\Gamma_2(0)$; $(\partial\Omega(0)) \cap B = \Gamma_1(0) \cup \Gamma_2(0)$ and $\Gamma_1(0) \cap \Gamma_2(0) = \{x_0\}$, and assume further that both $\Gamma_1(0)$ and $\Gamma_2(0)$ are of class C^1 and regular up to the endpoint x_0. Then the intersection of $\Omega(0)$ and the circle with center at x_0 and with small radius is a circular arc. We say that x_0 is *a corner with interior angle* φ if the ratio of the length of the circular arc to the radius tends to φ as the radius tends to 0. It follows that $0 \leq \varphi \leq 2\pi$. If $\varphi = \pi$, we interpret x_0 as a smooth boundary point of $\Omega(0)$. If $\varphi = \pi/2$, we say that x_0 is *a corner with right angle.*

If x_0 is a corner with interior angle φ, we can give a more accurate discussion than whether it is a stationary point or not. We introduce the following notion.

The corner x_0 is called *a laminar-flow stationary corner with interior angle* φ, if there is a small disk B_0 with center at x_0 and small $t_0 > 0$ such that $(\partial\Omega(t)) \cap B_0$ is a continuous simple arc for every t with $0 < t < t_0$ and $(\partial\Omega(t)) \cap B_0$ can be expressed as the union of two continuous simple arcs $\Gamma_1(t)$ and $\Gamma_2(t)$; $(\partial\Omega(t)) \cap B_0 = \Gamma_1(t) \cup \Gamma_2(t)$ and $\Gamma_1(t) \cap \Gamma_2(t) = \{x_0\}$, and both $\Gamma_1(t)$ and $\Gamma_2(t)$ are of class C^1 and regular up to the endpoint x_0, and real-analytic except for x_0. Furthermore x_0 is a corner of $\partial\Omega(t)$ with interior angle φ, and φ does not depend on t satisfying $0 < t < t_0$. It follows that $(\partial\Omega(s) \cap B_0) \setminus \{x_0\} \subset \Omega(t) \cap B_0$ for every s with $0 \leq s < t$.

The corner x_0 is called *a laminar-flow point*, if there is a small disk B_0 with center at x_0 and small $t_0 > 0$ such that $(\partial\Omega(t)) \cap B_0$ is a regular real-analytic simple arc for every t with $0 < t < t_0$. In this case, $(\partial\Omega(s) \cap B_0) \subset \Omega(t) \cap B_0$ for every s with $0 \leq s < t$.

We have already announced the following theorems:

Theorem A. *Let $x_0 \in \partial\Omega(0)$ be a corner with interior angle φ.*

(1) *If $0 \leq \varphi < \pi/2$, then x_0 is a laminar-flow stationary corner with interior angle φ.*

(2) *If $\varphi = \pi/2$, then x_0 is a laminar-flow stationary corner with*

right angle or a laminar-flow point.

(3) *If $\pi/2 < \varphi < 2\pi$, then x_0 is a laminar-flow point.*

Theorem B. *Let $x_0 \in \partial\Omega(0)$ be a corner with right angle.*

(1) *There is an example of corner x_0 which is a laminar-flow stationary corner with right angle.*

(2) *If $\Gamma_1(0)$ and $\Gamma_2(0)$ are of class $C^{1,\alpha}$ or x_0 is a Lyapunov-Dini corner with right angle, then x_0 is a laminar-flow point.*

In this paper, we give a more detailed discussion and give a sufficient condition for a corner with right angle to be a laminar-flow stationary corner with right angle and also give a sufficient condition to be a laminar-flow point. Each of them is not a necessary and sufficient condition, but very close to a necessary and sufficient condition.

2. GENERAL ARGUMENTS

We have already interpreted $\Omega(t)$ as the quadrature domain of $\lambda|\Omega(0) + t\delta_{p_0}$. For the sake of simplicity, we write $\Omega(0)$ for $\lambda|\Omega(0)$, that is to say, $\Omega(t)$ is a quadrature domain of $\Omega(0) + t\delta_{p_0}$. Now we introduce the restricted quadrature domain and measure of $D + \mu$, where D is a bounded domain and μ is a finite positive measure supported in D. Let R be a domain, which may not be bounded, with smooth boundary. We call this domain *a restriction domain.* For the sake of simplicity, we assume that $\mathrm{supp}\mu \subset D \cap R$ and $D \cap R$ is connected.

