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Large time behaviour of a generalized mean curvature flow

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1. Introduction. We are interested in a motion of a hypersurface by its mean curvature with right angle boundary condition in a cylindrical domain. In particular, we would like to know how does behave the surface as time tends to infinity.

Let Ω' be a convex bounded domain in $\boldsymbol{R}^{N-1}$ with smooth boundary, where $N \geq 2$. We set a cylindrical domain $\Omega := \Omega' \times \boldsymbol{R}$. Suppose that $\Omega_+(t)$ and $\Omega_-(t)$ are open sets in Ω at time t and $\Omega_+(t) \cap \Omega_-(t) = \emptyset$. We set a hypersurface $\Gamma_t := \partial\Omega_+(t) \cap \partial\Omega_-(t) \subset \overline{\Omega}$ at time t; Γ_t intersects the lateral boundary of Ω. Let $\boldsymbol{n}$ be a unit normal vector on Γ_t from $\Omega_+(t)$ to $\Omega_-(t)$; of course $\boldsymbol{n}$ depends on time t. We consider the mean curvature flow equation

$$V = \kappa \quad \text{on} \quad \Gamma_t, \tag{1.1a}$$

$$< \boldsymbol{n}, \nu >= 0 \quad \text{on} \quad b\Gamma_t := \partial\Omega \cap \Gamma_t, \tag{1.1b}$$

where V is normal velocity on Γ_t in the direction $\boldsymbol{n}$, κ is mean curvature on Γ_t and ν is an outward unit normal vector on $\partial\Omega$. We are interested in the behaviour of Γ_t as time tends to infinity. If Γ_0 is the graph of a function on Ω', then there is a global-in-time graph-like smooth solution Γ_t of the mean curvature flow equation with right angle boundary condition starting from Γ_0. Moreover, the solution Γ_t converges to a hyperplane perpendicular to $\partial\Omega$ in C^∞ topology. These results are due to Huisken [H]. It is interesting to study the large time behaviour of generalized interface evolution with a given initial (compact) hypersurface Γ_0 not necessarily a graph-like surface. It is too naive to guess that the limit of Γ_t as $t \to \infty$ is always a single hyperplane. Consider an initial hypersurface Γ_0 given by $r = r(x_N)$ where r is a distance from x_N-axis and Ω' is a ball in $\boldsymbol{R}^{N-1}$ centered at the origin. If $r = r(x_N)$ is an even convex function, we expect that Γ_t pinches in a finite time

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if $r(0)$ is very small so that Γ_0 has a thin neck near the origin of $\boldsymbol{R}^N$ provided that $N \geq 3$. Then it is natural to guess that Γ_t becomes two pieces and each piece converges to a different hyperplane. This suggests that the limit of Γ_t may consist of several hyperplanes perpendicular to $\partial\Omega$. As already pointed out in [ES] Γ_t may have interior even if Γ_0 has no interior; see also [G1], [G2] for the boundary value problems and references therein. This suggests that the limit of Γ_t may have interior. So the best we conjecture for general initial Γ_0 is that the limit of Γ_t as $t \to \infty$ is a closed set in $\overline{\Omega}$ and that the boundary of Γ_∞ consists of hyperplanes parallel to Ω'.

To treat a hypersurface Γ_t we apply the level set approach as in [CGG] and [ES]. Roughly speaking, the level set approach is to regard Γ_t as the zero-level set of an auxiliary function $u : (0, \infty) \times \overline{\Omega} \to \boldsymbol{R}$; say

$$\begin{aligned} \Gamma_t =& \{x \in \overline{\Omega}; u(t,x) = 0\}, \\ \Omega_\pm(t) =& \{x \in \overline{\Omega}; \pm u(t,x) > 0\} \end{aligned}$$

and each level set of u moves by (1.1a)-(1.1b). Then we obtain the level set equation of (1.1a)-(1.1b)

$$u_t - |\nabla u| \text{ div } (\nabla u / |\nabla u|) = 0 \quad \text{in } (0,\infty) \times \Omega, \tag{1.2a}$$

$$\partial u / \partial \nu = 0 \quad \text{on } (0,\infty) \times \partial\Omega.. \tag{1.2b}$$

