Title: Hele-Shaw flows moving boundary problem whose initial domain has a corner with right angle (Potential Theory and its Related Fields)

Author(s): Sakai, Makoto

Citation: 数理解析研究所講究録 (1999), 1116: 77-86

Issue Date: 1999-11

URL: http://hdl.handle.net/2433/63430

Type: Departmental Bulletin Paper

Textversion: publisher

Kyoto University
Hele-Shaw flows moving boundary problem whose initial domain has a corner with right angle

東京都立大学理学研究科 酒井 良 (Makoto Sakai)

1. HELE-SHAW FLOWS

We discuss a flow which is produced by injection of fluid into the narrow gap between two parallel planes. We call it a Hele-Shaw flow.

A mathematical description of the flow is the following: Let $\Omega(0)$ be a bounded connected open set in the plane and let $p_0$ be a point in $\Omega(0)$. We define $\Omega(0)$ and $p_0$ as the projection of the averaged initial blob of fluid and the injection point of fluid into one of the two parallel planes, respectively. The Hele-Shaw flow $\{\Omega(t)\}_{t>0}$ is the monotone increasing family of bounded connected open sets $\Omega(t)$ such that

$$\frac{1}{2\pi} \frac{\partial G(x,p_0,\Omega(t))}{\partial n_x} = v_{n_x}$$

for every $t \geq 0$ and every point $x$ on the boundary $\partial \Omega(t)$ of $\Omega(t)$, where $G(x,p_0,\Omega(t))$ denotes the Green function (of the Dirichlet problem for the Laplace operator) for $\Omega(t)$ with pole at $p_0$, $\partial/\partial n_x$ denotes the outer normal derivative at $x \in \partial \Omega(t)$ and $v_{n_x}$ denotes the velocity of $\partial \Omega(t)$ at $x$ in the direction of outer normal. Here we have assumed that $\partial \Omega(t)$ is smooth for every $t \geq 0$ and the function $t = t(x)$ which is defined by $x \in \partial \Omega(t)$ is also smooth. Thus, the problem of the Hele-Shaw flows with a free boundary is to find $\{\Omega(t)\}_{t>0}$ which satisfies the equation above for given $\Omega(0)$ and $p_0$.

It is very hard to discuss the problem as formulated above, because we do not know a priori the smoothness of $\partial \Omega(t)$ and $t(x)$ even if
the boundary $\partial \Omega(0)$ of the initial domain $\Omega(0)$ is sufficiently smooth. Therefore, we need another formulation of the problem. If we assume that $\partial \Omega(t)$ and $t(x)$ are sufficiently smooth, then we can easily prove that, for every $t > 0$, $\Omega(t)$ satisfies

$$\int_{\Omega(0)} s(x) dx + ts(p_0) \leq \int_{\Omega(t)} s(x) dx$$

for every integrable and subharmonic function $s$ in $\Omega(t)$. That is to say, the Hele-Shaw flow is a family $\{\Omega(t)\}_{t>0}$ of quadrature domains $\Omega(t)$ of $\lambda|\Omega(0)| + t\delta_{p_0}$, where $\lambda$ denotes the two-dimensional Lebesgue measure and $\delta_{p_0}$ denotes the unit one-point measure at $p_0$. In this formulation, we do not need the smoothness of $\partial \Omega(t)$ and $t(x)$. The existence and uniqueness of the solution are known. For more detailed discussions, see e.g. Gustafsson and Sakai [2] and Sakai [6].

We take a point $x_0$ on $\partial \Omega(0)$ and discuss the shape of $\Omega(t)$ around $x_0$ for small $t > 0$. If $x_0 \in \partial \Omega(t)$ for some $t > 0$, then $x_0 \in \partial \Omega(s)$ for every $s$ satisfying $0 < s < t$. We call such a point $x_0$ a stationary point. If $x_0$ is not a stationary point, then $x_0 \in \Omega(t)$ for every $t > 0$, In other words, $x_0$ is contained in $\Omega(t)$ right immediately after the initial time.

