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# On the Two-Phase Obstacle Problem

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## 1 Introduction

Although the regularity in one-phase free boundary problems has by now been extensively studied, the methods used there prove in many cases to be unsuitable for the corresponding two-phase problems.
Here we announce a result concerning the two-phase obstacle problem

$$\Delta u=\frac{\lambda_{+}}{2}\chi_{\{u>0\}}-\frac{\lambda_{-}}{2}\chi_{\{u<0\}} \quad . \tag{1}$$

The nonlinearities of this equation suggest that the solution should be locally a $H^{2,\infty}$-function. We obtain this regularity in the form of a growth estimate (Proposition 3.1). The proof uses new ideas as well as a monotonicity formula introduced by the author in [7]. A consequence is that the Hausdorff dimension of the free boundary $\partial\{u>0\}\cup\partial\{u<0\}$ is less than or equal to $n-1$ (Corollary 4.1).
Note that our approach can also be used to derive Lipschitz continuity of minimizers of the functional $v\mapsto\int_{\Omega}(|\nabla v|^{2}+\lambda_{+}\chi_{\{v>0\}}+\lambda_{-}\chi_{\{v<0\}})$ (Remark 4.1); Lipschitz continuity of minimizers of this functional has been proven

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in [1] using a result on optimal Poincaré constants with respect to spherical domains ([2]).

## 2 The equation

Let $n \geq 2$ and let $\Omega$ be a bounded open subset of $\mathbf{R}^n$ with Lipschitz boundary, assume that $u_D \in H^{1,2}(\Omega)$ and let $A := \{v \in H^{1,2}(\Omega) \,:\, v - u_D \in H_0^{1,2}(\Omega)\}$. Then the functional $E(v) := \int_\Omega (|\nabla v|^2 + \lambda_+ \max(v,0) - \lambda_- \min(v,0))$, being real-valued, non-negative, convex and weakly lower semicontinuous, attains its infimum on the affine subspace $A$ of $H^{1,2}(\Omega)$ at the point $u \in A$.
Throughout the whole paper $u$ shall denote this minimizer, however the reader may replace the boundary condition in the definition of $A$ at his own convenience, since from now on everything we do will be completely local.
Let us compute the first variation of the energy $E$ at the point $u$. Using $v := u + \epsilon\phi$ as test function for the minimality of $u$, where $\epsilon > 0$ and $\phi \in H_0^{1,2}(\Omega) \cap L^\infty(\Omega)$, we obtain that

$$\int_\Omega (2\nabla u \cdot \nabla\phi \;+\; \phi\,\lambda_+\,\chi_{\{u \geq -\epsilon\phi\}} \;-\; \phi\,\lambda_-\,\chi_{\{u \leq -\epsilon\phi\}}) \;\geq\; -\epsilon \int_\Omega |\nabla\phi|^2 ,$$

and, as $\epsilon \to 0$, that

$$\begin{aligned} &\textstyle\int_{\Omega\cap\{u=0\}}(-\lambda_+ \max(\phi,0) \;+\; \lambda_- \min(\phi,0)) \;\leq \\ &\textstyle\int_\Omega (2\nabla u \cdot \nabla\phi \;+\; \phi\,\lambda_+\,\chi_{\{u>0\}} \;-\; \phi\,\lambda_-\,\chi_{\{u<0\}}) \\ &\leq\; \textstyle\int_{\Omega\cap\{u=0\}}(\lambda_+ \max(-\phi,0) \;-\; \lambda_- \min(-\phi,0)) \end{aligned} \tag{2}$$

for every $\phi \in H_0^{1,2}(\Omega)$. By the characterization of non-negative distributions this implies that $v \mapsto \int(\nabla u \cdot \nabla\phi \;+\; \frac{\lambda_+}{2}\phi)$ is locally in $\Omega$ represented by a finite regular measure. Hence, (2) yields by Radon-Nikodym's theorem that $\Delta u \in L^1_{\text{loc}}(\Omega)$ and it follows that $\Delta u = \frac{\lambda_+}{2}\,\chi_{\{u>0\}} \;-\; \frac{\lambda_-}{2}\,\chi_{\{u<0\}}$ a.e. in $\Omega$.
At this point we observe that any other function $v \in H^{1,2}(\Omega)$ with boundary data $u_D$ on $\partial\Omega$ that satisfies the weak equation

