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Degenerate Elliptic Equation with Logistic Reaction*

Shingo Takeuchi (竹内 慎吾)†

Abstract

Degenerate elliptic equation $\lambda\Delta_p u + u^{q-1}(1-u^r) = 0$ with zero Dirichlet boundary condition, where λ is a positive parameter, $2 < p < q$ and $r > 0$, is studied in three aspects: existence of maximal solution, λ-dependence of maximal solution and multiplicity of solutions. We will show that there exists a positive number Λ such that if $\lambda > \Lambda$, then the problem has no solution; if $\lambda \leq \Lambda$, then it has a maximal solution, which possesses a flat core for sufficiently small $\lambda > 0$. It is also proved that $(\mathrm{P})_\lambda$ admits at least two solutions if $\lambda \in (0, \Lambda)$.

1 Introduction and Results

Let Ω be a connected, bounded open subset of $\mathbb{R}^N$, $N \geq 2$, with $C^{2,\alpha}$-boundary $\partial\Omega$ for some $\alpha \in (0,1)$. We consider the following degenerate elliptic equation:

$$(\mathrm{P})_{\lambda,\Omega}\begin{cases}\lambda\Delta_p u + f(u) = 0 & \text{in } \Omega,\\ u \geq 0,\ \not\equiv 0 & \text{in } \Omega,\\ u = 0 & \text{on } \partial\Omega,\end{cases}$$

where λ is a positive parameter and Δ_p is the p-Laplace operator defined by

$$\Delta_p u = \mathrm{div}(|\nabla u|^{p-2}\nabla u)$$

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†Department of Mathematics, Waseda University, 3-4-1 Ohkubo, Shinjuku-ku, Tokyo 169-8555, JAPAN.
E-mail address: 697m5041@mn.waseda.ac.jp

with $p > 2$ and f is given by

$$f(u) = u^{q-1}(1 - u^r)$$

with $q \geq 2$ and $r > 0$. We often write '$(\mathrm{P})_\lambda$' instead of '$(\mathrm{P})_{\lambda,\Omega}$'.

A function $u = u_\lambda \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ is called a *solution of* $(\mathrm{P})_\lambda$ if $u \geq 0$ a.e. in Ω, u does not vanish in a set of positive measure, and

$$-\lambda \int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi dx + \int_\Omega f(u) \varphi dx = 0 \tag{1.1}$$

for all $\varphi \in W_0^{1,p}(\Omega)$. A solution u of $(\mathrm{P})_\lambda$ is called a *maximal solution of* $(\mathrm{P})_\lambda$ if $u \geq v$ a.e. in Ω for all solutions v of $(\mathrm{P})_\lambda$. Obviously, a maximal solution is decided uniquely. If a function $u \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ satisfies $u \geq 0$ (resp. $u \leq 0$) on $\partial\Omega$ and

$$-\lambda \int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi dx + \int_\Omega f(u) \varphi dx \leq 0 \quad (\text{resp. } \geq 0)$$

for all $\varphi \in W_0^{1,p}(\Omega)$ satisfying $\varphi \geq 0$ a.e. in Ω, then it is called an *upper* (resp. a *lower*) *solution of* $(\mathrm{P})_\lambda$.

With respect to $(\mathrm{P})_\lambda$, there are a few works on the *equidiffusive case* $p = q$ as follows. Let λ_0 be the first eigenvalue of $-\Delta_p$ under zero Dirichlet boundary condition. In one-dimensional case $N = 1$, Guedda and Véron [7] have shown by phase plane analysis that if $\lambda < 1/\lambda_0$, then $(\mathrm{P})_\lambda$ has a unique solution u_λ, and that a set called *flat core of* u_λ:

$$\mathcal{O}_\lambda = \mathcal{O}_\lambda(u_\lambda) := \{x \in \Omega; u_\lambda(x) = 1\}$$

is nonempty for sufficiently small λ. Since the length of $\mathcal{O}_\lambda$ can be indicated explicitly, we can see that as $\lambda \to 0$, $\mathcal{O}_\lambda$ spreads out toward the whole of Ω with the growth as

$$\lim_{\lambda \to 0} \lambda^{-1/p} \operatorname{dist}(\mathcal{O}_\lambda, \partial\Omega) = C(f, p), \tag{1.2}$$

where $C(f,p) = \left(\frac{p-1}{p}\right)^{1/p} \int_0^1 (F(1) - F(s))^{-1/p} ds$ and $F(s) = \int_0^s f(t) dt$. In higher dimensional case $N \geq 2$, phase plane analysis is no longer useful and one has to approach by other methods. Constructing a suitable lower solution with use of the eigenfunction for λ_0, Kamin and Véron [9] have proved that the unique solution of $(\mathrm{P})_\lambda$ has a flat core for sufficiently small λ and extended the results of [7]. However, they have given only an estimate $\operatorname{dist}(\mathcal{O}_\lambda, \partial\Omega) \leq C\lambda^{1/p}$ as $\lambda \to 0$, where C is a constant independent of λ, without explicit information about C and any estimate of dist$(\mathcal{O}_\lambda, \partial\Omega)$ from below. In virtue of an exact estimate for $\mathcal{O}_\lambda$, García-Melián and Sabina de Lis [6] have utilized the solutions for $N = 1$, whose dependence on λ is understood well, to make upper- and lower solutions and concluded that (1.2) also holds true in the case $N \geq 2$.

The *subdiffusive case* $p > q$ can be also investigated in the same way as the equidiffusive case. One can observe that there exists a unique solution u_λ for every $\lambda > 0$ and that as the equidiffusive case, $\mathcal{O}_\lambda(u_\lambda)$ is nonempty for sufficiently small $\lambda > 0$ and it grows as (1.2). See the author and Yamada [18] for $N = 1$ and [6] with its Remarks 2.2 b for $N \geq 2$. For uniqueness, see also Diaz and Saa [4].