This is a degenerate parabolic equation. So we consider this equation in viscosity sense. This equation (1.2a)-(1.2b) was initially studied by [S] then by [GS]. They established a comparison principle to (1.2a)-(1.2b). Moreover, for each given bounded uniformly continuous function g such that

$$u(0,x) = g(x) \quad \text{on} \quad \overline{\Omega}, \tag{1.2c}$$

they proved existence of global-in-time solution and uniqueness of solution to (1.2a)-(1.2c). Instead of studying Γ_t directly, we study the large time behaviour of solution of (1.2a)-(1.2c). Then we have two sub problems:

(i) Does $u(t,x)$ converge as $t \to +\infty$?

(ii) What is property of the limit function?

2. Results. Before to state our results, we have to say assumptions on $g(x)$.

Assumptions on g. We assume that $g(x)$ is constant where $|x_N|$ is sufficiently large; i.e., there exist constants c_1, c_2 and positive constant $m > 0$ so that

$$\begin{aligned} g(x', x_N) &= c_1 \quad \text{for all} \quad x_N \geq m,\ x' \in \overline{\Omega'}, \\ g(x', x_N) &= c_2 \quad \text{for all} \quad x_N \leq -m,\ x' \in \overline{\Omega'}. \end{aligned} \tag{2.1}$$

For a compact Γ_0 this condition is not restrictive. Now we shall state our results.

Theorem 2.1(Convergence). *Assume that Ω' is a smoothly bounded convex domain in $\mathbf{R}^{N-1}$. Assume that $g \in C(\overline{\Omega})$ is as above. Then the unique viscosity solution $u \in C([0,\infty) \times \overline{\Omega})$ of (1.2a)-(1.2c) satisfying (2.1) with the same m, c_1, c_2 at each time converges uniformly on $\overline{\Omega}$ to a function $v \in C(\overline{\Omega})$ as $t \to \infty$ that satisfies the level set minimal surface equation with the Neumann condition*

$$-|\nabla v| \ \text{div}\ (\nabla v / |\nabla v|) = 0 \quad \text{in } \Omega, \tag{2.2a}$$

$$\partial v / \partial \nu = 0 \quad \text{on } \partial\Omega \tag{2.2b}$$

in the viscosity sense. (If g is Lipschitz continuous, so is v). Moreover, v fulfills (2.1) with the same m, c_1 and c_2.

Remark 2.2. The uniqueness of solution of (1.2a)-(1.2c) satisfying (2.1) is proved by the comparison theorem [S], [GS]. We take continuous functions g^-, g^+ independent of x' such that

$$\begin{aligned} g^-(x) \leq g(x) &\leq g^+(x) \quad \text{on} \quad \overline{\Omega}, \\ g^-(x) = g(x) &= g^+(x) \quad \text{for all} \quad |x_N| \geq m,\ x' \in \overline{\Omega'}. \end{aligned}$$

Since g^- and g^+ are stationary solution of (1.2a)-(1.2b), comparison yields $g^- \leq u(t, \cdot) \leq g^+$ for all $t \geq 0$. This implies u satisfies (2.1) at each time.

Remark 2.3. For the Dirichlet boundary condition motion of Γ_t was studied by [SZ] and [ISZ] when Ω is bounded, mean convex. The same convergence theorem was proved by [ISZ] except the statement related to (2.1).

Remark 2.4. The assertion is still valid for arbitrary smoothly bounded convex domain Ω not necessarily a cylinder in $\boldsymbol{R}^N$ except the statement related to (2.1). The proof goes as well as that of [ISZ].

Theorem 2.5 (Strong maximum principle). *Let Ω' be a smoothly bounded domain in $\mathbf{R}^{N-1}$. Assume that $v \in C(\overline{\Omega})$ is a viscosity solution of (2.2a)-(2.2b). If $v(x', x_N)$ is a constant for sufficiently large x_N (or $-x_N$), then v is independent of x' as a function in $\overline{\Omega}$.*

Remark 2.6. We cannot completely remove that v is a constant for sufficiently large x_N. If we remove this condition, we can make a counter example. Let $N = 2$, $\Omega' = (0,1)$ and $v(x) = x_1$. We easily see that v is a viscosity solution of (2.2a)-(2.2b). However, each level set of v is parallel to x_2-axis. This means v is not a constant where x_2 is sufficiently large.

