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A generalization of the Liouville theorem to polyharmonic functions

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1 Introduction

Let $\mathbb{R}^n$ be the $n$-dimensional Euclidean space with a point $x = (x_1, x_2, \ldots, x_n)$. For a multi-index $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$, we set

$$|\lambda| = \lambda_1 + \lambda_2 + \cdots + \lambda_n,$$

$$x^\lambda = x_1^{\lambda_1}x_2^{\lambda_2}\cdots x_n^{\lambda_n}$$

and

$$D^\lambda = \left(\frac{\partial}{\partial x_1}\right)^{\lambda_1}\left(\frac{\partial}{\partial x_2}\right)^{\lambda_2}\cdots\left(\frac{\partial}{\partial x_n}\right)^{\lambda_n}.$$ 

We denote by $rB^n$ the open ball centered at the origin with radius $r > 0$, whose boundary is denoted by $rS^{n-1}$.

A real valued function $u$ is called polyharmonic of order $m$ on $\mathbb{R}^n$ if $u \in C^{2m}$ and $\Delta^m u = 0$, where $m$ is a positive integer, $\Delta$ denotes the Laplacian and $\Delta^m u = \Delta^{m-1}(\Delta u)$. We denote by $H^m(\mathbb{R}^n)$ the space of polyharmonic functions of order $m$ on $\mathbb{R}^n$. In particular, $u$ is harmonic on $\mathbb{R}^n$ if $u \in H^1(\mathbb{R}^n)$.

The Liouville theorem for polyharmonic functions is known in several forms (cf. [1, 3, 4]).

**Theorem A.** Let $u \in H^m(\mathbb{R}^n)$ and $s > 2(m - 1)$. Then $u$ is a polynomial of degree less than $s$ if one of the following conditions holds:

(i) $\lim_{r \to \infty} \frac{1}{r^{s+n-1}} \int_{rS^{n-1}} u^+ dS = 0$ (see [1]);

(ii) $\lim_{r \to \infty} \frac{1}{r^{s+n}} \int_{rB^n} u^+ dx = 0$ (see [3]);

(iii) $\limsup_{r \to \infty} \left(\max_{x \in rS^{n-1}} \frac{u(x)}{|x|^s}\right) \leq 0$ (see [4]).

Now we propose the following theorem.
THEOREM. Let \( u \in H^m(\mathbb{R}^n) \) and \( s > 2(m - 1) \). Then \( u \) is a polynomial of degree at most \( s \) if and only if

\[
\lim_{r \to \infty} \inf_{S^{n-1}} \frac{1}{r^{s+n-1}} \int_{rS^{n-1}} u^+ \, dS < \infty.
\]

(1)

We here note that each condition of Theorem A implies (1), so that our theorem gives an improvement of Theorem A.

2 The main lemmas

Let us begin with the following lemma, which expands polyharmonic functions to harmonic functions (cf. [2], [5]).

**Lemma 1 (The Finite Almansi Expansion).** A real valued function \( u \) on \( \mathbb{R}^n \) belongs to \( H^m(\mathbb{R}^n) \), then there exists a unique family \( \{h_i\}_{i=1}^m \subset H^1(\mathbb{R}^n) \) such that

\[
u(x) = \sum_{i=1}^m |x|^{2(i-1)}h_i(x)
\]

for every \( x \in \mathbb{R}^n \).

**Proof.** We prove this lemma by induction on \( m \). For \( m = 1 \) the conclusion is trivial. Suppose the conclusion is true for \( m = k \), and let \( u \in H^{k+1}(\mathbb{R}^n) \). Then there exists a family \( \{g_i\}_{i=1}^k \subset H^1(\mathbb{R}^n) \) such that

\[
\Delta u = \sum_{i=1}^k |x|^{2(i-1)}g_i(x)
\]

(3)

If a family \( \{h_i\}_{i=1}^{k+1} \subset H^1(\mathbb{R}^n) \) satisfies (2), then we should have

\[
\Delta u = \sum_{i=1}^{k+1} \Delta (|x|^{2(i-1)}h_i(x)) = \sum_{i=1}^k \Delta (|x|^{2i}h_{i+1}(x)).
\]

If we write \( r = |x| \), then

\[
\Delta (|x|^{2i}h_{i+1}(x)) = \sum_{j=1}^n \left( \frac{\partial}{\partial x_j} \right)^2 (r^{2i}h_{i+1}(x)) = \sum_{j=1}^n \left( \partial_{x_j}^2 h_{i+1}(x) + 2r^{2i} \frac{\partial r^{2i}}{\partial x_j} \frac{\partial h_{i+1}(x)}{\partial x_j} + r^{2i} \frac{\partial^2 h_{i+1}(x)}{\partial x_j^2} \right)
\]
\[
= 2ir^{2(i-1)} \left\{ (2i - 2 + n)h_{i+1}(x) + 2r \frac{\partial h_{i+1}}{\partial r}(x) \right\}
\]
\[
= |x|^{2(i-1)} \left\{ 2i(2i - 2 + n)h_{i+1}(x) + 4ir \frac{\partial h_{i+1}}{\partial r}(x) \right\}.
\]