On the other hands, the structure of solution-set in the *superdiffusive case* $p < q$ is essentially different from those in the other cases. For $N = 1$, using time-map, the author and Yamada [18] have shown that $(\mathrm{P})_\lambda$ produces a spontaneous bifurcation for λ, that is, there exists $\Lambda > 0$ such that if $\lambda > \Lambda$, then $(\mathrm{P})_\lambda$ has no solution; if $\lambda = \Lambda$, then $(\mathrm{P})_\lambda$ has a unique solution; if $\lambda < \Lambda$, then $(\mathrm{P})_\lambda$ has exactly two distinct solutions $\overline{u}_\lambda$ and $\underline{u}_\lambda$ satisfying $\overline{u}_\lambda > \underline{u}_\lambda$ in Ω. It also follows from our analysis that as $\lambda \to 0$, $\mathcal{O}_\lambda(\overline{u}_\lambda)$ spreads out toward the whole of Ω with (1.2) and $\underline{u}_\lambda \to 0$ uniformly in Ω. For $N \geq 2$, Guo [8] has studied the case that there exists $\beta > 0$ such that $f(0) = f(\beta) = 0$, $(\beta - x)f(x) > 0$ in $(0,\beta) \cup (\beta, +\infty)$, $\lim_{s\to 0} f(s)/s^{p-1} = 0$ and $(f(s)/s^{p-1})'' < 0$ in $(0,\beta)$ (the condition '$f''(x) < 0$' in [8, Theorem 3.3] is a misprint and should be replaced by '$(f(x)/x^{p-1})'' < 0$'), and has found two distinct solutions for sufficiently small $\lambda > 0$. However, this is a particular case in our problem, and no information about the shape of solutions and about the λ-range of multiple existence of solutions, is given.

In the present paper, we will discuss $(\mathrm{P})_\lambda$ in the case $2 < p < q$, $N \geq 2$, and study $(\mathrm{P})_\lambda$ in three aspects: (a) existence of solution, especially maximal solution; (b) λ-dependence of maximal solution; and (c) multiplicity of solutions. As for (a), we can prove the following theorem by so-called barrier method:

Theorem 1.1. *Let $2 < p < q$ and $r > 0$. Then there exists a positive number Λ such that*
(i) *if $\lambda > \Lambda$, then $(\mathrm{P})_\lambda$ has no solution;*
(ii) *if $\lambda \leq \Lambda$, then $(\mathrm{P})_\lambda$ has a maximal solution $\overline{u}_\lambda$;*
(iii) *if $\lambda_1 < \lambda_2 \leq \Lambda$, then $\overline{u}_{\lambda_2} \leq \overline{u}_{\lambda_1}$;*
(iv) *the mapping $\lambda \mapsto \overline{u}_\lambda$ is left-continuous on $(0,\Lambda]$ in $C^{1,\beta'}(\overline{\Omega})$ for any $\beta' \in (0,\beta)$, where β is the constant appearing in Proposition 2.1.*

Remark 1.1. Theorem 1.1 (i) has been obtained by Véron [19, Theorem 3] for the p-Laplace operator on a compact Riemannian manifold without boundary.

We will state our result on (b). The proof essentially consists in constructing suitable upper- and lower solutions by the idea of García-Melián and Sabina de Lis [6] and the one-dimensional result in [18].

Theorem 1.2. *Let $2 < p < q$ and $r > 0$. There exists a positive number $\lambda^* \in (0, \Lambda]$ such that*
(i) *if $\lambda \leq \lambda^*$, then $\mathcal{O}_\lambda = \mathcal{O}_\lambda(\overline{u}_\lambda)$ is nonempty;*
(ii) *if $\lambda_1 < \lambda_2 \leq \lambda^*$, then $\mathcal{O}_{\lambda_2} \subset \mathcal{O}_{\lambda_1}$;*
(iii) *for sufficiently small $\varepsilon > 0$, there exists $\lambda \leq \lambda^*$ such that $\Omega \setminus \Omega_\varepsilon \subset \mathcal{O}_\lambda$.*
Furthermore, $\mathcal{O}_\lambda$ satisfies (1.2) as $\lambda \to 0$.

Remark 1.2. From the last assertion of Theorem 1.2, we can see that the growth order of maximal solution of $(\mathrm{P})_\lambda$ when $\lambda \to 0$ is the same as that of case $p \geq q$.

To mention (c), we define the functional Φ on $W_0^{1,p}(\Omega)$ corresponding with $(\mathrm{P})_\lambda$:

$$\Phi(u) = \frac{\lambda}{p}\|\nabla u\|_p^p - \int_\Omega \overline{F}(u)dx, \tag{1.3}$$

where $\overline{F}(u) = \int_0^u \overline{f}(s)ds$ and $\overline{f}(s) := f(s)$ in $[0, \xi]$, $:= 0$ in $(-\infty, 0)$ and $:= f(\xi)$ in $(\xi, +\infty)$ for any $\xi > 1$ fixed. Here, $\|\cdot\|_p$ denotes L^p-norm. Solutions of $(\mathrm{P})_\lambda$ satisfy $0 < u \leq 1$ in Ω (see Proposition 2.1); so that they coincide with critical points of Φ. We can show that Φ satisfies the Palais-Smale condition (see the proof of [16, Theorem 1.3]). Our strategy is to apply an extended Mountain pass theorem by Pucci and Serrin, which asserts that, if Φ has a pair of local minima, then Φ possesses a third critical point (see Pucci and Serrin [13, Theorem 4]). We will prove that the trivial solution $u = 0$ is a local minimizer of Φ in $W_0^{1,p}$ for every $\lambda > 0$, and that if the maximal solution $\overline{u}_\lambda$ is isolated, then $\overline{u}_\lambda$ is also a local minimizer of Φ in $W_0^{1,p}$ for $\lambda \in (0, \Lambda)$. Finally we can conclude the following theorem:

Theorem 1.3. *Let $2 < p < q$ and $r > 0$. Then, for any $\lambda \in (0, \Lambda)$, $(\mathrm{P})_\lambda$ has a solution u_λ satisfying $u_\lambda \leq \overline{u}_\lambda$, $\not\equiv \overline{u}_\lambda$.*

Remark 1.3. For the linear diffusion case $2 = p < q$, Rabinowitz [14] has studied $(\mathrm{P})_\lambda$ by combining critical point theory and the Leray-Schauder degree theory, and obtained Theorem 1.3. Especially, when Ω is a ball, Ouyang and Shi [12] have obtained precise global bifurcation diagram and concluded that there exist *exactly* two solutions for a certain range of λ by using a bifurcation theorem of Crandall and Rabinowitz.

2 Proofs of Theorems 1.1 and 1.2

In this section, we will prove Theorems 1.1 and 1.2. The following proposition is fundamental in this paper (for the proof, see [16, Proposition 2.1]).

Proposition 2.1. *Let u be a solution of* $(\mathrm{P})_\lambda$. *Then* $u \in C_0^{1,\beta}(\overline{\Omega}) \cap C^{2,\alpha}(\overline{\Omega_\varepsilon})$, *where* $\Omega_\varepsilon := \{x \in \Omega; \mathrm{dist}(x, \partial\Omega) < \varepsilon\}$, *for some* $\beta \in (0,1)$ *and sufficiently small* $\varepsilon > 0$. *Furthermore,* $0 < u \leq 1$ *in* Ω *and* $\partial u/\partial \boldsymbol{n} < 0$ *on* $\partial\Omega$, *where* $\boldsymbol{n}$ *denotes an outer normal at* $\partial\Omega$.

Lemma 2.1. *For sufficiently small* $\lambda > 0$, *there exists a maximal solution* $\overline{u}_\lambda$ *such that* $\mathcal{O}_\lambda$ *is nonempty and*

$$\limsup_{\lambda \to 0} \lambda^{-1/p}\, \mathrm{dist}(\mathcal{O}_\lambda, \partial\Omega) \leq C(f,p). \tag{2.1}$$

Proof. Take $R > 0$ and $x_0 \in \Omega$ satisfying $B_R(x_0) \subset \Omega$, where $B_R(x_0)$ is the ball with radius R and center at x_0. To obtain a lower solution of $(\mathrm{P})_{\lambda,\Omega}$, we will construct a lower solution v_{R,x_0} of $(\mathrm{P})_{\lambda, B_R(x_0)}$. It suffices to find a radially symmetric one, i.e., $v(\rho) = v_{R,x_0}(x)$ satisfying

$$\begin{cases} \lambda(\rho^{N-1}|v_\rho|^{p-2}v_\rho)_\rho + \rho^{N-1} f(v) \geq 0 & \text{in } (0,R), \\ v_\rho(0) = v(R) = 0, \end{cases} \tag{2.2}$$

where $\rho = |x - x_0|$. By a change of variable $\xi = g(\rho)$ such that

$$\xi = g(\rho) = \begin{cases} \frac{R^{1-\theta} - \rho^{1-\theta}}{1-\theta} & \text{if } \theta \neq 1, \\ \log \frac{R}{\rho} & \text{if } \theta = 1, \end{cases}$$

where $\theta := (N-1)/(p-1)$, (2.2) can be rewritten as follows:

$$\begin{cases} \lambda(|w_\xi|^{p-2}w_\xi)_\xi + g^{-1}(\xi)^{p\theta} f(w) \geq 0 & \text{in } (0,T), \\ w(0) = w_\xi(T) = 0, \end{cases} \tag{2.3}$$

where $w(\xi) = v(g^{-1}(\xi))$ and $T = +\infty$ if $\theta \geq 1$, $= \frac{R^{1-\theta}}{1-\theta}$ if $\theta < 1$. In order to find a function w satisfying (2.3), we take any $b \in (0,T)$ and consider the following auxiliary boundary value problem:

$$\begin{cases} \lambda(|\phi_\xi|^{p-2}\phi_\xi)_\xi + g^{-1}(b)^{p\theta} f(\phi) = 0 & \text{in } (0,b), \\ \phi(0) = \phi(b) = 0. \end{cases} \tag{2.4}$$

A change of scale $\xi = b\eta$ gives

$$\begin{cases} \lambda(|\psi_\eta|^{p-2}\psi_\eta)_\eta + \{bg^{-1}(b)^\theta\}^p f(\psi) = 0 & \text{in } (0,1), \\ \psi(0) = \psi(1) = 0, \end{cases} \tag{2.5}$$

where $\psi(\eta) = \phi(b\eta)$. Take λ sufficiently small as

$$\lambda \leq \left\{ \frac{bg^{-1}(b)^\theta}{2C(f,p)} \right\}^p .$$

Then, we have already known from [18, Theorem 3.3] that (2.5) has a solution ψ such that $\psi(x) = 1$ in $[C_{\lambda,b}/b, 1 - C_{\lambda,b}/b]$ and $0 \leq \psi(x) < 1$ otherwise, where

$$C_{\lambda,b} = \frac{C(f,p)}{g^{-1}(b)^\theta}\lambda^{1/p} \ (\leq b/2). \tag{2.6}$$