To give a more concrete discussion, we treat a corner with interior angle $\varphi$. Assume that $(\partial \Omega(0)) \cap B$ is a continuous simple arc passing through $x_0$ for a small disk $B$ with center at $x_0$. Assume further that $B \setminus (\partial \Omega(0))$ consists of two connected components and $\Omega(0) \cap B$ is one of them. We express $(\partial \Omega(0)) \cap B$ as the union of two continuous simple arcs $\Gamma_1(0)$ and $\Gamma_2(0)$; $(\partial \Omega(0)) \cap B = \Gamma_1(0) \cup \Gamma_2(0)$ and $\Gamma_1(0) \cap \Gamma_2(0) = \{x_0\}$, and assume further that both $\Gamma_1(0)$ and $\Gamma_2(0)$ are of class $C^1$ and regular up to the endpoint $x_0$. Then the intersection of $\Omega(0)$ and the circle with center at $x_0$ and with small radius is a
circular arc. We say that \( x_0 \) is a corner with interior angle \( \varphi \) if the ratio of the length of the circular arc to the radius tends to \( \varphi \) as the radius tends to 0. It follows that \( 0 \leq \varphi \leq 2\pi \). If \( \varphi = \pi \), we interpret \( x_0 \) as a smooth boundary point of \( \Omega(0) \). If \( \varphi = \pi/2 \), we say that \( x_0 \) is a corner with right angle.

If \( x_0 \) is a corner with interior angle \( \varphi \), we can give a more accurate discussion than whether it is a stationary point or not. We introduce the following notion.

The corner \( x_0 \) is called a laminar-flow stationary corner with interior angle \( \varphi \), if there is a small disk \( B_0 \) with center at \( x_0 \) and small \( t_0 > 0 \) such that \( (\partial\Omega(t)) \cap B_0 \) is a continuous simple arc for every \( t \) with \( 0 < t < t_0 \) and \( (\partial\Omega(t)) \cap B_0 \) can be expressed as the union of two continuous simple arcs \( \Gamma_1(t) \) and \( \Gamma_2(t) \); \( (\partial\Omega(t)) \cap B_0 = \Gamma_1(t) \cup \Gamma_2(t) \) and \( \Gamma_1(t) \cap \Gamma_2(t) = \{x_0\} \), and both \( \Gamma_1(t) \) and \( \Gamma_2(t) \) are of class \( C^1 \) and regular up to the endpoint \( x_0 \), and real-analytic except for \( x_0 \). Furthermore \( x_0 \) is a corner of \( \partial\Omega(t) \) with interior angle \( \varphi \), and \( \varphi \) does not depend on \( t \) satisfying \( 0 < t < t_0 \). It follows that \( (\partial\Omega(s) \cap B_0) \setminus \{x_0\} \subset \Omega(t) \cap B_0 \) for every \( s \) with \( 0 \leq s < t \).

The corner \( x_0 \) is called a laminar-flow point, if there is a small disk \( B_0 \) with center at \( x_0 \) and small \( t_0 > 0 \) such that \( (\partial\Omega(t)) \cap B_0 \) is a regular real-analytic simple arc for every \( t \) with \( 0 < t < t_0 \). In this case, \( (\partial\Omega(s) \cap B_0) \subset \Omega(t) \cap B_0 \) for every \( s \) with \( 0 \leq s < t \).

We have already announced the following theorems:

**Theorem A.** Let \( x_0 \in \partial\Omega(0) \) be a corner with interior angle \( \varphi \).

1. If \( 0 \leq \varphi < \pi/2 \), then \( x_0 \) is a laminar-flow stationary corner with interior angle \( \varphi \).

2. If \( \varphi = \pi/2 \), then \( x_0 \) is a laminar-flow stationary corner with
right angle or a laminar-flow point.

(3) If $\pi/2 < \varphi < 2\pi$, then $x_0$ is a laminar-flow point.

**Theorem B.** Let $x_0 \in \partial\Omega(0)$ be a corner with right angle.

(1) There is an example of corner $x_0$ which is a laminar-flow stationary corner with right angle.

(2) If $\Gamma_1(0)$ and $\Gamma_2(0)$ are of class $C^{1,\alpha}$ or $x_0$ is a Lyapunov-Dini corner with right angle, then $x_0$ is a laminar-flow point.

In this paper, we give a more detailed discussion and give a sufficient condition for a corner with right angle to be a laminar-flow stationary corner with right angle and also give a sufficient condition to be a laminar-flow point. Each of them is not a necessary and sufficient condition, but very close to a necessary and sufficient condition.

2. GENERAL ARGUMENTS

We have already interpreted $\Omega(t)$ as the quadrature domain of $\lambda|\Omega(0) + t\delta_{p_0}$. For the sake of simplicity, we write $\Omega(0)$ for $\lambda|\Omega(0)$, that is to say, $\Omega(t)$ is a quadrature domain of $\Omega(0) + t\delta_{p_0}$. Now we introduce the restricted quadrature domain and measure of $D + \mu$, where $D$ is a bounded domain and $\mu$ is a finite positive measure supported in $D$. Let $R$ be a domain, which may not be bounded, with smooth boundary. We call this domain a *restriction domain*. For the sake of simplicity, we assume that $\text{supp}\mu \subset D \cap R$ and $D \cap R$ is connected.