Combining Theorems 2.1 and 2.5 we have:

Theorem 2.7. *Under the same hypothesis of Theorem 2.1 the solution $u(t,x)$ converges to a function $v = v(x_N)$ (satisfying (2.1)) uniformly in $\overline{\Omega}$ as $t \to \infty$. In particular for each $c \in \mathbf{R}$*

$$\lim_{t\to\infty} \sup\{\text{dist }(x,\ \Gamma_\infty);\ x \in \Gamma_t\} = 0 \tag{2.3}$$

with

$$\Gamma_\infty = \{(x', x_N) \in \mathbf{R}^N;\ v(x_N) = c,\ x' \in \overline{\Omega'}\},$$
$$\Gamma_t = \{(x', x_N) \in \mathbf{R}^N;\ u(t, x', x_N) = c\},$$

where dist $(x, A) = \inf\{|x - y|;\ y \in A\}$.

We conjecture that

$$\lim_{t\to\infty} \sup\{\text{dist }(y, \Gamma_t);\ y \in \Gamma_\infty\} = 0. \tag{2.4}$$

We can prove (2.4) when Γ_∞ consists of a finite collection of parallel hyperplanes (perpendicular to x_N-axis). If (2.4) is proved, combining (2.3) and (2.4) implies that Γ_t converges to Γ_∞ in the topology of the Hausdorff distance as $t \to \infty$.

3. Sketch of proof of Theorem 2.5. To prove Theorem 2.5 we establish a kind of strong maximum principle for (2.2a)-(2.2b).

Lemma 3.1 (Propagation of maximum, interior version). *Let D' be a domain in $\mathbf{R}^{N-1}$ and let $D = D' \times (\alpha, \beta)$ with $\alpha, \beta \in \mathbf{R}$. Let w be an upper semicontinuous viscosity subsolution of*

$$-|\nabla w| \text{div } (\nabla w / |\nabla w|) = 0 \quad \text{in } D.$$

Assume that w attains its maximum K in D.

Let $M \in \mathbf{R}$ be of form

$$M = \sup\{x_N \in (\alpha, \beta);\ w(x', x_N) = K \quad \text{for some } x' \in D'\}$$

If $M < \beta$ and $w(\cdot, M)$ attains its maximum K at some (interior) point $\xi' \in D'$, then $w(x', M) = K$ for all $x' \in D'$.

Lemma 3.2 (Boundary version). *Let D and D' be as in Lemma 3.1. Assume that $\partial D'$ is C^2. Let w be an upper semicontinuous viscosity subsolution of*

$$-|\nabla w| \operatorname{div}(\nabla w/|\nabla w|) = 0 \quad \text{in} \quad D,$$
$$\partial w/\partial \nu = 0 \quad \text{on} \quad \partial D' \times (\alpha, \beta).$$

Assume that w attains its maximum K in $\overline{D}$. Let $M \in \mathbf{R}$ be of form

$$M = \sup\{x_N \in (\alpha, \beta);\ w(x', x_N) = K \quad \text{for some} \quad x' \in \overline{D'}\}.$$

If $M < \beta$ and $w(\cdot, M)$ attains its maximum K at some point $\xi' \in \partial D'$, then $w(x', M) = K$ for all $x' \in \overline{D'}$.

Sketch of proof of Theorem 2.5. We may assume that $v = v(x', x_N)$ is a constant c_1 for sufficiently large x_N, say $x_N \geq m$. We set

$$A_\lambda^+ = \{x \in \overline{\Omega};\ v(x) \geq \lambda\},\ A_\lambda^- = \{x \in \overline{\Omega};\ v(x) \leq \lambda\}.$$

To show that v is independent of x', it suffices to prove that A_λ^+ and A_λ^- are perpendicular to x_N-axis for all $\lambda > c_1$ and $\lambda < c_1$, respectively. Here a set A in $\overline{\Omega}$ is called *perpendicular* to x_N-axis if $(x', x_N) \in A$ for some $x' \in \overline{\Omega'}$ implies $(z, x_N) \in A$ for all $z \in \overline{\Omega'}$.