From the uniqueness of Almansi expansion for \( \Delta u \), it is necessary and sufficient to find a unique solution \( h_{i+1} \) for the equation
\[
g_i(x) = 2i(2i - 2 + n)h_{i+1}(x) + 4ir \frac{\partial h_{i+1}}{\partial r}(x)
\]
for each \( i = 1, \cdots, k \). We see that the unique solution for (4) is given by
\[
h_{i+1}(x) = \frac{1}{4ir^{i-1/2}+n} \int_0^r t^{i-1} \int_0^r t^{i-2+n/2}g_i(tx/r) dt.
\]

Here we have only to show that \( h_{i+1} \) is harmonic on \( \mathbb{R}^n \). Actually, putting \( x = r\zeta \), where \( r = |x| \) and \( \zeta = x/|x| = x/r \), we have
\[
h_{i+1}(x) = h_{i+1}(r\zeta) = \frac{r^{1-i-n/2}}{4i} \int_0^r t^{i-2+n/2}g_i(r\zeta) dt
\]
\[
= \frac{r^{1-i-n/2}}{4i} \int_0^1 (rs)^{i-2+n/2}g_i(rs\zeta) rds
\]
\[
= \frac{1}{4i} \int_0^1 s^{i-2+n/2}g_i(xs) ds.
\]

Since \( g_i \) is harmonic, we see that \( \Delta h_{i+1}(x) = 0 \) for \( i = 1, \cdots, k \). Now put
\[
h_1(x) = u(x) - \sum_{i=2}^{k+1} |x|^{2(i-1)}h_i(x).
\]

Then \( \Delta h_1(x) = 0 \) by (3), and the induction is completed.

Next we prepare the following lemma, which gives a relation between spherical means and derivatives for harmonic functions.

**Lemma 2.** Suppose \( u \in H^1(\mathbb{R}^n) \). For each multi-index \( \lambda \), there exists a positive constant \( C = C(\lambda) \) such that
\[
\int_{rS^{n-1}} u x^\lambda \ dS = Cr^2|\lambda|n^{-1}D^\lambda u(0) + P_{2|\lambda|+n-3}(r)
\]
for every \( r > 0 \), where \( P_k(r) \) is a polynomial of degree at most \( k \) depends on \( u \).

**Proof.** We prove this lemma by induction on the length of \( \lambda \). Assume first that \( \lambda_n = 1 \) and \( \lambda_i = 0 \) (\( i = 1, \ldots, n-1 \)). Using Green's formula and the mean-value property for harmonic functions, we have
\[ \int_{rS^{n-1}} u x^\lambda \, dS = \int_{rS^{n-1}} u x_n \, dS = r \int_{rS^{n-1}} u \frac{x_n}{r} \, dS = r \int_{rB^n} \frac{\partial u}{\partial x_n} \, dx = \sigma_n r^{n+1} \frac{\partial u}{\partial x_n}(0), \]

where \( \sigma_n \) is the \( n \)-dimensional volume of the unit ball. Hence (5) holds for \(|\lambda| = 1\).

Next suppose that (5) holds for \(|\lambda| \leq k\), where \( k \) is a positive integer. Let \( \mu = (\mu_1, \ldots, \mu_n) \) such that \(|\mu| = k + 1\). We may assume without loss of generality that \( \mu_n \geq 2 \), and set \( \mu' = (\mu_1, \ldots, \mu_{n-1}, \mu_n - 1) \). Then we write

\[ \int_{rS^{n-1}} u x^\mu \, dS = r \int_{rS^{n-1}} u x^\mu x_n \frac{x_n}{r} \, dS. \]

From Green's formula we obtain

\[ \int_{rS^{n-1}} u x^\mu \, dS = r \int_{rB^n} \frac{\partial (ux^\mu)}{\partial x_n} \, dx = r \int_{rB^n} \left( x^\mu \frac{\partial u}{\partial x_n} + (\mu_n - 1)ux_1^{\mu_1} \cdots x_n^{\mu_n - 2} \right) \, dx = (*). \]

Set \( \mu'' = (\mu_1, \ldots, \mu_{n-1}, \mu_n - 2) \). Since \(|\mu'| = k\) and \(|\mu''| = k - 1\), we find

\[ (* ) = r \int_0^r \left( \int_{tS^{n-1}} \left( x^\mu \frac{\partial u}{\partial x_n} + (\mu_n - 1)ux^{\mu''} \right) \, dS \right) \, dt \]

\[ = r \int_0^r \left( \int_{tS^{n-1}} \frac{\partial u}{\partial x_n} x^{\mu'} \, dS \right) \, dt + (\mu_n - 1)r \int_0^r \left( \int_{tS^{n-1}} u x^{\mu''} \, dS \right) \, dt \]

\[ = r \int_0^r \left( C(\mu') r^{2|\mu'|+n-1} D^{\mu'} \left( \frac{\partial u}{\partial x_n} \right)(0) + P_{2|\mu'|+n-3}(t) \right) \, dt \]

\[ + (\mu_n - 1)r \int_0^r \left( C(\mu'') r^{2|\mu''|+n-1} D^{\mu''} u(0) + P_{2|\mu''|+n-3}(t) \right) \, dt \]

\[ = C(\mu)r^{2k+n+1} D^{\mu} u(0) + P_{2k+n-1}(r), \]

where \( C(\mu) = \frac{C(\mu')}{2k+n} > 0 \) and \( P_\ell \) denotes various polynomials of degree at most \( \ell \) which may change from one occurrence to the next; throughout this note, we use this convention. Hence (5) also holds for \(|\mu| = k + 1\). The induction is completed.
3 Proof of the theorem

First we show that our theorem is valid under the two sided condition on spherical means for polyharmonic functions.

