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Kyoto University
Hyperplane integrals of BLD and monotone BLD functions

1 Introduction

Let $D$ denote the half space

$$D = \{ x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}^1 : x_n > 0 \}$$

and set

$$H = \partial D;$$

we sometimes identify $x' \in \mathbb{R}^{n-1}$ with $(x', 0) \in H$. We define the $q$th hyperplane integral $H_q(u)$ over $H$ by

$$H_q(u) = \left( \int_H |u(x')|^q dx' \right)^{1/q}$$

for a measurable function $u$ on $H$.

Our main aim in this note is to study the existence of limits of $H_q(U_r)$ at $r = 0$, where

$$U_r(x') = u(x', r) - \sum_{k=0}^{m-1} \frac{r^k}{k!}[(\partial/\partial x_n)^ku](x', 0)$$

for quasicontinuous Sobolev functions $u$ on $D$, where the vertical limits

$$[(\partial/\partial x_n)^ku](x', 0) = \lim_{x_n \to 0} (\partial/\partial x_n)^ku(x', x_n)$$

exist for almost all $x' = (x', 0) \in H$ and $0 \le k \le m - 1$ (see [15, Theorem 2.4]). More precisely, we show (in Theorem 3.1 below) that

$$\lim_{r \to 0} r^{-\omega}H_q(U_r) = 0$$

for some $\omega > 0$.

Consider the Dirichlet problem for polyharmonic operator

$$\Delta^m u(x) = 0$$

with the boundary conditions

$$(\partial/\partial x_n)^k u(x', 0) = f_k(x') \quad (k = 0, 1, ..., m - 1).$$
We show (in Corollary 3.1 below) that if $1 < p \leq q < \infty$, $n/p - (n-1)/q < 1$ and $u \in W^{m,p}(D)$ is a solution of the Dirichlet problem with $f_k(x') = (\partial/\partial x_n)^k u(x', 0)$ for $0 \leq k \leq m-1$, then
\[
\lim_{r \to 0} r^{n/p - (n-1)/q - m} H_q(U_r) = 0,
\]
where $U_r(x') = u(x', r) - \sum_{k=0}^{m-1} \frac{r^k}{k!} f_k(x')$.

To prove our results, we apply the integral representation in [12, 15]. For this purpose, we are concerned with $K$-potentials $U_K f$ defined by
\[
U_K f(x) = \int K(x - y) f(y) dy
\]
for functions $f$ on $\mathbb{R}^n$ satisfying the weighted $L^p$ condition:
\[
\int_{\mathbb{R}^n} |f(y)|^p |y_n|^\beta dy < \infty, \quad -1 < \beta < p - 1.
\]

In connection with our integral representation, $K(x)$ is of the form $x^\lambda |x|^{-n}$ for a multi-index $\lambda$ with length $m$. Our basic fact is stated as follows (see Theorem 2.1 below):
\[
\lim_{r \to 0} r^{n/p - (n-1)/q - m} H_q(u_r) = 0,
\]
where $u_r(x') = U_K f(x', r) - \sum_{k=0}^{m-1} \frac{r^k}{k!} [(\partial/\partial x_n)^k U_K f](x')$.

In Section 4, we give growth estimates of higher differences of Sobolev functions. In the final section, we study the existence of limits of hyperplane integrals for monotone BLD functions $u$ on $D$ satisfying
\[
(1.1) \quad \int_D |\nabla u(x)|^p x_n^\beta dx < \infty, \quad p > n - 1,
\]
where $\nabla$ denotes the gradient, $1 < p < \infty$ and $-1 < \beta < p - 1$; see Section 5 for the definition of monotone functions. We here note that harmonic functions are monotone, $A$-harmonic functions and hence coordinate functions of quasiregular mappings are monotone (see [4] and [20]), and thus the class of monotone functions is considerably wide.

For related results, see Gardiner [2], Stoll [24, 25, 26], the first author [11, 12, 16] and the authors [17, 18].

2 Hyperplane integrals of potentials

For a multi-index $\lambda$ and $\ell > 0$, set
\[
K(x) = \frac{x^\lambda}{|x|^\ell}.
\]
We define the $K$-potential $U_Kf$ by

$$U_Kf(x) = \int_{\mathbb{R}^n} K(x-y)f(y)dy$$

for a measurable function $f$ on $\mathbb{R}^n$ satisfying

(2.1) \[ \int_{\mathbb{R}^n} (1+|y|)^{|\lambda|-\ell}|f(y)|dy < \infty \]

and

(2.2) \[ \int_{\mathbb{R}^n} |f(y)|^p|y_n|^{\beta}dy < \infty, \quad y = (y_1, \cdots, y_n). \]

In particular, $K$ is the Riesz $\alpha$-kernel when $\lambda = 0$ and $\ell = n-\alpha$. In this case, $U_Kf$ is written as $U_\alpha f$ with $\alpha = |\lambda| - \ell + n$. Note here that (2.1) is equivalent to the condition that

(2.3) \[ U_\alpha|f| \not\equiv \infty. \]

Throughout this paper, let $M$ denote various constants independent of the variables in question.

For a nonnegative integer $m$, consider

$$K_m(x, y) = K(x-y) - \sum_{k=0}^{m} \frac{x_n^k}{k!}[(\partial/\partial_{x_n})^kK](x'-y),$$

where $x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$; we sometimes identify $x'$ with $(x', 0)$.