We call $(\Omega_R, \nu_R)$ the *restricted quadrature domain and measure in* $R$ of $D \cap R + \mu$ if
(i) \( \Omega_R \) is a bounded domain containing \( D \cap R \);
(ii) \( \nu_R \) is a finite positive measure on \( (\partial \Omega_R) \setminus (R \cap \partial \Omega_R) \);
(iii) 
\[
\int_{D \cap R} s(x)dx + \int \sigma d\mu \leq \int_{\Omega_R} s(x)dx + \int \sigma d\nu_R
\]
for every integrable and subharmonic function \( s \) on \( \overline{\Omega_R} \setminus (R \cap \partial \Omega_R) \).

Here we interpret \( \nu_R \) as 0 if \( (\partial \Omega_R) \setminus (R \cap \partial \Omega_R) \) is empty and we say that \( s \) is subharmonic on \( \overline{\Omega_R} \setminus (R \cap \partial \Omega_R) \) if \( s \) is subharmonic in some open set containing \( \overline{\Omega_R} \setminus (R \cap \partial \Omega_R) \). If \( \mu > 0 \), then there exists a smallest \( \Omega_R \). We always treat the case that \( (\Omega_R, \nu_R) \) is determined uniquely. For the properties of the restricted quadrature domain and measure \( (\Omega_R, \nu_R) \), see Gustafsson and Sakai [2, Sect.2] and Sakai [6, Chap.I, Sect.4]. Simple facts which we use afterward are

\[
D \cap R \subset \Omega_R \subset \Omega \cap R,
\]

where \( \Omega \) denotes the quadrature domain of \( D + \mu \) and

\[
\beta(\mu, D \cap R)|\partial R \leq \nu_R \leq \beta(\mu, \Omega_R)|\partial R,
\]

where \( \beta(\mu, D \cap R) \) denotes the balayage measure of \( \mu \) onto the boundary of \( D \cap R \).

Let \( x_0 \) be a corner with right angle and let \( R_a = \{ y \in \mathbb{R}^2 : |y - x_0| > a \} \) be a restriction domain. Let \((\Omega_a(t), \nu_a(t))\) be the restricted quadrature domain and measure in \( R_a \) of \( \Omega(0) \cap R_a + t\delta_{p_0} \). Then we obtain the following proposition:
Proposition 1. $x_0$ is a laminar-flow stationary corner with right angle if and only if
\[
\lim_{a \to 0} \inf \frac{||\nu_a(t)||}{a^2} = 0
\]
for some $t > 0$.

Replacing $D$ with $\Omega(0)$, $R$ with $R_a$, $\mu$ with $t \delta_{p_0}$ and $\nu_R$ with $\nu_a(t)$ in the first inequality before Proposition 1, we obtain
\[
\beta(t \delta_{p_0}, \Omega(0) \cap R_a) |\partial R_a \leq \nu_a(t).
\]
Since
\[
\beta(t \delta_{p_0}, \Omega(0) \cap R_a) = t \beta(\delta_{p_0}, \Omega(0) \cap R_a),
\]
we obtain the following corollary:

Corollary 2. If
\[
\lim_{a \to 0} \inf \frac{||\beta(\delta_{p_0}, \Omega(0) \cap R_a) |\partial R_a||}{a^2} > 0,
\]
then $x_0$ is a laminar-flow point.

3. CONCRETE RESULTS

From now on, we discuss very concrete cases. We assume that $x_0 = 0$, $p_0 = (1, 0) \in \Omega(0)$ and
\[
\Omega(0) \cap \{(r, \theta) : 0 < r < 1\} = \{(r, \theta) : 0 < r < 1, -\frac{\pi}{4} + \delta_2(r) < \theta < \frac{\pi}{4} + \delta_1(r)\},
\]
where $\delta_j$ is a function on the interval $[0, 1]$ such that
(i) $\delta_j$ is continuous on $[0, 1]$ and of class $C^1$ on $]0, 1[$;
(ii) $\delta_j(0) = 0$ and $|\delta_j(r)| < \frac{\pi}{8}$ on $[0, 1]$;
(iii) \( \lim_{r \to 0} r \delta_j'(r) = 0 \).