Claim. If A_λ^+ and A_λ^- are perpendicular to x_N-axis for all $\lambda > c_1$ and $\lambda < c_1$, respectively, then $A_{c_1}^+$ and $A_{c_1}^-$ are perpendicular to x_N-axis.

We can check this by contradiction. There would exist $\hat{x}_N \in A_{c_1}^+$ such that $v(\bar{x}', \hat{x}_N) \neq v(\bar{y}', \hat{x}_N)$ for some $\bar{x}', \bar{y}' \in \overline{\Omega'}$ with $\bar{x}' \neq \bar{y}'$. We may assume that $v(\bar{y}', \hat{x}_N) = c_1$ and we set $\mu = v(\bar{x}', \hat{x}_N)$. We consider the case $\mu < c_1$. Since $\hat{x}_N \in A_\mu^-$ and $\mu < c_1$, we see that A_μ^- is perpendicular to x_N-axis; i.e., $v(x', \hat{x}_N) = \mu$ for all $x' \in \overline{\Omega'}$. However, this contradicts that there exists $\bar{y}'$ such that $v(\bar{y}', \hat{x}_N) = c_1$. We can prove the case $\mu > c_1$ similarly.

We shall only give a proof that A_λ^+ is perpendicular to x_N-axis for all $\lambda > c_1$ since the proof for A_λ^- is symmetric by taking $-v$ instead of v. We may assume that $v \leq \lambda$ on $\overline{\Omega}$ by replacing v by min (v, λ) since (1.2a)-(1.2b) is geometric so that min (v, λ) is still a viscosity solution of (1.2a)-(1.2b) [CGG, S]. By these reduction it suffices to prove that

$$A_\lambda^+ = \{x \in \overline{\Omega};\ v(x) = \lambda\}$$

is perpendicular to x_N-axis, when $v \leq \lambda$ on $\overline{\Omega}$ and $v = c_1 < \lambda$ for $x_N \geq m$. We may assume that A_λ^+ is nonempty.

Let Σ be the projection of A_λ^+ on x_N-axis, i.e.,

$$\Sigma = \{x_N \in \mathbf{R};\ (x', x_N) \in A_\lambda^+\}.$$

Since $\overline{\Omega'}$ is compact and A_λ^+ is closed by continuity of v, it is easy to see that Σ is a closed set in $\mathbf{R}$. Since $v = c_1 < \lambda$ for $x_N \geq m$, Σ is bounded from above. We have to take care of the case Σ is like a cantor set. For simplicity, we consider the case Σ is a bounded closed interval.

Step 1. At the boundary of Σ. If $v(x', x_N)$ is a viscosity subsolution of (2.2a)-(2.2b) then so is $v(x', -x_N)$. We apply Lemmas around the maximum of Σ and the minimum of Σ. We see that $v(x', x_N) = \lambda$ for all $x_N \in \partial\Sigma,\ x' \in \overline{\Omega'}$.

Step 2. On the interior of Σ. There would exist a set

$$A_{-\lambda_0}^- := \{x \in \overline{\Omega};\ v(x) \leq -\lambda_0\} \subset \overline{\Omega'} \times \Sigma \quad \text{with} \quad -\lambda_0 < \lambda.$$

We may assume that $v(x) \geq -\lambda_0$ in $\overline{\Omega}$ by replacing v by $\max(v, -\lambda_0)$. We set $w(x) := -v(x)$ then $w(x) \leq \lambda_0$ in $\overline{\Omega}$. We see w is a viscosity subsolution of (2.2a)-(2.2b) since v is a viscosity supersolution of (2.2a)-(2.2b). Let Σ^- be the projection of $A_{-\lambda_0}^-$ on x_N-axis. Applying Lemmas on the boundary of Σ^- implies that $v(x', x_N) = -\lambda_0$ for all $x_N \in \partial\Sigma^-,\ x' \in \overline{\Omega'}$. This is a contradiction. $\square$

We only give the proof of Lemma 3.1. Then we can prove Lemma 3.2. However, we do not give it here.