**Lemma 2.1** ([19, Lemma 2.1]). Let $m$ be a nonnegative integer such that $|\lambda| - \ell < m + 1$.

1. If $|x'-y| \geq x_n/2 > 0$ and $|x-y| \geq x_n/2 > 0$, then

$$|K_m(x, y)| \leq Mx_n^{m+1}|x'-y|^{|\lambda| - \ell - m - 1}.$$

2. If $|x-y| < x_n/2$, then $|K_m(x, y)| \leq M(x_n^{|\lambda| - \ell} + |x-y|^{|\lambda| - \ell}).$

3. If $|x'-y| < x_n/2$, then $|K_m(x, y)| \leq M(x_n^{|\lambda| - \ell} + x_n^m|x'-y|^{|\lambda| - \ell - m}).$

For a point $x \in \mathbb{R}^n$ and $r > 0$, we denote by $B(x, r)$ the open ball with center at $x$ and radius $r$.

**Lemma 2.2** (cf. [16, Lemma 3.2]). Let $\beta > -1$, $q > 0$ and $|\lambda| - \ell + n/q > 0$. Let $m$ be a nonnegative integer such that

$$m < |\lambda| - \ell + \frac{n + \beta}{q} < m + 1.$$
Then
\[ \left( \int |K_m(x, y)|^q |y_n|^{\beta} dy \right)^{1/q} \leq M x_n^{\lambda - \ell + (n+\beta)/q} \]
for all \( x = (x', x_n) \in \mathbf{D} \).

**Proof.** For fixed \( x \in \mathbf{D} \), consider the sets
\[ E_1 = B(x, x_n/2), \quad E_2 = B(x', x_n/2), \quad E_3 = \mathbf{D} - (E_1 \cup E_2). \]
Since \(|\lambda| - \ell + (n + \beta)/q - m - 1 < 0\), applying the polar coordinates about \( x' \), we have by Lemma 2.1(1)
\[
\left( \int_{E_3} |K_m(x, y)|^q |y_n|^{\beta} dy \right)^{1/q} \leq M x_n^{\lambda - \ell + (n+\beta)/q}\left( \int_{E_3} |x' - y|^m q |y_n|^{\beta} dy \right)^{1/q} \leq M x_n^{\lambda - \ell + (n+\beta)/q}.
\]
Similarly, since \(|\lambda| - \ell + n/q > 0\), we have by Lemma 2.1(2)
\[
\left( \int_{E_1} |K_m(x, y)|^q |y_n|^{\beta} dy \right)^{1/q} \leq M x_n^{\lambda - \ell + (n+\beta)/q} \left( \int_{E_1} |x_n^\lambda + |y_n|^\beta|y_n|^{\beta} dy \right)^{1/q} \leq M x_n^{\lambda - \ell + (n+\beta)/q}.
\]
Finally, since \(|\lambda| - \ell + (n + \beta)/q - m > 0\), we obtain by Lemma 2.1(3)
\[
\left( \int_{E_2} |K_m(x, y)|^q |y_n|^{\beta} dy \right)^{1/q} \leq M \left( \int_{E_2} |x_n^\lambda + x_m^\lambda |x' - y|^m q |y_n|^{\beta} dy \right)^{1/q} \leq M x_n^{\lambda - \ell + (n+\beta)/q} + M x_n^{\lambda - \ell + (n+\beta)/q} \left( \int_{0}^{x_n/2} r^{(\lambda - \ell - m)q + \beta} r^{-1} dr \right)^{1/q} \leq M x_n^{\lambda - \ell + (n+\beta)/q}.
\]
The required inequality now follows.

**Lemma 2.3** (cf. [16, Lemma 3.4]). Let \( q > 0 \) and \( m \) be a nonnegative integer such that
\[ m < |\lambda| - \ell + \frac{n-1}{q} < m + 1. \]
If \( x = (x', x_n) \in \mathbf{D} \) and \( y = (y', y_n) \in \mathbf{R}^n \), then
\[
\left( \int_{\mathbf{R}^{n-1}} |K_m(x, y)|^q dx' \right)^{1/q} \leq M x_n^{m+1} (x_n + |y_n|)^{|\lambda| - \ell - m - 1 + (n-1)/q}.
\]
PROOF. Let \( x = (x', x_n) \in D \) and \( y = (y', y_n) \in R^n \). If \( |y_n| \geq 2x_n \), then, since \( |\lambda| - \ell - m - 1 + (n - 1)/q < 0 \), we have by Lemma 2.1(1)

\[
\left( \int_{R^{n-1}} |K_m(x, y)|^q dx' \right)^{1/q} \leq M x_n^{m+1} \left( \int_{R^{n-1}} |x' - y|^{(|\lambda| - \ell - m - 1)q} dx' \right)^{1/q}
\]

\[
= M x_n^{m+1} \left( \int_{0}^{\infty} (r^2 + y_n^2)^{(|\lambda| - \ell - m - 1)/2} r^{n-2} dr \right)^{1/q}
\]

\[
= M x_n^{m+1} |y_n|^{(|\lambda| - \ell - m - 1 + (n - 1)/q)}.
\]