We call (Ω_R, ν_R) *the restricted quadrature domain and measure in R of $D \cap R + \mu$* if

(i) Ω_R is a bounded domain containing $D \cap R$;

(ii) ν_R is a finite positive measure on $(\partial\Omega_R) \setminus (R \cap \partial\Omega_R)$;

(iii)

$$\int_{D \cap R} s(x)dx + \int s d\mu \leq \int_{\Omega_R} s(x)dx + \int s d\nu_R$$

for every integrable and subharmonic function s on $\overline{\Omega_R} \setminus (R \cap \partial\Omega_R)$.

Here we interpret ν_R as 0 if $(\partial\Omega_R) \setminus (R \cap \partial\Omega_R)$ is empty and we say that s is subharmonic on $\overline{\Omega_R} \setminus (R \cap \partial\Omega_R)$ if s is subharmonic in some open set containing $\overline{\Omega_R} \setminus (R \cap \partial\Omega_R)$. If $\mu > 0$, then there exists a smallest Ω_R. We always treat the case that (Ω_R, ν_R) is determined uniquely. For the properties of the restricted quadrature domain and measure (Ω_R, ν_R), see Gustafsson and Sakai [2, Sect.2] and Sakai [6, Chap.I, Sect.4]. Simple facts which we use afterward are

$$D \cap R \subset \Omega_R \subset \Omega \cap R,$$

where Ω denotes the quadrature domain of $D + \mu$ and

$$\beta(\mu, D \cap R)|\partial R \leq \nu_R \leq \beta(\mu, \Omega_R)|\partial R,$$

where $\beta(\mu, D \cap R)$ denotes the balayage measure of μ onto the boundary of $D \cap R$.

Let x_0 be a corner with right angle and let $R_a = \{y \in \mathbf{R}^2 : |y - x_0| > a\}$ be a restriction domain. Let $(\Omega_a(t), \nu_a(t))$ be the restricted quadrature domain and measure in R_a of $\Omega(0) \cap R_a + t\delta_{p_0}$. Then we obtain the following proposition:

Proposition 1. *x_0 is a laminar-flow stationary corner with right angle if and only if*

$$\liminf_{a\to 0} \frac{||\nu_a(t)||}{a^2} = 0$$

for some $t > 0$.

Replacing D with $\Omega(0)$, R with R_a, μ with $t\delta_{p_0}$ and ν_R with $\nu_a(t)$ in the first inequality before Proposition 1, we obtain

$$\beta(t\delta_{p_0}, \Omega(0) \cap R_a)|\partial R_a \leq \nu_a(t).$$

Since

$$\beta(t\delta_{p_0}, \Omega(0) \cap R_a) = t\beta(\delta_{p_0}, \Omega(0) \cap R_a),$$

we obtain the following corollary:

Corollary 2. *If*

$$\liminf_{a\to 0} \frac{||\beta(\delta_{p_0}, \Omega(0) \cap R_a)|\partial R_a||}{a^2} > 0,$$

then x_0 is a laminar-flow point.

3. CONCRETE RESULTS

From now on, we discuss very concrete cases. We assume that $x_0 = 0$, $p_0 = (1,0) \in \Omega(0)$ and

$$\Omega(0) \cap \{(r,\theta) : 0 < r < 1\} = \{(r,\theta) : 0 < r < 1, -\frac{\pi}{4} + \delta_2(r) < \theta < \frac{\pi}{4} + \delta_1(r)\},$$

where δ_j is a function on the interval $[0,1[$ such that

(i) δ_j is continuous on $[0,1[$ and of class C^1 on $]0,1[$;

(ii) $\delta_j(0) = 0$ and $|\delta_j(r)| < \frac{\pi}{8}$ on $[0,1[$;

(iii) $\lim_{r\to 0} r\delta_j'(r) = 0$.