Proof of Lemma 3.1. We may assume that $K = 0$ since w plus a constant is still a subsolution when w is a subsolution. We may also assume that $M = 0$ by a translation.

We argue by contradiction. Suppose that there would exist $\zeta' \in D'$ such that $w(\zeta', 0) < 0 = K$. The basic strategy for the proof is to find a domain E in D and a test function $\varphi \in C^2(E)$ that satisfies

$$\max_E (w - \varphi) = (w - \varphi)(\hat{x}', \hat{x}_N), \tag{3.1}$$
$$- |\nabla \varphi| \operatorname{div}(\nabla \varphi / |\nabla \varphi|) > 0 \quad \text{at} \quad (\hat{x}', \hat{x}_n) \tag{3.2}$$

for some $\hat{x} = (\hat{x}', \hat{x}_N) \in E$. This evidently contradicts the assumption that w is a subsolution in D. Our construction of φ and E reflects the proof of the classical strong maximum principle in [PW], [GT].

1. Choice of a test function. Let w_0 be a function on D' of form

$$w_0(x') = w(x', 0).$$

Since w_0 is upper semicontinuous, there is an open ball B_0 with $\overline{B_0} \subset D'$ that satisfies

$$w_0 < 0 \quad \text{in} \quad B_0 \quad \text{and}$$
$$w_0(y') = 0 \quad \text{for some} \quad y' \in \partial B_0.$$

This is standard; see e.g. [PW]. (Indeed, we take a curve γ starting from ζ' to ξ' and denote by η' the first point attaining $w_0 = 0$ on γ starting from ζ'. Then there exists a point ζ_1' on the arc $\zeta'\eta'$ such that

$$\zeta_1' \in B(\eta', d/2) \subset D',$$

where

$$d = \operatorname{dist}(\gamma, \partial D')$$

and $B(\eta', \sigma)$ denotes the open ball in $\mathbf{R}^{N-1}$ of radius σ centered at η'. We set

$$r_0 = \sup\{r;\ w_0(x') < 0 \quad \text{for all} \quad x' \in B(\zeta_1', r) \subset D'\}$$

so that

$$r_0 < |\zeta_1' - \eta| < d/2.$$

If we set $B_0 = B(\zeta_1', r_0)$, then B_0 satisfies all desired properties.)

Let B_1 be a little bit smaller open ball in B_0 such that $\partial B_0 \cap \partial B_1 = \{y'\}$. Let a be the center of B_1 and $r_1(< r_0)$ be the radius of B_1. We take

$$\varphi(x', x_N) = -\varepsilon_1\, z(x') - \varepsilon_2\, x_N,$$
$$z(x') = e^{-\gamma|x'-a|^2} - e^{-\gamma r_1^2}$$

with positive parameters $\varepsilon_1, \varepsilon_2$ and γ to be determined later. By definition one observe that

$$\begin{aligned} 0 < z(x') < 1 \quad &\text{in} \quad B_1 = B(a, r_1), \\ z(x') = 0 \quad &\text{on} \quad \partial B_1, \\ -1 < z(x') < 0 \quad &\text{outside} \quad \overline{B_1}. \end{aligned} \tag{3.3}$$

2. Choice of γ. For each $\mu = \varepsilon_2/\varepsilon_1$ there is $\gamma_0 = \gamma_0(\mu)$ such that for $\gamma \geq \gamma_0$ it holds

$$-|\nabla\varphi|\ \mathrm{div}(\nabla\varphi/|\nabla\varphi|) > 0 \quad \text{at all} \quad (x', x_N) \tag{3.4}$$

with

$$\frac{r_1}{2} \leq |x' - a| \leq \frac{3r_1}{2},\ x_N \in \mathbf{R}.$$

Since

$$-|\nabla\varphi|\ \mathrm{div}(\nabla\varphi/|\nabla\varphi|) = \varepsilon_1(|\nabla' z(x')|^2 + \mu^2)^{1/2}\ H(z)$$

with $H(z) = \mathrm{div}'\{\nabla' z(x')/(\mu^2 + |\nabla' z(x')|^2)^{1/2}\}$, it suffices to prove that $H(z)(x') > 0$ for x' with $r_1 \leq 2|x' - a| \leq 3r_1$ when γ is sufficiently large. Here ∇' denotes the gradient in x' and div' denotes the divergence in x'.