If \( |y_n| < 2x_n \), then we have as in the proof of Lemma 2.1

\[
\left( \int_{R^{n-1}} |K_m(x, y)|^q dx' \right)^{1/q} \leq M \left( \int_{\{x': (x', x_n) \in E_1\}} (x_n^{|\lambda| - \ell} + |x - y|^{(|\lambda| - \ell))q} dx' \right)^{1/q}
\]

\[
+ M \left( \int_{\{x': (x', x_n) \in E_2\}} (x_n^{|\lambda| - \ell} + x_n |x' - y|^{(|\lambda| - \ell - m))q} dx' \right)^{1/q}
\]

\[
+ M x_n^{m+1} \left( \int_{\{x': (x', x_n) \in E_3\}} |x' - y|^{(|\lambda| - \ell - m - 1)q} dx' \right)^{1/q}
\]

\[
\leq M x_n^{|\lambda| - \ell + (n - 1)/q} + M \left( \int_{B(y', x_n/2)} |x' - y|^{(|\lambda| - \ell)q} dx' \right)^{1/q}
\]

\[
+ M x_n^m \left( \int_{B(y', x_n/2)} |x' - y|^{(|\lambda| - \ell - m)q} dx' \right)^{1/q}
\]

\[
+ M x_n^{m+1} \left( \int_{R^{n-1}} (x_n + |x' - y|^{(|\lambda| - \ell - m - 1)q} dx' \right)^{1/q}
\]

\[
= M x_n^{|\lambda| - \ell + (n - 1)/q}.
\]

Therefore the required inequality now follows.

**Lemma 2.4** (cf. [1, Theorem 13.5], [15, Section 6.5]). Let \( \alpha = |\lambda| - \ell + n \), \( \alpha p > 1 \), \( \alpha p > 1 + \beta \) and \( -1 < \beta < p - 1 \). If \( f \) is a measurable function on \( R^n \) satisfying (2.2) and (2.3), then \( U_K f \) has the (ACL) property; in particular, \( U_K f(x', x_n) \) is absolutely continuous on \( R \) for almost every \( x' \in R^{n-1} \). Moreover, in case \( m \) is a positive integer such that \( (\alpha - m)p > 1 \) and \( (\alpha - m)p > 1 + \beta \),

\[
(\partial/\partial x_n)^m U_K f(x', x_n) = \int (\partial/\partial x_n)^m K(x - y) f(y) dy
\]

is absolutely continuous on \( R \) for almost every \( x' \in R^{n-1} \).

**Theorem 2.1** (cf. [11, Theorem 2.1] and [16, Theorem 2.1]). Let \( \alpha = |\lambda| - \ell + n \) satisfy \( m + 1/p < \alpha < m + n \). Let \( 1 < p \leq q < \infty \), \( \beta < p - 1 \) and

\[
\frac{n - \alpha p}{p(n - \alpha)} < \frac{n - 1}{q(n - \alpha + m) \quad \text{when } n - \alpha > 0.}
\]
Further suppose \( m < \omega < m + 1 \), where \( \omega = (n - 1)/q - (n - \alpha p + \beta)/p \). If \( f \) is a nonnegative measurable function on \( \mathbb{R}^n \) satisfying (2.2) and (2.3), then

\[
\lim_{r \to 0} r^{-\omega} H_q(u_r) = 0,
\]

where \( u_r(x') = U_K f(x', r) - \sum_{k=0}^{m} \frac{r^k}{k!} [(\partial/\partial x_n)^k U_K f](x', 0) \).

**Proof.** Under the assumptions on \( p, \alpha, \beta, q \) and \( m \) in Theorem 2.1, we can take \((\delta, \gamma)\) such that

(2.4) \[ \beta \leq \gamma < p(n - \alpha + m + 1)\delta + \beta - p(n - 1)/q, \]

(2.5) \[ p(n - \alpha + m + 1)\delta + (\alpha - m - 1)p - n < \gamma < p(n - \alpha + m)\delta + (\alpha - m)p - n, \]

(2.6) \[ \beta < \gamma < p - 1, \quad 0 < \delta < 1, \]

(2.7) \[ \delta p(n - \alpha) > n - \alpha p \]

and

(2.8) \[ \frac{n - 1}{q(n - \alpha + m + 1)} < \delta < \frac{n - 1}{q(n - \alpha + m)}. \]

Set \( a = (1 - \delta)p' \) and \( b = -\gamma p'/p \). Then, by (2.6) and (2.7),

\[ b > -1 \quad \text{and} \quad \alpha - n + \frac{n + b}{a} > 0. \]

Further, (2.5) implies

\[ m < \alpha - n + \frac{n + b}{a} < m + 1. \]

We first note from Lemma 2.4 that

\[
u_n(x') = U_K f(x) - \sum_{k=0}^{m} \frac{a_n^k}{k!} [(\partial/\partial x_n)^k U_K f](x', 0) = \int K_m(x, y) f(y) \, dy.
\]