We need the last condition, because it holds if and only if \( \Gamma_j(0) \) is of class \( C^1 \) up to the origin. We set \( \delta(r) = \delta_1(r) - \delta_2(r) \). It follows that

\[
\left( \frac{\pi}{4} + \delta_1(r) \right) - \left( -\frac{\pi}{4} + \delta_2(r) \right) = \frac{\pi}{2} + \delta(r) \longrightarrow \frac{\pi}{2} \quad (r \to 0).
\]

Hence the origin is a corner with right angle.

Now, we apply estimates of harmonic measure which were given originally by Ahlfors [1] and improved by Warschawski [7] and others. By using our notation, we express them as follows:

\[
|| \beta(\delta_{p_0}, \Omega(0) \cap R_a) \partial R_a || \leq C_1 \exp \left( -\pi \int_a^1 \frac{dr}{r \theta(r)} \right),
\]

where \( C_1 \) denotes an absolute constant and \( \theta(r) = \frac{\pi}{2} + \delta(r) \) and

\[
|| \beta(\delta_{p_0}, \Omega(0) \cap R_a) \partial R_a || \geq C_2 \exp \left( -\pi \int_a^1 \frac{dr}{r \theta(r)} \right),
\]

where \( C_2 \) denotes a constant which depends on the total variations of \( \delta_1 \) and \( \delta_2 \).

Substituting \( \frac{\pi}{2} + \delta(r) \) for \( \theta(r) \), we obtain

\[
\pi \int_a^1 \frac{dr}{r \theta(r)} = -2 \log a - \frac{4}{\pi} \int_a^1 \frac{\delta(r)}{1 + \frac{2}{\pi} \delta(r)} \frac{dr}{r}.
\]

We set

\[
\Delta(r) = \frac{4 \pi \delta(r)}{1 + \frac{2}{\pi} \delta(r)}.
\]

We denote by \( V(I; \delta_j) \) the total variation on an interval \( I \) of \( \delta_j \) and set

\[
V(r) = V([r, 1]; \delta_1) + V([r, 1]; \delta_2).
\]

Then we obtain the following main theorem:
Theorem 3. Let the origin be a corner with right angle.

(1) If there is a positive constant $\epsilon$ such that
\[
\int_0^1 \exp \left(\int_r^1 \Delta(s) \frac{ds}{s} + \epsilon V(r)\right) \frac{dr}{r} < +\infty,
\]
then the origin is a laminar-flow stationary corner with right angle.

(2) If there is a positive constant $\epsilon$ such that
\[
\int_0^1 \exp \left(\int_r^1 \Delta(s) \frac{ds}{s} - \epsilon V(r)\right) \frac{dr}{r} = +\infty,
\]
then the origin is a laminar-flow point.

Example. Let
\[
\delta(r) = \delta_1(r) - \delta_2(r) = \frac{A}{\log \left(\frac{1}{r}\right)}
\]
for small $r$, where $A$ denotes a constant, and $\delta_1$ and $\delta_2$ are monotone functions satisfying (i) through (iii). Then $\int_0^1 \delta(r)^2 \frac{dr}{r} < +\infty$, and so
\[
\int_0^1 \exp \left(\int_r^1 \Delta(s) \frac{ds}{s}\right) \frac{dr}{r} < +\infty
\]
if and only if
\[
\int_0^1 \exp \left(\frac{4}{\pi} \int_r^1 \delta(s) \frac{ds}{s}\right) \frac{dr}{r} < +\infty.
\]
Since the last inequality holds if and only if
\[
\int_0^{r_0} \left(\log \left(\frac{1}{r}\right)\right)^{\frac{4}{\pi} A} \frac{dr}{r} < +\infty
\]
for some \( r_0 < 1 \), the origin is a laminar-flow stationary corner with right angle if and only if \( A < -\frac{\pi}{4} \).

The proof of Theorem 3 is complicated and long. We prove the first assertion by applying the Ahlfors distortion theorem which we have already mentioned before Theorem 3 as the first estimate of harmonic measure. Ahlfors [1] called it *Die erste Hauptungleichung*. In the paper he also discussed the opposite inequality, which he called *Die zweite Hauptungleichung*. This second inequality was improved extensively by Warschawski [7], Lelong-Ferrand [4], Jenkins and Oikawa [3] and Rodin and Warschawski [5]. We prove the second assertion by applying the second inequality formulated and proved by Warschawski.

REFERENCES


Department of Mathematics
Tokyo Metropolitan University
Minami-Ohsawa 1-1, Hachioji-shi
Tokyo, 192-0397 JAPAN