Thus, (2.4) also has a solution ϕ such that $\phi(x) = 1$ in $[C_{\lambda,b}, b - C_{\lambda,b}]$ and $0 \leq \phi(x) < 1$ otherwise. Using ϕ, we construct a function w satisfying (2.3) as follows: $w = \phi$ in $[0, C_{\lambda,b})$, $= 1$ in $[C_{\lambda,b}, T)$. Indeed, since g^{-1} is monotone decreasing,

$$\lambda(|w_\xi|^{p-2}w_\xi)_\xi + g^{-1}(\xi)^{p\theta} f(w) = \{g^{-1}(\xi)^{p\theta} - g^{-1}(b)^{p\theta}\} f(\phi) \geq 0 \quad \text{in } [0, C_{\lambda,b})$$

and the boundary conditions are obviously satisfied. Therefore $v(\rho) = w(g(\rho))$ satisfies (2.2), hence the function

$$v_{R,x_0}(x) = \begin{cases} 1 & \text{if } 0 \leq |x - x_0| \leq g^{-1}(C_{\lambda,b}), \\ \phi(g(|x - x_0|)) & \text{if } g^{-1}(C_{\lambda,b}) < |x - x_0| \leq R \end{cases} \tag{2.7}$$

is a lower solution of $(\mathrm{P})_{\lambda, B_R(x_0)}$.

Now, we define $\tilde{v}_{R,x_0}(x) = v_{R,x_0}(x)$ in $B_R(x_0)$, $= 0$ in $\Omega \setminus B_R(x_0)$. Then, one can observe that $\tilde{v}$ is a lower solution of $(\mathrm{P})_{\lambda,\Omega}$. Taking the function $u \equiv 1$ as an upper solution, we obtain a maximal solution $\overline{u}_\lambda$ of $(\mathrm{P})_\lambda$ such that $\tilde{v}_{R,x_0}(x) \leq \overline{u}_\lambda(x) \leq 1$ for all $x \in \Omega$ by the monograph of Diaz [3, Theorem 4.14]. In particular, it follows from (2.7) that $\overline{u}_\lambda(x) = 1$ in $B_{g^{-1}(C_{\lambda,b})}(x_0)$. By the arbitrariness of x_0 satisfying $B_R(x_0) \subset \Omega$ and the uniqueness of maximal solution, it holds that $\overline{u}_\lambda(x) = 1$ in $\Omega \setminus \Omega_{R'}$, where $R' = R'(\lambda, b) = R - g^{-1}(C_{\lambda,b})$. Thus $\operatorname{dist}(\mathcal{O}_\lambda, \partial\Omega) \leq R'$. It follows from (2.6) and l'Hospital's theorem that $R'(\lambda, b) = R^\theta C_{\lambda,b} + o(\lambda^{1/p})$ as $\lambda \to 0$; so we obtain

$$\limsup_{\lambda \to 0} \lambda^{-1/p} \operatorname{dist}(\mathcal{O}_\lambda, \partial\Omega) \leq \lim_{\lambda \to 0} \lambda^{-1/p} R'(\lambda, b) = \left\{ \frac{R}{g^{-1}(b)} \right\}^\theta C(f,p).$$

Passing to the limit as $b \to 0$, we conclude (2.1). □

Proof of Theorem 1.1. Define

$$\Lambda = \sup\{\lambda > 0; (\mathrm{P})_\lambda \text{ has a solution.}\}.$$

Since Lemma 2.1 implies $\Lambda > 0$, we will show $\Lambda < +\infty$ to see (i). Suppose that there exists a sequence $\{\lambda_m\}_{m=1}^{\infty}$ such that $\lambda_m \to \infty$ as $m \to \infty$ and $(\mathrm{P})_{\lambda_m}$ has a solution $u_m = u_{\lambda_m}$. Putting $\lambda = \lambda_m$ and $u = \varphi = u_m$ in (1.1), we have $\lambda_m \|\nabla u_m\|_p^p = \int_\Omega u_m f(u_m)dx$. Since $sf(s) \le s^p$ for $s \in [0,1]$ if $p < q$, it follows that $\lambda_m \|\nabla u_m\|_p^p \le \|u_m\|_p^p$. Combining this inequality and the Poincaré inequality, we obtain $C\lambda_m \|u_m\|_p^p \le \|u_m\|_p^p$, where C is a positive constant. Since $\|u_m\|_p^p > 0$, the inequality is a contradiction for sufficiently large m.

Next, we will prove (ii) and (iii). Consider the case $\lambda < \Lambda$. From the definition of Λ, for $\lambda < \Lambda$ there exists $\mu \in (\lambda, \Lambda]$ such that $(\mathrm{P})_\mu$ has a solution u_μ. By an easy calculation, u_μ is a lower solution of $(\mathrm{P})_\lambda$. Since $u \equiv 1$ is an upper solution of $(\mathrm{P})_\lambda$, it follows from [3, Theorem 4.14] that $(\mathrm{P})_\lambda$ admits a maximal solution $\overline{u}_\lambda$ satisfying $\overline{u}_\lambda \ge u_\mu$. (Note that the same arguments give the proof of (iii).) The case $\lambda = \Lambda$ is treated as follows. Let $\{\lambda_m\}_{m=1}^{\infty}$ be a positive increasing sequence satisfying $0 < \lambda_m < \Lambda$ and $\lambda_m \to \Lambda$ as $m \to \infty$, and let $\overline{u}_m$ be the maximal solution of $(\mathrm{P})_{\lambda_m}$. From [11, Theorem 1], we know that $\{\overline{u}_m\}$ is uniformly bounded in $C^{1,\beta}(\overline{\Omega})$ for some $\beta \in (0,1)$. Thus, the Ascoli-Arzelà theorem assures that there exist u_Λ and a subsequence of $\{\overline{u}_m\}$, still denoted by $\{\overline{u}_m\}$, such that $\overline{u}_m \to u_\Lambda$ in $C^{1,\beta'}(\overline{\Omega})$ for each $\beta' \in (0, \beta)$. It is easy to see that $u_\Lambda \ge 0$ in Ω and that u_Λ satisfies (1.1). To observe that $u_\Lambda \not\equiv 0$, we assume $u_\Lambda \equiv 0$. Since $\{\overline{u}_m\}$ converges to 0 uniformly in Ω as $m \to \infty$, it follows from $p < q$ that for sufficiently large m