Using Hölder’s inequality and Lemma 2.2, we have

\[
|u_n(x')| \leq \left( \int |K_m(x, y)|^a |y_n|^b dy \right)^{(1-\delta)/a} \left( \int |K_m(x, y)|^{\delta p} f(y)^p |y_n|^\gamma dy \right)^{1/p} \\
\leq M x_n^{(\alpha-n)(1-\delta)+n/p' - \gamma/p} \left( \int |K_m(x, y)|^{\delta p} f(y)^p |y_n|^\gamma dy \right)^{1/p}.
\]
In view of Minkowski's inequality for integral we have

$$H_q(u_{x_n}) \leq Mx_n^{(\alpha-n)(1-\delta)+n/p'-\gamma/p}$$

$$\times \left\{ \int \left( \int_{\mathbb{R}^{n-1}} |K_m(x, y)|^\delta dx' \right)^{p/q} f(y)^p |y_n|^{\gamma} \right\}^{1/p}.$$ 

Here, noting (2.8), we have by Lemma 2.3

$$\left( \int_{\mathbb{R}^{n-1}} |K_m(x, y)|^\delta dx' \right)^{p/q} \leq M[x_n^{m+1}(x_n + |y_n|)\alpha-n-m-1+(n-1)/\delta q]^\delta.$$ 

Consequently

$$H_q(u_r) \leq Mr^{(\alpha-n)(1-\delta)+n/p'-\gamma/p+(m+1)\delta}$$

$$\times \left\{ \int [(r + |y_n|)\alpha-n-m-1+(n-1)/\delta q]^\delta |y_n|^{\gamma-\beta} f(y)^p |y_n|^\beta \right\}^{1/p}.$$ 

Consider the function

$$k(r, y_n) = r^{[n-\alpha p+\beta]/p-(n-1)/q,r^{(\alpha-n)(1-\delta)+n/p'-\gamma/p+(m+1)\delta}}$$

$$\times [((r + |y_n|)\alpha-n-m-1+(n-1)/\delta q]^\delta |y_n|^{\gamma-\beta}. $$

Then

$$r^{-\omega} H_q(u_r) \leq M \left\{ \int k(r, y_n)f(y)^p |y_n|^\beta dy \right\}^{1/p},$$

where $\omega = (n-1)/q - (n - \alpha p + \beta)/p$. It follows from (2.4) that

$$r^{-\omega} r^{(\alpha-n)(1-\delta)+n/p'-\gamma/p+(m+1)\delta} = r^{(n-\alpha+m+1)\delta+(\beta-\gamma)/p-(n-1)/q} \to 0$$

as $r \to 0$. If $r < |y_n|$, then

$$k(r, y_n) \leq M(r/|y_n|)^{(n-\alpha+m+1)\delta+(\beta-\gamma)-p(n-1)/q} \leq M;$$

if $|y_n| \leq r$, then

$$k(r, y_n) \leq M(|y_n|/r)^{\gamma-\beta} \leq M.$$ 

Hence Lebesgue’s dominated convergence theorem implies that

$$\lim_{r \to 0} r^{-\omega} H_q(u_r) = 0.$$ 

Now the proof of Theorem 2.1 is completed.
3 Sobolev functions

For an open set $G \subset \mathbb{R}^n$, we denote by $BL_m(L^p_{loc}(G))$ the Beppo Levi space

$$BL_m(L^p_{loc}(G)) = \{ u \in L^p_{loc}(G) : D^\lambda u \in L^p_{loc}(G) \mid |\lambda| = m \}$$

(see [15]). In view of [15], each $u \in BL_m(L^p_{loc}(D))$ satisfying

$$(3.1) \quad \int_D |\nabla_m u(x)|^p x^\beta dx < \infty$$

has an $(m, p)$-quasicontinuous representative $\tilde{u}$, where $|\nabla_m u(x)| = (\sum_{|\mu|=m} |D^\mu u(x)|^2)^{1/2}$, $1 < p < \infty$ and $-1 < \beta < p - 1$. Moreover, $\tilde{u}$ is given by

$$\tilde{u}(x) = \sum_{|\lambda|=m} a_\lambda \int \tilde{K}_\lambda(x, y) D^\lambda \overline{u}(y) dy + P(x),$$

where $\overline{u}$ is an extension of $u$ to $\mathbb{R}^n$, $P(x)$ is a polynomial of degree at most $m - 1$, $K_\lambda(x) = x^\lambda |x|^{-n}$ and

$$\tilde{K}_\lambda,m(x, y) = \begin{cases} K_\lambda(x - y), & y \in B(0, 1), \\ K_\lambda(x - y) - \sum_{|\mu| \leq m-1} \frac{x^\mu}{|\mu|!} [(\partial/\partial x_1)^\mu K_\lambda](-y), & y \in \mathbb{R}^n - B(0, 1). \end{cases}$$

Note further from Lemma 2.4 that for each $k$ with $0 \leq k \leq m - 1$ and for almost every $x' \in \mathbb{R}^{n-1}$,

$$(\partial/\partial x_n)^k \int \tilde{K}_\lambda,m(x, y) D^\lambda \overline{u}(y) dy = \int (\partial/\partial x_n)^k \tilde{K}_\lambda,m(x, y) D^\lambda \overline{u}(y) dy$$

holds for $x_n \in \mathbb{R}$, where $x = (x', x_n)$.