$$\int_\Omega (2\nabla v \cdot \nabla\phi \;+\; \phi\,\lambda_+\,\chi_{\{v>0\}} \;-\; \phi\,\lambda_-\,\chi_{\{v<0\}}) \;=\; 0 \;\text{ for every } \phi \in H_0^{1,2}(\Omega)$$

must coincide with $u$ : subtracting the weak equation for $u$ and inserting $\phi := v - u$ as test function we obtain that

$$\int_\Omega 2|\nabla(v-u)|^2 \leq$$

$$\int_\Omega (2\nabla(v-u)\cdot\nabla(v-u)+\lambda_+(\chi_{\{v>0\}}-\chi_{\{u>0\}})(v-u)-\lambda_-(\chi_{\{v<0\}}-\chi_{\{u<0\}})(v-u))$$

$= 0$ . Thus the weak solution is *unique* and it is therefore no restriction to confine our study to the minimizer $u$ .

In what follows, the term "solution" shall always denote a $H^{2,1}$-function solving the strong equation $\Delta v = \frac{\lambda_+}{2}\chi_{\{v>0\}} - \frac{\lambda_-}{2}\chi_{\{v<0\}}$ a.e. in a given open set.

A powerful tool is now a monotonicity formula introduced in [7] by the author for a class of semilinear free boundary problems. For the sake of completeness let us state the two-phase obstacle problem case here:

**Theorem 2.1 (the monotonicity formula)** *Suppose that* $B_\delta(x_0) \subset \Omega$ . *Then for all* $0 < \rho < \sigma < \delta$ *the function*

$$\Phi_{x_0}(r) := r^{-n-2}\int_{B_r(x_0)} \Big(|\nabla u|^2 + \lambda_+ \max(u,0) + \lambda_- \max(-u,0)\Big)$$

$$- 2\, r^{-n-3} \int_{\partial B_r(x_0)} u^2 \, d\mathcal{H}^{n-1} ,$$

*defined in* $(0,\delta)$ , *satisfies the monotonicity formula*

$$\Phi_{x_0}(\sigma) - \Phi_{x_0}(\rho) = \int_\rho^\sigma r^{-n-2}\int_{\partial B_r(x_0)} 2\left(\nabla u\cdot\nu - 2\,\frac{u}{r}\right)^2 d\mathcal{H}^{n-1}\, dr \geq 0 .$$

# 3 Pointwise regularity and non-degeneracy

By $L^p$-theory the solution $u \in C^{1,\alpha}_{\mathrm{loc}}(\Omega)$ for every $\alpha \in (0,1)$ . The set $R := \Omega\cap\{u=0\}\cap\{\nabla u \neq 0\}$ is therefore open relative to $\Omega\cap(\partial\{u>0\}\cup\partial\{u<0\})$ and the implicit function theorem implies that $R$ is a $C^{1,\alpha}$-surface for every $\alpha \in (0,1)$. The set of interest is therefore the set $S := \Omega\cap\{\nabla u = 0\}\cap(\partial\{u > 0\} \cup \partial\{u<0\})$ .

**Lemma 3.1** *Let $\alpha - 1 \in \mathbf{N}$, let $w \in H^{1,2}(B_1(0))$ be a harmonic function in $B_1(0)$ and assume that $D^j w(0) = 0$ for $0 \le j \le \alpha - 1$.*

$$\textit{Then } \int_{B_1(0)} |\nabla w|^2 \; - \; \alpha \int_{\partial B_1(0)} w^2 \, d\mathcal{H}^{n-1} \; \ge \; 0 \, ,$$

*and equality implies that $w$ is homogeneous of degree $\alpha$ in $B_1(0)$.*

The proof is based on the well-known fact that the mean frequency of a harmonic function is a non-decreasing function of the radius.