$$C\|\overline{u}_m\|_p^p \le \|\nabla \overline{u}_m\|_p^p = \frac{1}{\lambda_m}\int_\Omega \overline{u}_m f(\overline{u}_m)dx \le \frac{C}{2}\|\overline{u}_m\|_p^p;$$

which contradicts to $\|\overline{u}_m\|_p^p > 0$. Therefore, u_Λ is a solution of $(\mathrm{P})_\Lambda$. We have to show the maximality of u_Λ. Suppose that u_Λ is not maximal. Then, $(\mathrm{P})_\Lambda$ has a maximal solution $v_\Lambda \ge u_\Lambda$, $(\not\equiv u_\Lambda)$ and there exists $x_0 \in \Omega$ such that $u_\Lambda(x_0) < v_\Lambda(x_0)$. By (iii), since $\overline{u}_m$ decreases toward u_Λ as $m \to \infty$, it holds that $u_\Lambda(x_0) \le \overline{u}_m(x_0) < v_\Lambda(x_0)$ for sufficiently large m. On the other hand, it follows from (iii) and the fact $\lambda_m < \Lambda$ that $v_\Lambda(x_0) \le \overline{u}_m(x_0)$. These inequalities contradict each other; so u_Λ is maximal, which can be written as $\overline{u}_\Lambda$. Finally, one can observe (iv) in the similar way as the proof for maximality of u_Λ. □

Proof of Theorem 1.2. The existence of λ^* satisfying (i) is directly induced from Lemma 2.1 and (ii) follows from (iii) of Theorem 1.1. From the proof of Lemma 2.1, (iii) is obvious for sufficiently small $\varepsilon > 0$ such that $\Omega \setminus \Omega_\varepsilon \ne \emptyset$. It remains to show (1.2), i.e., growth-order of $\mathcal{O}_\lambda$ as $\lambda \to 0$ near $\partial\Omega$.

Take any $x_0 \in \partial\Omega$. Let $a > 0$ (resp. $R > 0$) be sufficiently small (resp. large) such that the annulus $A := \{x \in \mathbb{R}^N; a < |x - y_0| < R\}$, where $y_0 := x_0 + a\boldsymbol{n}$ and $\boldsymbol{n}$ denotes the outer normal at x_0, satisfies $\Omega \subset A$. Define $\tilde{u}_\lambda$ by $\tilde{u}_\lambda := \overline{u}_\lambda$ in Ω, $= 0$ in $A \setminus \Omega$. Then $\tilde{u}_\lambda$ is a lower solution of $(\mathrm{P})_{\lambda,A}$; so a maximal solution $\overline{v}_{\lambda,A}$ of $(\mathrm{P})_{\lambda,A}$ exists and particularly

$$\overline{u}_\lambda(x) \leq \overline{v}_{\lambda,A}(x) \quad \text{in } \Omega. \tag{2.8}$$

From the maximality, $\overline{v}_{\lambda,A}$ is radially symmetric on A; hence $v(\rho) = \overline{v}_{\lambda,A}(x)$ satisfies

$$\begin{cases} \lambda(\rho^{N-1}|v_\rho|^{p-2}v_\rho)_\rho + \rho^{N-1}f(v) = 0 & \text{in } (a, R), \\ v(a) = v(R) = 0, \end{cases} \tag{2.9}$$

where $\rho = |x - y_0|$. As in the proof of Lemma 2.1, we introduce a change of variable

$$\xi = h(\rho) = \begin{cases} \frac{\rho^{1-\theta} - a^{1-\theta}}{1-\theta} & \text{if } \theta \neq 1, \\ \log \frac{\rho}{a} & \text{if } \theta = 1, \end{cases}$$

where $\theta := (N-1)/(p-1)$; then (2.9) can be rewritten as

$$\begin{cases} \lambda(|w_\xi|^{p-2}w_\xi)_\xi + h^{-1}(\xi)^{p\theta}f(w) = 0 & \text{in } (0, T), \\ w(0) = w(T) = 0, \end{cases}$$

where $w(\xi) = v(h^{-1}(\xi))$ and $T = h(R)$. It is easy to see that w is a lower solution of

$$\begin{cases} \lambda(|\phi_\xi|^{p-2}\phi_\xi)_\xi + h^{-1}(b)^{p\theta}f(\phi) = 0 & \text{in } (0, b), \\ \phi(0) = 0, \ \phi(b) = 1 \end{cases} \tag{2.10}$$

for any $b \in (0, T)$. Thus, (2.10) has a maximal solution $\overline{\phi}$ such that