Since $Q(x) - \sum_{k=0}^{m-1} \frac{x_n^k}{k!} [(\partial/\partial x_1)^k Q](x') = 0$ for any polynomial $Q$ of degree at most $m - 1$, we have

$$U(x) \equiv \tilde{u}(x) - \sum_{k=0}^{m-1} \frac{x_n^k}{k!} (\partial/\partial x_n)^k \tilde{u}(x')$$

$$= \sum_{|\lambda|=m} a_\lambda \int K_\lambda,m(x, y) D^\lambda \overline{u}(y) dy = \tilde{u}(x) - P(x)$$

for q.e. $x \in D$, where $K_\lambda,m(x, y) = K_\lambda(x - y) - \sum_{k=0}^{m-1} \frac{x_n^k}{k!} [(\partial/\partial x_n)^k K_\lambda](x' - y)$.

Theorem 2.1 now gives the following result.

**THEOREM 3.1** (cf. [19, Theorem 3.1]). Let $1 < p \leq q < \infty$,

$$\frac{n - mp}{p(n - m)} < \frac{1}{q} \quad \text{when } n - m > 0$$
and
\[
\frac{n-p+\beta}{p(n-1)} < \frac{1}{q} < \frac{n+\beta}{p(n-1)}.
\]
If \( u \in BL_{m}(L^{p}_{\text{loc}}(D)) \) satisfying (3.1) for \(-1 < \beta < p-1\) is \((m,p)\)-quasicontinuous on \( D \), then
\[
\lim_{r \to 0} r^{(n-mp+\beta)/p-(n-1)/q} H_{q}(U_{r}) = 0,
\]
where \( U_{r}(x') = u(x', r) - \sum_{k=0}^{m-1} \frac{r^{k}}{k!} \left[(\partial/\partial x_{n})^{k}u\right](x', 0) \).

Consider the Dirichlet problem for polyharmonic operator:
\[
\Delta^{m}u(x) = 0
\]
with the boundary conditions
\[
(\partial/\partial x_{n})^{k}u(x', 0) = f_{k}(x') \quad (k = 0, 1, ..., m-1).
\]
We denote by \( W^{m,p}(G) \) the Sobolev space
\[
W^{m,p}(G) = \{ u \in L^{p}(G) : D^{\lambda}u \in L^{p}(G) \ (|\lambda| \leq m) \}
\]
(see Stein [23, Chapter 6]). If \( u \in W^{m,p}(D) \) is a solution of the Dirichlet problem, then the vertical limit \( (\partial/\partial x_{n})^{k}u(x', 0) \) exists for almost every \( x' = (x', 0) \in \partial D \) and \( 0 \leq k \leq m-1 \) (see [12], [14]).

We also see that every function in \( W^{m,p}(D) \) can be extended to a function in \( W^{m,p}(\mathbb{R}^{n}) \) (see Stein [23]). Hence Theorem 3.1 gives the following result.

**Corollary 3.1.** Let \( 1 < p \leq q < \infty \) and
\[
(0 < \frac{n}{p} - \frac{n-1}{q} < 1.
\]
If \( u \in W^{m,p}(D) \) is a solution of the Dirichlet problem with \( f_{k}(x') = (\partial/\partial x_{n})^{k}u(x', 0) \) for \( 0 \leq k \leq m-1 \), then
\[
\lim_{r \to 0} r^{m/p-(n-1)/q-m} H_{q}(U_{r}) = 0,
\]
where \( U_{r}(x') = u(x', r) - \sum_{k=0}^{m-1} \frac{r^{k}}{k!} f_{k}(x') \).

### 4 Higher differences

For \( r > 0 \) and a function \( u \), we define the first difference
\[
\Delta_{r}u(t) = \Delta_{r}^{1}u(t) = u(t+r) - u(t).
\]
and the $m$-th difference
\[ \Delta_r^m u(t) = \Delta_r^{m-1} (\Delta_r u(\cdot))(t). \]

It is easy to see that
\[ \Delta_r^m u(t) = \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} u(t + kr). \]

As in Section 2, we consider
\[ K(x) = \frac{x^\lambda}{|x|^\ell} \]
and define
\[ u_r(x') = \Delta_r^m U_K f(x', \cdot)(0) = \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} U_K f(x', kr). \]

**THEOREM 4.1.** Let $\alpha = |\lambda| - \ell + n$, $1 < p \leq q < \infty$, $\beta < p - 1$ and
\[ \frac{n - \alpha p}{p} < \frac{n - 1}{q} \quad \text{(when } n - \alpha > 0). \]

Further suppose $0 < \omega < m$, where $\omega = (n-1)/q - (n-\alpha p+\beta)/p$. If $f$ is a nonnegative measurable function on $\mathbb{R}^n$ satisfying (2.2) and (2.3), then
\[ \lim_{r \to 0} r^{-\omega} H_q(u_r) = 0, \]
where $u_r(x') = \Delta_r^m U_K f(x', \cdot)(0)$.

To prove this, we have only to prepare the following two lemmas instead of Lemmas 2.2 and 2.3.

**LEMMA 4.1 ([19, Lemma 4.1]).** Let $\beta > -1$, $q > 0$ and $|\lambda| - \ell + n/q > 0$. Let $m$ be a positive integer such that
\[ 0 < |\lambda| - \ell + \frac{n+\beta}{q} < m. \]

Then
\[ \left( \int |K_m^*(x, y)|^q |y_n|^\beta \, dy \right)^{1/q} \leq M x_n^{\lambda|\ell + (n+\beta)/q}
\]
for all $x = (x', x_n) \in D$, where $K_m^*(x, y) = \Delta_x^m K(x' - y', \cdot - y_n)(0)$ for $x = (x', x_n) \in D$ and $y = (y', y_n) \in \mathbb{R}^n$. 