Since $z(x')$ is radial, i.e.,

$$z(x') = g(|x' - a|) \quad \text{with} \quad g(\rho) = e^{-\gamma\rho^2} - e^{-\gamma r_1^2},$$
$$H(z) = \left(\frac{g'}{((g')^2 + \mu^2)^{1/2}}\right)' + \frac{N-2}{\rho}\,\frac{g'}{((g')^2 + \mu^2)^{1/2}}\Bigg|_{\rho = |x'-a|}$$

Since $g'(\rho) = -2\,\gamma\rho e^{-\gamma\rho^2}$, $g''(\rho) = -2\gamma e^{-\gamma\rho^2} + 4\gamma^2\rho^2 e^{-\gamma\rho^2}$, we obtain

$$H(z) = \frac{\{4\mu^2\gamma^2\rho^2 - 2(N-1)\mu^2\gamma - 8(N-2)\gamma^3\rho^2 e^{-2\gamma\rho^2}\}\ e^{-\gamma\rho^2}}{(4\gamma^2\rho^2 e^{-2\gamma\rho^2} + \mu^2)((g')^2 + \mu^2)^{1/2}}$$

with $\rho = |x' - a|$. The quantity in $\{\ \}$ is uniformly positive for ρ, $r_1 \leq 2\rho \leq 3r_1$ provided that γ is sufficiently large say $\gamma > \gamma_0(\mu)$.

3. Choice of the domain E, $\varepsilon_1, \varepsilon_2$. Let y' be the point as in Step 1. By definition

$$w_0 < 0 \quad \text{in} \quad \overline{B_1} \backslash \{y'\} \quad \text{and} \quad w_0(y') = 0.$$

We set $B_2 = B(y', r_1/2)$. Since $r_1 < r_0 < d/2$, B_2 is contained in D'. We take $\delta > 0$ so small that

$$\partial(B(a, r_1 + \delta)) \cap \partial B_2 \subset B_0.$$

We then divide the boundary of B_2 into two pieces:

$$C_2' = \partial B_2 \cap \overline{B(a, r_1 + \delta)}, \ C_2'' = \partial B_2 \backslash \overline{B(a, r_1 + \delta)};$$

clearly ∂B_2 is a disjoint union of C_2' and C_2''. Since $w_0 < 0$ on a compact set C_2', there exists a constant $\ell > 0$ that satisfies $w_0 \leq -\ell$ on C_2' by upper semicontinuity of w_0. Since w is upper semicontinuous,

$$w \leq -\ell/2 \quad \text{on} \quad C_2' \times [\alpha', \beta'], \ [\alpha', \beta'] \subset (\alpha, \beta)$$

for $\alpha' < 0 < \beta'$ sufficiently close to zero. We first fix $\alpha' < 0$ since $|z(x')|$ on $\overline{B_2}$ is bounded by 1 by (3.3), we take $\mu > (-\alpha')^{-1}$ so that

$$\sup\{z(x');\ x' \in B_2\}(-\alpha')^{-1} < \mu \tag{3.5}$$

for all $\gamma > 0$. We fix γ with $\gamma > \gamma_0(\mu)$ so that (3.4) holds. We then take β' smaller so that

$$-\sup\{z(x');\ x' \in C_2''\}/\beta' > \mu. \tag{3.6}$$

We set

$$\sigma_1 = \sup\{w(x', x_N);\ x' \in C_2',\ \alpha' < x_N < \beta'\},$$
$$\sigma_2 = \sup\{w(x', \beta');\ x' \in \overline{B_2}\}.$$

By definition of C_2' and $M = 0$ we see that $\sigma_1 \leq -\ell/2$, $\sigma_2 < 0$. Choose $\varepsilon_1, \varepsilon_2$ sufficiently small so that

$$\max\{\sigma_1, \sigma_2\} + \varepsilon_1 + \varepsilon_2 \beta' < 0 \tag{3.7}$$

keeping $\mu = \varepsilon_2/\varepsilon_1$. We take $E = B_2 \times (\alpha', \beta')$ and fix $\alpha', \mu, \gamma, \beta'$, $\varepsilon_1, \varepsilon_2$ satisfying (3.5)-(3.7) with $\gamma > \gamma_0(\mu)$.