We need the last condition, because it holds if and only if $\Gamma_j(0)$ is of class C^1 up to the origin. We set $\delta(r) = \delta_1(r) - \delta_2(r)$. It follows that

$$\left(\frac{\pi}{4} + \delta_1(r)\right) - \left(-\frac{\pi}{4} + \delta_2(r)\right) = \frac{\pi}{2} + \delta(r) \longrightarrow \frac{\pi}{2} \quad (r \to 0).$$

Hence the origin is a corner with right angle.

Now, we apply estimates of harmonic measure which were given originally by Ahlfors [1] and improved by Warschawski [7] and others. By using our notation, we express them as follows:

$$||\beta(\delta_{p_0}, \Omega(0) \cap R_a)|\partial R_a|| \leq C_1 \exp\left(-\pi \int_a^1 \frac{dr}{r\theta(r)}\right),$$

where C_1 denotes an absolute constant and $\theta(r) = \frac{\pi}{2} + \delta(r)$ and

$$||\beta(\delta_{p_0}, \Omega(0) \cap R_a)|\partial R_a|| \geq C_2 \exp\left(-\pi \int_a^1 \frac{dr}{r\theta(r)}\right),$$

where C_2 denotes a constant which depends on the total variations of δ_1 and δ_2.

Substituting $\frac{\pi}{2} + \delta(r)$ for $\theta(r)$, we obtain

$$\pi \int_a^1 \frac{dr}{r\theta(r)} = -2\log a - \frac{4}{\pi} \int_a^1 \frac{\delta(r)}{1 + \frac{2}{\pi}\delta(r)} \frac{dr}{r}.$$

We set

$$\Delta(r) = \frac{\frac{4}{\pi}\delta(r)}{1 + \frac{2}{\pi}\delta(r)}.$$

We denote by $V(I; \delta_j)$ the total variation on an interval I of δ_j and set

$$V(r) = V([r, 1]; \delta_1) + V([r, 1]; \delta_2).$$

Then we obtain the following main theorem:

Theorem 3. *Let the origin be a corner with right angle.*

(1) *If there is a positive constant ϵ such that*

$$\int_0^1 \exp\left(\int_r^1 \Delta(s)\frac{ds}{s} + \epsilon V(r)\right)\frac{dr}{r} < +\infty,$$

then the origin is a laminar-flow stationary corner with right angle.

(2) *If there is a positive constant ϵ such that*

$$\int_0^1 \exp\left(\int_r^1 \Delta(s)\frac{ds}{s} - \epsilon V(r)\right)\frac{dr}{r} = +\infty,$$

then the origin is a laminar-flow point.

Example. Let

$$\delta(r) = \delta_1(r) - \delta_2(r) = \frac{A}{\log\left(\frac{1}{r}\right)}$$

for small r, where A denotes a constant, and δ_1 and δ_2 are monotone functions satisfying (i) through (iii). Then $\int_0^1 \delta(r)^2\frac{dr}{r} < +\infty$, and so

$$\int_0^1 \exp\left(\int_r^1 \Delta(s)\frac{ds}{s}\right)\frac{dr}{r} < +\infty$$

if and only if

$$\int_0^1 \exp\left(\frac{4}{\pi}\int_r^1 \delta(s)\frac{ds}{s}\right)\frac{dr}{r} < +\infty.$$

Since the last inequality holds if and only if

$$\int_0^{r_0} \left(\log\left(\frac{1}{r}\right)\right)^{\frac{4}{\pi}A}\frac{dr}{r} < +\infty$$

for some $r_0 < 1$, the origin is a laminar-flow stationary corner with right angle if and only if $A < -\frac{\pi}{4}$.

The proof of Theorem 3 is complicated and long. We prove the first assertion by applying the Ahlfors distortion theorem which we have already mentioned before Theorem 3 as the first estimate of harmonic measure. Ahlfors [1] called it *Die erste Hauptungleichung.* In the paper he also discussed the opposite inequality, which he called *Die zweite Hauptungleichung.* This second inequality was improved extensively by Warschawski [7], Lelong-Ferrand [4], Jenkins and Oikawa [3] and Rodin and Warschawski [5]. We prove the second assertion by applying the second inequality formulated and proved by Warschawski.

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