LEMMA 4.2 ([19, Lemma 4.2]). Let \( q > 0 \) and \( m \) be a positive integer such that

\[
0 < |\lambda| - \ell + \frac{n-1}{q} < m.
\]

If \( x = (x', x_n) \in \mathcal{D} \) and \( y = (y', y_n) \in \mathbb{R}^n \), then

\[
\left( \int_{\mathbb{R}^{n-1}} |K_m^*(x, y)|^q dx' \right)^{1/q} \leq M x_n^m (x_n + |y_n|)^{|\lambda| - \ell - m + (n-1)/q}.
\]

THEOREM 4.2. Let \( 1 < p \leq q < \infty \) and

\[
(0 < ) \frac{n}{p} - \frac{n-1}{q} < m.
\]

If \( u \in BL_m(L^p(\mathbb{R}^n)) \) is \((m, p)\)-quasicontinuous on \( \mathbb{R}^n \), then

\[
\lim_{r \to 0} r^{n/p-(n-1)/q} m H_q(U_r) = 0,
\]

where \( U_r(x') = \Delta_r^m u(x', \cdot)(0) \) for \( r > 0 \).

In fact, since \( \Delta_r^m Q = 0 \) for any polynomial \( Q \) of degree at most \( m - 1 \), we have

\[
U(x) \equiv \Delta_r^m u(x', \cdot)(0) = \sum_{|\lambda|=m} a_\lambda \int K^{\ast}_{\lambda, m}(x, y) D^\lambda u(y) dy,
\]

where \( K^{\ast}_{\lambda, m}(x, y) = \Delta_r^m K_{\lambda}(x' - y', \cdot - y_n)(0) \) with \( K_{\lambda}(x) = x^\lambda |x|^{-n} \).

5 Monotone functions

We say that a continuous function \( u \) is monotone in an open set \( G \), in the sense of Lebesgue ([6]), if both

\[
\max_{\partial D} u(x) = \max_{\partial D} u(x) \quad \text{and} \quad \min_{\partial D} u(x) = \min_{\partial D} u(x)
\]

hold for every relatively compact open set \( D \) with the closure \( \overline{D} \subset G \).

The class of monotone functions is considerably wide. We give some examples of monotone functions.

EXAMPLE 1. Let \( f(r) \) be a non-increasing (or non-decreasing) continuous function on \((0, \infty)\). If we define

\[
u(x) = f(|x - \xi|)
\]

for \( x \in G \) and \( \xi \in \partial G \), then \( u \) is monotone in \( G \).
EXAMPLE 2. Harmonic functions on an open set $G$ are monotone in $G$. More generally, solutions of elliptic partial differential equations of second order may be monotone (see Gilbarg-Trudinger [3]).

EXAMPLE 3. Weak solutions for variational problems may be monotone; in particular, weak solutions of the $p$-Laplacian are monotone. Moreover, if $f$ is a quasi-regular mapping on $G$, then the coordinate functions of $f$ are monotone in $G$. For these facts, see Heinonen-Kilpeläinen-Martio [4], Reshetnyak [20], Serrin [21] and Vuorinen [27], [28].

A key result for monotone BLD functions is the following fact.

**Lemma 5.1** (cf. [5, Lemma 7.1], [7, Remark, p.9], [28, Sect. 16]). Let $p > n - 1$. If $u$ is a monotone BLD function on $B(x_0, 2r)$, then

$$|u(x) - u(y)|^p \leq M r^{p-n} \int_{B(x_0, 2r)} |\nabla u(z)|^p \, dz \quad \text{whenever } x, y \in B(x_0, r).$$

This lemma is shown by the application of Sobolev's inequality on the spherical surfaces, so that the restriction $p > n - 1$ is needed; for a proof of Lemma 5.1, see for example [5, Lemma 7.1] or [15, Theorem 5.2, Chap.8].

6 Hyperplane integrals of monotone functions

We define the $q$th integral $H_{q,N}(u)$ over $\{x' : |x'| < N\}$ by

$$H_{q,N}(u) = \left( \int_{\{x' : |x'| < N\}} |u(x')|^q \, dx' \right)^{1/q}$$

for a measurable function $u$ on $\{x' : |x'| < N\}$.

**Theorem 6.1** (cf. [17, Theorem 2]). Let $u$ be a monotone function on $D$ satisfying (1.1). If $n - 1 < p < n + \beta$, $p \leq q < \infty$ and

$$\frac{1}{q} < \frac{n-p+\beta}{p(n-1)},$$

then

$$\lim_{r \to 0} r^{(n-p+\beta)/p-(n-1)/q} H_{q,N}(u_r) = 0,$$

where $u_r(x') = u(x', r)$ for $r > 0$.