**Lemma 3.** Let \( u \in H^m(\mathbb{R}^n) \) and \( s > 2(m-1) \). Then \( u \) is a polynomial of degree at most \( s \) if

\[
\lim_{r \to \infty} \inf \frac{1}{r^{s+n-1}} \int_{rS^{n-1}} |u| \, dS < \infty. \tag{6}
\]

**Proof.** By (6) we can find a sequence \( \{r_j\}_{j=1}^\infty \) such that \( r_j \to \infty \) and

\[
\sup_j \left( r_j^{-s-n+1} \int_{r_jS^{n-1}} |u| \, dS \right) < \infty. \tag{7}
\]

Using (2) and Lemma 2, we have

\[
\int_{rS^{n-1}} u x^\lambda \, dS = \int_{rS^{n-1}} \left( \sum_{i=1}^{m} |x|^{2(i-1)} h_i(x) \right) x^\lambda \, dS
= \sum_{i=1}^{m} r^{2(i-1)} \int_{rS^{n-1}} h_i(x) x^\lambda \, dS
= \sum_{i=1}^{m} r^{2(i-1)} \left( C_i r^{2|\lambda|+n-1} D^\lambda h_i(0) + P_{i,2|\lambda|+n-3}(r) \right),
\]

where \( C_i = C_i(\lambda) \) is a positive constant and \( P_{i,k} \) denotes various polynomials of degree at most \( k \) depends on \( h_i \). Hence it follows that

\[
r^{|\lambda|} \int_{rS^{n-1}} |u| \, dS \geq \left| \sum_{i=1}^{m} r^{2(i-1)} \left( C_i r^{2|\lambda|+n-1} D^\lambda h_i(0) + P_{i,2|\lambda|+n-3}(r) \right) \right|
\]

so that we obtain

\[
r_j^{-s-n+1} \int_{r_jS^{n-1}} |u| \, dS \geq r_j^{s+2(m-1)} \left| C_m D^\lambda h_m(0) + O(r_j^{-2}) \right|
\]

as \( r_j \to \infty \). By (7), we find

\[D^\lambda h_m(0) = 0\]

for all \( |\lambda| > s - 2(m-1) \). By analyticity of harmonic functions, we see that \( h_m \) is a polynomial of degree at most \( s - 2(m-1) \). Hence we note that

\[
r^{2(m-1)} \int_{rS^{n-1}} h_m(x) x^\lambda \, dS = O(r^{s+|\lambda|+n-1}) \quad \text{as } r \to \infty.
\]

Consequently,

\[
r_j^{-s-n+1} \int_{r_jS^{n-1}} |u| \, dS \geq r_j^{s+2(m-2)} \left| C_{m-1} D^\lambda h_{m-1}(0) + O(r_j^{-2}) \right| + O(1)
\]
as $r_j \to \infty$. This implies that $D^\lambda h_{m-1}(0) = 0$ for $|\lambda| > s - 2(m - 2)$, so that $h_{m-1}$ is a polynomial of degree at most $s - 2(m - 2)$. By repeating this arguments, we see that each $h_i$ is a polynomial of degree at most $s - 2(i - 1) (i = 1, \ldots, m)$. Thus it follows that $u$ is a polynomial. In view of (2), the degree of $u$ is at most $2(i - 1) + s - 2(i - 1) = s$.

PROOF OF THE THEOREM. If $u \in H^m(\mathbb{R}^n)$, then we see from (2) that
\[
\frac{1}{\omega_n r^{n-1}} \int_{rS^{n-1}} u \, dS = \sum_{i=1}^m r^{2(i-1)} h_i(0),
\]
where $\omega_n$ denotes the surface measure of $S^{n-1}$.

Since $|u| = 2u^+ - u$, we have
\[
\liminf_{r \to \infty} r^{-s-n+1} \int_{rS^{n-1}} |u| \, dS \\
= \liminf_{r \to \infty} \left( 2r^{-s-n+1} \int_{rS^{n-1}} u^+ \, dS - r^{-s-n+1} \int_{rS^{n-1}} u \, dS \right) \\
= \liminf_{r \to \infty} \left( 2r^{-s-n+1} \int_{rS^{n-1}} u^+ \, dS - r^{-s} P_{2(m-1)}(r) \right).
\]
Hence (1) implies (6) since $s > 2(m - 1)$, so that the present theorem follows from Lemma 3.

References