$$w(\xi) \leq \overline{\phi}(\xi) \quad \text{in } (0, b). \tag{2.11}$$

In fact, we know from [18, Theorem 3.3] that $0 < \overline{\phi}(\xi) < 1$ in $(0, D_{\lambda,b})$, $\overline{\phi}(\xi) = 1$ otherwise, where $D_{\lambda,b} = C(f,p)\lambda^{1/p}/h^{-1}(b)^\theta$ ($\leq b/2$). Hence, it follows from (2.8) and (2.11) that $\overline{u}_\lambda(x) \leq \phi(h(|x - y_0|)) < 1$ if $x \in \Omega$ and $a < |x - y_0| < h^{-1}(D_{\lambda,b})$. This means that $\mathrm{dist}(x_0, \mathcal{O}_\lambda) \geq h^{-1}(D_{\lambda,b}) - a$ for each $x_0 \in \partial\Omega$. Making $a > 0$ (resp. $R > 0$) sufficiently small (resp. large), one can get an uniform estimate $\mathrm{dist}(\mathcal{O}_\lambda, \partial\Omega) \geq h^{-1}(D_{\lambda,b}) - a$. Since $h^{-1}(D_{\lambda,b}) - a = a^\theta D_{\lambda,b} + o(\lambda^{1/p})$ as $\lambda \to 0$, it is possible to obtain that

$$\liminf_{\lambda \to 0} \lambda^{-1/p}\,\mathrm{dist}(\mathcal{O}_\lambda, \partial\Omega) \geq \left\{\frac{a}{h^{-1}(b)}\right\}^\theta C(f,p).$$

Passing to the limit $b \to 0$, we have

$$\liminf_{\lambda \to 0} \lambda^{-1/p} \operatorname{dist}(\mathcal{O}_\lambda, \partial\Omega) \geq C(f,p); \tag{2.12}$$

so combining (2.12) and (2.1) of Lemma 2.1, we conclude (1.2). □

Remark 2.1. From (2.12) and more delicate analyses of (2.1), we can see

$$\lim_{\lambda \to 0} \lambda^{-1/p} \sup_{x \in \partial\Omega} \operatorname{dist}(x, \mathcal{O}_\lambda) = C(f,p),$$

which implies that $\mathcal{O}_\lambda$ uniformly spreads out toward the whole of Ω as the order of $\lambda^{1/p}$.

3 Proof of Theorem 1.3

In this section, we will show Theorem 1.3. Let us prepare some lemmas.

Lemma 3.1. *For every $\lambda > 0$, the trivial solution $u = 0$ is a local minimizer of Φ in $W_0^{1,p}$.*

Proof. Since $p < q$, for any $\delta > 0$ there exists $C_\delta > 0$ such that $\overline{f}(s) \leq \delta s^{p-1} + C_\delta s^{q^*-1}$, where q^* is any number satisfying $p < q^* < p^*$ and $p^* := Np/(N-p)$ if $p < N$, $:= +\infty$ if $p \geq N$. Then $\overline{F}(u) \leq \delta u^p/p + C_\delta u^{q^*}/q^*$. Thus, the Sobolev inequality assures

$$\begin{aligned}\Phi(u) &\geq \frac{\lambda}{p}\|\nabla u\|_p^p - \frac{\delta}{p}\|u\|_p^p - \frac{C_\delta}{q^*}\|u\|_{q^*}^{q^*} \\ &\geq \left(\frac{\lambda - C_1\delta}{p} - \frac{C_2 C_\delta}{q^*}\|\nabla u\|_p^{q^*-p}\right)\|\nabla u\|_p^p,\end{aligned}$$

where C_1, C_2 are positive constants and $\delta \in (0, \lambda/C_1)$. Therefore, we see that there exists a positive number ρ such that $\Phi(u) \geq 0 = \Phi(0)$ if $\|\nabla u\|_p \leq \rho$. □

Fix $\lambda \in (0, \Lambda)$ and let λ_i, ε_i $(i = 1,2)$ be numbers satisfying that $0 < \lambda_2 < \lambda < \lambda_1 \leq \Lambda$ and $(\lambda/\lambda_1)^{1/(q-p)} < \varepsilon_1 < 1 < \varepsilon_2 < \min\{\xi, (\lambda/\lambda_2)^{1/(q-p)}\}$, where ξ is the number appearing in the definition of $\overline{f}$. Then we can see that $u_1 := \varepsilon_1 \overline{u}_{\lambda_1}$ is a lower solution and $u_2 := \varepsilon_2 \overline{u}_{\lambda_2}$ is an upper solution of $(\mathrm{P})_\lambda$, respectively, where $\overline{u}_{\lambda_i}$ $(i = 1,2)$ is the maximal solution of $(\mathrm{P})_{\lambda_i}$. Note that $\overline{u}_\lambda$ is an interior point of

$$\mathcal{A} := \{u \in C_0^1(\overline{\Omega}); u_1 \leq u \leq u_2 \text{ in } \Omega\}, \tag{3.1}$$

with respect to C^1-topology by Proposition 2.1, and that $f(u) = \overline{f}(u)$ for all $u \in \mathcal{A}$.

Lemma 3.2. *Let $\lambda \in (0, \Lambda)$ and assume that $\overline{u}_\lambda$ is a unique solution of* $(\mathrm{P})_\lambda$ *in $\mathcal{A}$. Then $\overline{u}_\lambda$ is a local minimizer of Φ in C_0^1.*

Proof. Let $\tilde{f}$ be a truncated function of $\overline{f}$ defined by

$$\tilde{f}(x,s) := \begin{cases} \overline{f}(u_1(x)) & \text{if } s < u_1(x), \\ \overline{f}(s) & \text{if } u_1(x) \le s \le u_2(x), \\ \overline{f}(u_2(x)) & \text{if } s > u_2(x) \end{cases}$$

and set $\tilde{F}(x,u) := \int_0^u \tilde{f}(x,s)ds$. Using $\tilde{F}$, we consider the following auxiliary functional $\tilde{\Phi}$ associated with Φ:

$$\tilde{\Phi}(u) = \frac{\lambda}{p}\|\nabla u\|_p^p - \int_\Omega \tilde{F}(x,u)dx.$$