**Proof.** Let $u$ be a monotone function on $D$ satisfying (1.1) with $n - 1 < p < n + \beta$. If $|s - t| \leq r < t/2$, then Lemma 5.1 yields

$$|H_{q,N}(u_s) - H_{q,N}(u_t)| \leq \left( \int_{\{x' : |x'| < N\}} |u_s(x') - u_t(x')|^q \, dx' \right)^{1/q}$$

$$\leq M r^{(p-n)/p} \left( \int_{\{x' : |x'| < N\}} \left( \int_{B(x', 2r)} |\nabla u(z)|^p \, dz \right)^{q/p} \, dx' \right)^{1/q},$$
so that Minkowski’s inequality for integral yields

$$|H_{q,N}(u_s) - H_{q,N}(u_t)| \leq M r^{(p-n)/p(2r)^{(n-1)/q}} \left( \int_{\{z=(z',z_n): 2r < z_n < 3r \}} |\nabla u(z)|^p \, dz \right)^{1/p}.$$  

Let \( r_j = 2^{-j-1}, t_j = r_{j-1} \) and \( A_j = \{ z = (z', z_n): r_j < z_n < 3r_j \} \) for \( j = 1, 2, \ldots \). Set
\[
\omega = (n - p + \beta)/p - (n - 1)/q > 0.
\]
Then we find

$$|H_{q,N}(u_{t_j}) - H_{q,N}(u_r)| \leq M r_j^{-\omega} \left( \int_{A_j} |\nabla u(z)|^p z_n^\beta \, dz \right)^{1/p}$$

for \( t_j - r_{j+1} < r \leq t_j \),

$$|H_{q,N}(u_{t_j-r_{j+1}}) - H_{q,N}(u_r)| \leq M r_{j+2}^{-\omega} \left( \int_{A_{j+1}} |\nabla u(z)|^p z_n^\beta \, dz \right)^{1/p}$$

for \( t_j - r_{j+1} - r_{j+2} < r \leq t_j - r_{j+1} \) and

$$|H_{q,N}(u_r) - H_{q,N}(u_{t_{j+1}})| \leq M r_{j+1}^{-\omega} \left( \int_{A_{j+1}} |\nabla u(z)|^p z_n^\beta \, dz \right)^{1/p}$$

for \( t_{j+1} < r \leq t_j - r_{j+1} - r_{j+2} \). Collecting these results, we have

$$|H_{q,N}(u_{t_j}) - H_{q,N}(u_r)| \leq M \sum_{\ell=1}^{j+m} r_{\ell+1}^{-\omega} \left( \int_{A_{\ell+1}} |\nabla u(z)|^p z_n^\beta \, dz \right)^{1/p}.$$  

Since \( A_\ell \cap A_k = \emptyset \) when \( \ell \geq k + 2 \), Hölder’s inequality gives

$$|H_{q,N}(u_{t_j}) - H_{q,N}(u_{t_{j+m}})| \leq M \left( \sum_{\ell=1}^{j+m} r_{\ell+1}^{-\omega} \right)^{1/p'} \left( \sum_{\ell=1}^{j+m} \int_{A_{\ell+1}} |\nabla u(z)|^p z_n^\beta \, dz \right)^{1/p}$$

\[
\leq M r_{j+m}^{-\omega} \left( \int_{\{z=(z',z_n): 2r_{j+m} < z_n < 3r_j \}} |\nabla u(z)|^p z_n^\beta \, dz \right)^{1/p}.
\]
More generally, if $0 < r \leq t_j$, then we take $m$ such that $t_{j+m} < r \leq t_{j+m-1}$, and establish

$$|H_{q,N}(u_{t_j}) - H_{q,N}(u_r)| \leq Mr^{-\omega} \left( \int_{\{z=(z',z_n):0<z_n<3r_j\}} |\nabla u(z)|^p z_n^\beta \, dz \right)^{1/p},$$

which implies that

$$\limsup_{r \to 0} r^\omega H_{q,N}(u_r) \leq M \left( \int_{\{z=(z',z_n):0<z_n<3r_j\}} |\nabla u(z)|^p z_n^\beta \, dz \right)^{1/p}$$

for all $j$. Therefore it follows that

$$\lim_{r \to 0} r^\omega H_{q,N}(u_r) = 0,$$

as required.

In case $1/q = (n - p + \beta)/p(n - 1) > 0$, one might expect that $H_{q,N}(u_r)$ is bounded. In fact, we can show that this is true only in case $0 \leq \beta < p - 1$ without assuming the monotonicity. We refer the reader to the result by Yamashita [29] who showed affirmatively the case $p = 2$ and $\beta = 1$ for harmonic functions. In the hyperplane case, we refer to [16, Theorem 2.2]. The case $\beta = p - 1$ remains open.

For Sobolev functions, we have a weak limit result as follows.

**Theorem 6.2** (cf. [16, Theorem 2.1]). Let $-1 < \beta < p - 1$, $1 < p \leq q < \infty$ and

$$\frac{n - p - 1}{p(n - 1)} < \frac{1}{q} < \frac{n - p + \beta}{p(n - 1)}.$$

If $u$ is a $(1, p)$-quasicontinuous function on $\mathbf{D}$ satisfying (1.1), then there exists a number $A$ such that

$$\liminf_{r \to 0} r^{(n-p+\beta)/p-(n-1)/q} H_q(u_r - A) = 0,$$

where $u_r(x') = u(x', r)$ for $r > 0$.

By this together with Theorem 6.1, we can prove the following result.