The following proposition gives an estimate on the growth of the solution near $S$ :

**Proposition 3.1** *There exists for each $\delta > 0$ a constant $C < \infty$ such that*

$$\int_{\partial B_r(x_0)} u^2 \, d\mathcal{H}^{n-1} \; \le \; C \, r^{n-1+4}$$

*for every $r \in (0, \delta)$ and every $x_0 \in S$ satisfying $B_{2\delta}(x_0) \Subset \Omega$.*
*Furthermore the estimate*

$$r^{1-n-4} \int_{\partial B_r(x_0)} u^2 \, d\mathcal{H}^{n-1}$$

$$\le \; \frac{1}{2} {r_0}^{-n-2} \int_{B_{r_0}(x_0)} \Big( |\nabla u|^2 \; + \; \lambda_+ \max(u, 0) \; + \; \lambda_- \max(-u, 0) \Big)$$

*holds for every $0 < r < r_0$ and $x_0 \in S$ satisfying $B_{r_0}(x_0) \subset \Omega$.*

**Remark 3.1** *Note that in the one-phase case $\lambda_- = 0$, $u_D \ge 0$ the first estimate of Proposition 3.1 can be proved via a Harnack inequality argument: introducing for $r > 0$ the scaled function $u_r(x) := \frac{u(x_0 + rx)}{r^2}$ and supposing that $u(x_0) = 0$ and $B_{r_0}(x_0) \subset\subset \Omega$ we obtain that $\Delta u_r = \frac{1}{2} \chi_{\{u_r > 0\}}$ in $B_1(0)$ for $r \in (0, r_0)$. Now the fact that $u \in H^{2,p}(B_{r_0}(x_0))$ allows us to apply Harnack's inequality Theorem 8.18 of [3] to deduce that $\sup_{B_1(0)} u_r \le C(n)$ and, in the original scaling, that $\sup_{B_r(x_0)} u \le C(n) \, r^2$.*

**Lemma 3.2 (non-degeneracy)** *For every $x_0 \in \overline{\{u>0\}} \cup \overline{\{u<0\}}$ and every $B_{2r}(x_0) \subset \Omega$ the estimate*

$$\sup_{\partial B_r(x_0)} |u| \geq \frac{1}{4n} \min(\lambda_+, \lambda_-)\, r^2 \quad \textit{holds.}$$

*Proof:* We observe that it is sufficient to prove the statement for every $x_0 \in \{u>0\}$ such that $B_{2r}(x_0) \subset \Omega$. Assuming that $\sup_{\partial B_r(x_0)} u \leq \frac{1}{4n}\lambda_+ r^2$, the comparison principle yields that $u(x) \leq v(x) := \frac{1}{4n}\lambda_+ |x-x_0|^2$ in $B_r(x_0)$. This, however, contradicts the assumption $u(x_0) > 0$.

# 4 A Hausdorff dimension estimate

From now on we assume that $\min(\lambda_+, \lambda_-) > 0$. The results of the previous section lead to the following consequences.

**Lemma 4.1** *Let $x_0 \in S$ and let $u_k(x) := \frac{u(x_0+\rho_k x)}{{\rho_k}^2}$ be a* blow-up sequence, *i.e. assume that $\rho_k \to 0$ as $k \to \infty$. Then $(u_k)_{k\in\mathbf{N}}$ is for each open $D \subset\subset \mathbf{R}^n$ and each $p \in (1,\infty)$ bounded in $H^{2,p}(D)$, and each limit $u_0$ with respect to a subsequence $k \to \infty$ is a nontrivial homogeneous solution of degree 2 in $\mathbf{R}^n$ and satisfies the following:*
*for each compact set $K \subset \mathbf{R}^n$ and each open set $U \supset K \cap S_0$ there exists $k_0 < \infty$ such that $S_k \cap K \subset U$ for $k \geq k_0$; here $S_0 := \{\nabla u_0 = 0\} \cap (\partial\{u_0 > 0\} \cup \partial\{u_0 < 0\})$ and $S_k := \{\nabla u_k = 0\} \cap (\partial\{u_k > 0\} \cup \partial\{u_k < 0\})$.*

Applying standard geometric measure theoretic tools we obtain the following theorem:

**Theorem 4.1** *The Hausdorff dimension of the set $S$ is less than or equal to $n-1$.*

**Corollary 4.1** *The Hausdorff dimension of $\partial\{u>0\} \cup \partial\{u<0\}$ is less than or equal to $n-1$.*

**Remark 4.1** *The procedure of Proposition 3.1 yields a new proof for the regularity of a minimizer $\tilde{u}$ of the functional* $v \mapsto \int_\Omega (|\nabla v|^2 + \lambda_+ \chi_{\{v>0\}} + \lambda_- \chi_{\{v<0\}})$ .

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