It follows from the direct method that $\tilde{\Phi}$ has a global minimizer $u_0 \in W_0^{1,p}$. Therefore u_0 satisfies

$$\lambda \Delta_p u_0 + \tilde{f}(x, u_0) = 0 \quad \text{in } \Omega \tag{3.2}$$

and we see $u_0 \in C_0^1(\overline{\Omega})$ by Lieberman's regularity result [11, Theorem 1]. Moreover, since u_1 and u_2 are a lower and an upper solution of $(\mathrm{P})_\lambda$, respectively, and u_0 is a solution of (3.2), we can prove $u_1 \le u_0 \le u_2$ in Ω (for details, see the proof of [17, Lemma 2.2]). Therefore, $u_0 \in \mathcal{A}$ and (3.2) becomes $\lambda\Delta_p u_0 + \overline{f}(u_0) = 0$ in Ω; consequently u_0 is a solution of $(\mathrm{P})_\lambda$, which belongs to $\mathcal{A}$. By the assumption, $u_0 = \overline{u}_\lambda$, hence $\overline{u}_\lambda$ is a global minimizer of $\tilde{\Phi}$ in $W_0^{1,p}$.

Now, if $\varepsilon > 0$ is sufficiently small, then any $u \in C_0^1(\overline{\Omega})$ with $\|u - \overline{u}_\lambda\|_{C^1} < \varepsilon$ satisfies $u \in \mathcal{A}$ because $\overline{u}_\lambda$ is an interior point of $\mathcal{A}$. Furthermore, for any $u \in \mathcal{A}$

$$\begin{aligned} \Phi(u) - \tilde{\Phi}(u) &= \int_\Omega \int_0^{u(x)} (\overline{f}(s) - \tilde{f}(x,s))dsdx \\ &= \int_\Omega \int_0^{u_1(x)} (\overline{f}(s) - \overline{f}(u_1(x)))dsdx \end{aligned}$$

is a constant independent of u. Since $\overline{u}_\lambda$ is a global minimizer of $\tilde{\Phi}$, it consequently becomes a local minimizer of Φ in C_0^1. □

Remark 3.1. The proof of Lemma 3.2 is essentially due to Brézis and Nirenberg [2].

Lemma 3.3. *Let $\lambda \in (0, \Lambda)$ and assume that $\overline{u}_\lambda$ is a unique solution of* $(\mathrm{P})_\lambda$ *in $\mathcal{A}$. Then $\overline{u}_\lambda$ is a local minimizer of Φ in $W_0^{1,p}$.*

Proof. Suppose that for any neighborhood O of $\overline{u}_\lambda$ in $W_0^{1,p}$, there exists $v \in O$ such that $\Phi(v) < \Phi(\overline{u}_\lambda)$. Then, for sufficiently small $\varepsilon > 0$ there exists $v_\varepsilon \in B_\varepsilon$ such that $\Phi(v_\varepsilon) < \Phi(\overline{u}_\lambda)$, where $B_\varepsilon := \{u \in W_0^{1,p}(\Omega); \|u - \overline{u}_\lambda\|_2 \le \varepsilon\}$, because Sobolev's inequality allows us to take a neighborhood $O \subset B_\varepsilon$ of $\overline{u}_\lambda$ in $W_0^{1,p}$. Moreover, we may assume that v_ε is a global minimizer of Φ in B_ε without loss of generality.

If $\|v_\varepsilon - \overline{u}_\lambda\|_2 < \varepsilon$, then v_ε becomes a local minimizer of Φ in $W_0^{1,p}$, and hence v_ε is a solution of $(\mathrm{P})_\lambda$ and $0 < v_\varepsilon \le 1$. We next consider the case $\|v_\varepsilon - \overline{u}_\lambda\|_2 = \varepsilon$. Then there exists Lagrange's multiplier $\mu_\varepsilon \le 0$ (for the non-positivity, see the proof of [17, Lemma 2.3]) such that

$$\lambda \int_\Omega |\nabla v_\varepsilon|^{p-2}\nabla v_\varepsilon \cdot \nabla\zeta dx - \int_\Omega \overline{f}(v_\varepsilon)\zeta dx = \mu_\varepsilon \int_\Omega (v_\varepsilon - \overline{u}_\lambda)\zeta dx$$

for all $\zeta \in W_0^{1,p}$, i.e., $\lambda\Delta_p v_\varepsilon + g(\mu_\varepsilon, x, v_\varepsilon) = 0$, where $g(a,x,s) := \overline{f}(s) + a(s - \overline{u}_\lambda(x))$. Noting $\overline{u}_\lambda \le 1$, we can observe that $g(a,x,s) \ge 0$ in $\{a \le 0\} \times \Omega \times \{s \le 0\}$ and $g(a,x,s) \le 0$ in $\{a \le 0\} \times \Omega \times \{s \ge 1\}$. These facts assure that $0 \le v_\varepsilon \le 1$. Therefore, in any case, Lieberman's regularity result [11, Theorem 1] yields $\|v_\varepsilon\|_{C^{1,\beta}} \le C$ for some constants $C > 0$ and $\beta \in (0,1)$ independent of ε. Thus, the Ascoli-Arzelà theorem allows us to take a subsequence $\{v_{\varepsilon'}\}$ of $\{v_\varepsilon\}$ satisfying $v_{\varepsilon'} \to \overline{u}_\lambda$ in C^1 (here, we have used $v_{\varepsilon'} \in B_{\varepsilon'}$). This result, together with $\Phi(v_{\varepsilon'}) < \Phi(\overline{u}_\lambda)$, contradicts Lemma 3.2. □

Remark 3.2. Brézis and Nirenberg [2] have shown that for a certain functional corresponding to semilinear elliptic equations, its local minimizer in C^1 becomes a local minimizer in $H^1 = W^{1,2}$. Lemma 3.3 is a partial extension of [2] to $W^{1,p}$ versus C^1.