**Corollary 6.1.** Let $u$ be a monotone function on $\mathbf{D}$ satisfying (1.1). If $n - 1 < p < n + \beta$, $-1 < \beta < p - 1$, $p \leq q < \infty$ and

$$\frac{1}{q} < \frac{n - p + \beta}{p(n - 1)},$$

then there exists a number $A$ such that

$$\lim_{r \to 0} r^{(n-p+\beta)/p-(n-1)/q} H_q(u_r - A) = 0,$$

where $u_r(x') = u(x', r)$ for $r > 0$. 
THEOREM 6.3 (cf. [18, Theorem 1]). Let $u$ be a monotone function on $D$ satisfying (1.1) with $n - 1 < p \leq n + \beta$. If $p \leq q < \infty$ and

$$\frac{1}{q} > \frac{n - p + \beta}{p(n - 1)},$$

then

$$\lim_{r \to 0} r^{(n-p+\beta)/p-(n-1)/q} H_q(U_r) = 0,$$

where $U_r(x') = u(x', r) - u(x', 0)$ for $r > 0$.

PROOF. Let $u$ be a monotone function on $D$ satisfying (1.1) with $n - 1 < p \leq n + \beta$. If $|s - t| \leq r < t/2$, then Lemma 5.1 gives

$$|H_q(u_s - u_t)| = \left( \int_{\mathbb{R}^{n-1}} |u_s(x') - u_t(x')|^q \, dx' \right)^{1/q}$$

$$\leq M r^{(p-n)/p} \left( \int_{\mathbb{R}^{n-1}} \left( \int_{B((x', t), 2r)} |\nabla u(z)|^p \, dz \right)^{q/p} \, dx' \right)^{1/q},$$

so that Minkowski's inequality for integral yields

$$|H_q(u_s - u_t)| \leq M r^{(p-n)/p} (2r)^{(n-1)/q} \left( \int_{\{z = (z', z_n) : r < z_n < 3r\}} |\nabla u(z)|^p \, dz \right)^{1/p}.$$ 

Let $r_j = 2^{-j-1}$, $t_j = r_{j-1}$ and $A_j = \{z = (z', z_n) : r_j < z_n < 3r_j\}$ for $j = 1, 2, \ldots$. For simplicity, set

$$\omega = (n - p + \beta)/p - (n - 1)/q < 0.$$ 

Then we find

$$|H_q(u_{t_j} - u_r)| \leq M r_{j+1}^{-\omega} \left( \int_{A_j} |\nabla u(z)|^p z_n^\beta \, dz \right)^{1/p}$$

for $t_j - r_{j+1} < r \leq t_j$,

$$|S_q(u_r - u_s)| \leq M r_{j+2}^{-\omega} \left( \int_{A_j} |\nabla u(z)|^p z_n^\beta \, dz \right)^{1/p}$$

for $t_j - r_{j+1} - r_{j+2} < r < s \leq t_j - r_{j+1}$, and

$$|S_q(u_s - u_{t_{j+1}})| \leq M r_{j+2}^{-\omega} \left( \int_{A_{j+1}} |\nabla u(z)|^p z_n^\beta \, dz \right)^{1/p}$$
for $t_{j+1} < s \leq t_j - r_{j+1} - r_{j+2}$. Collecting these results, we have

$$|H_q(u_{t_j} - u_r)| \leq Mr_j^{-\omega} \left( \int_{A_j} |\nabla u(z)|^p z_{n}^{\beta} dz \right)^{1/p}$$

$$+ Mr_{j+1}^{-\omega} \left( \int_{A_{j+1}} |\nabla u(z)|^p z_{n}^{\beta} dz \right)^{1/p}$$

for $t_{j+1} < r \leq t_j$. Hence it follows that

$$|H_q(u_r - u_{t_{j+m}})| \leq M \sum_{\ell=j}^{j+m} r_{\ell}^{-\omega} \left( \int_{A_{\ell}} |\nabla u(z)|^p z_{n}^{\beta} dz \right)^{1/p}$$

for $t_{j+m} < r \leq t_j$. Since $A_{\ell} \cap A_k = \emptyset$ for $\ell \geq k + 2$, Hölder’s inequality gives

$$|H_q(u_r - u_{t_{j+m}})| \leq M \left( \sum_{\ell=j}^{j+m} r_{\ell}^{-\omega} \left( \int_{A_{\ell}} |\nabla u(z)|^p z_{n}^{\beta} dz \right)^{1/p} \right. \left. \left( \int_{A_{\ell}} |\nabla u(z)|^p z_{n}^{\beta} dz \right)^{1/p} \right)$$

$$\leq Mr_j^{-\omega} \left( \int_{\{z=(z'),zn):r_{j+m}<zn<3r_j\}} |\nabla u(z)|^p z_{n}^{\beta} dz \right)^{1/p}$$

for $t_{j+m} < r \leq t_j$, where $1/p + 1/p' = 1$. Now, letting $m \to \infty$, we establish

$$|H_q(U_r)| \leq Mr^{-\omega} \left( \int_{\{z=(z'),zn):0<zn<3r_j\}} |\nabla u(z)|^p z_{n}^{\beta} dz \right)^{1/p}$$

for $t_{j+1} < r \leq t_j$, which implies that

$$\lim_{r \to 0} r^\omega H_q(U_r) = 0,$$

as required.

As to the condition on $q$, compare Theorem 6.3 with Theorem 3.1.
References


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