Proof of Theorem 1.3. As mentioned in Section 1, Φ satisfies the Palais-Smale condition. Let $\mathcal{A}$ be the set defined by (3.1). If there exists a solution distinct from $\overline{u}_\lambda$ in $\mathcal{A}$, then we have nothing to prove. Thus we may assume that there exists no solution in $\mathcal{A}$ except for $\overline{u}_\lambda$. Then, from Lemmas 3.1 and 3.3, we have obtained two local minimizers 0 and $\overline{u}_\lambda$ of Φ in $W_0^{1,p}$. Therefore, it follows from an extended Mountain pass theorem by Pucci and Serrin [13, Theorem 4] that there exists a third critical point of Φ, which is a solution of $(\mathrm{P})_\lambda$ distinct from 0 and $\overline{u}_\lambda$. □

Remark 3.3. With respect to multiplicity results for the p-Laplace operator, we have the results of Ambrosetti, Garcia Azorero and Peral [1], Drábek and Pohozaev [5], and [16]. Theorem 1.3 is a maximal extension of [16, Theorem 1.3].

4 Remarks and Open Problems

In this section, we give some remarks and open problems with respect to $(\mathrm{P})_\lambda$.

Shape of Flat Core. The relation between the flat core and Ω is unknown. We give an interesting question; when Ω is convex (resp. star-shaped), does the flat core of maximal solution u also become convex (resp. star-shaped) ? more generally, also the level set $\{x \in \Omega; u(x) \geq c\}$ for any number $c \in [0,1]$? Such problems have been considered for a class of semilinear equations (cf. Kawohl [10]).

Radially Symmetric Case. As we have pointed out in Remark 1.3, Ouyang and Shi [12] have considered the radially symmetric case when $2 = p < q$ and obtained the exact multiplicity result. In their studies, it is important to analyse the corresponding linearized equation at turning points in the bifurcation diagram. Since it seems that the idea of linearization for the p-Laplace operator has not established, we can not trace their proofs.

However, when Ω is the unit ball in $\mathbb{R}^N$, we can obtain an information about the shape of maximal solution on the boundary of its flat core. Then, the maximal solution (the unique solution if $p \geq q$) u becomes radially symmetric one; so that u satisfies

$$\begin{cases} \lambda(\rho^{N-1}|u_\rho|^{p-2}u_\rho)_\rho + \rho^{N-1}f(u) = 0 & \text{in } (0,1), \\ u_\rho(0) = 0, \ u(1) = 0. \end{cases}$$

For sufficiently small $\lambda > 0$, u has a flat core: there exists $\rho_0 \in (0,1)$ such that $u(\rho) = 1$ in $[0, \rho_0]$ and $u(\rho) < 1$, $u_\rho(\rho) < 0$ in $(\rho_0, 1]$. We can show the following proposition (when $N = 1$, the same result has been shown in [15, Lemma 2.1] by time-map method):

Proposition 4.1. *The maximal solution u satisfies that for any $\varepsilon \in (0,r)$, there exists $\delta > 0$ such that if $\rho_0 < \rho < \rho_0 + \delta$, then*

$$C_1(\rho - \rho_0)^{\frac{p}{p-2}} \leq u(\rho_0) - u(\rho) \leq C_2(\rho - \rho_0)^{\frac{p}{p-2}},$$

where

$$C_1 = \left(\frac{p-2}{p}\right)^{\frac{p}{p-2}} \left\{\frac{r - \varepsilon}{2\lambda C(p, N, \rho_0)}\right\}^{\frac{1}{p-2}},$$

$$C_2 = \left(\frac{p-2}{p}\right)^{\frac{p}{p-2}} \left\{\frac{p(r + \varepsilon)}{2\lambda(p-1)}\right\}^{\frac{1}{p-2}},$$

$$C(p, N, \rho_0) = \frac{p-1}{p} + (N-1)\frac{1-\rho_0}{\rho_0}.$$

Parabolic Problem. The solutions of $(\mathrm{P})_\lambda$ are regarded as positive stationary solutions of the following degenerate parabolic equation:

$$\begin{cases} u_t = \lambda \operatorname{div}(|\nabla u|^{p-2}\nabla u) + |u|^{q-2}u(1-|u|^r), & (x,t) \in \Omega \times (0,+\infty), \\ u(x,t) = 0, & (x,t) \in \partial\Omega \times (0,+\infty), \\ u(x,0) = u_0(x), & x \in \Omega. \end{cases} \tag{4.1}$$

In one-dimensional case, the author and Yamada [18] have shown the existence and uniqueness of global solution of (4.1) and the inclusion-relation of the ω-limit set into the set of stationary solutions ϕ, and studied stability properties of ϕ. In connection with stability property, the author [15] has investigated the local behavior (in the space-variable) of solutions. For any (x,t) where solution u of (4.1) intersects flat hats of ϕ, the reaction effect for u disappears and there exists only the diffusion effect for u, whose coefficient is $\lambda(p-1)|u_x|^{p-2}$. When u_0 touches ϕ anywhere in its flat hats, we can expect that $u(t;u_0)$ keeps on touching ϕ there and that the touching area does not spread out. We claim that this is right if u_0, which touches the flat hats, is very close to ϕ in a certain sense. In case u_0 crosses a flat hat transversely, the diffusion may cause their intersection points to change as a function of t along $u(t;u_0)$. The paper [15] assures that the area on which the intersections may change, is uniformly bounded for t. These phenomena will arise in high-dimensional case.

More generally, it is interesting to purchase the behavior of flat place of solutions, which has flat cores when $t=0$. This is related to the waiting time problem for the porous medium equation, and the studies which have been done for the equation will be useful to observe flat cores for the p-Laplace operator.

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