4. Completion of the proof. To show (3.1) it suffices to prove

$$\max_{\partial E}(w-\varphi)<0 \tag{3.8}$$

since $(w-\varphi)(y',0)=0$ and $(y',0)\in E$. We divide ∂E into four pieces

(a) $x'\in C_2'$ and $\alpha'<x_N<\beta'$,
(b) $x'\in C_2''$ and $\alpha'<x_N<\beta'$,
(c) $x'\in \overline{B_2}$ and $x_N=\alpha'$,
(d) $x'\in \overline{B_2}$ and $x_N=\beta'$.

On the part (a) because of a bound $w\le -\ell/2$ we conclude $w-\varphi$ is negative if $\varepsilon_1,\varepsilon_2$ is taken by (3.7); note that $|z|$ is bounded independent of γ by (3.3). On the part (b) by (3.3)

$$\sup\{z(x');\ x'\in C_2''\}<0.$$

The negativity of $w-\varphi$ follows from (3.6). On the part (c) the negativity of $w-\varphi$ follows from (3.5). On the part (d) since $\sigma_2<0$, (3.7) implies the negativity of $w-\varphi$. Thus we have proved (3.8). Since (3.4) holds on $B_2\times\mathbf{R}$, we get desired φ and E satisfying (3.1) and (3.2). □

Remark 3.3. Our Theorem 2.5 as well as Lemmas 3.1 and 3.2 applies more general equation than (2.2a). We may replace (2.2a) by

$$F(\nabla u,\ \nabla^2 u)=0 \tag{3.9}$$

with F satisfying

(i) $F:(\mathbf{R}^N\backslash\{0\})\times\mathbf{S}^N\to\mathbf{R}$ is continuous and geometric in the sense of [CGG].
(ii) $F(p,O)=0$ for all $p\in\mathbf{R}^N\backslash\{0\}$.
(iii) For each $\lambda_0>0$ there exists $N_0>0$ such that if $\lambda_{\max}(Q_{\overline{p}}(X))\le\lambda_0$ and $\lambda_{\min}(Q_{\overline{p}}(X))\le -N_0$ (resp. $\lambda_{\min}\ge-\lambda_0$, $\lambda_{\max}\ge N_0$) then $F(p,Q_{\overline{p}}(X))>0$ (resp. <0) for all $X\in\mathbf{S}^N$ and $p\in\mathbf{R}^N\backslash\{0\}$, where $Q_{\overline{p}}(X)=(I-\overline{p}\otimes\overline{p})X(I-\overline{p}\otimes\overline{p})$ with $\overline{p}=p/|p|$.

Here $\mathbf{S}^N$ denotes the space of all real symmetric matrices and $\lambda_{\min}(Y)$ and $\lambda_{\max}(Y)$ are the smallest and the largest eigenvalues of $Y\in\mathbf{S}^N$, respectively. Even if (2.2a) is replaced by (3.9) the proof of Lemma 3.1 is the same except step 2 where we have to replace (3.4) by

$$F(\nabla\varphi,\nabla^2\varphi)>0 \quad \text{at all} \quad (x',x_N) \tag{3.4'}$$

satisfying $r_1 \le 2|x' - a| \le 3r_1, x_N \in \mathbf{R}$. To prove (3.4′) for large $\gamma \ge \gamma_0(\mu)$ the property (iii) is invoked. For example,

$$F(p, X) = -\operatorname{trace}\{A(-\overline{p})Q_{\overline{p}}(X)\}$$

satisfies the above conditions (i)–(iii), where $A(\overline{p})$ is a given matrix in S^N and positive definite for $p \neq 0$. This F appear when we study a level set equation of the anisotropic mean curvature flow equation (for the anisotropic mean curvature equation see e.g. [Gur] and for its level set equation see e.g. [CGG].) Here we shall check the conditions (i)–(iii). For (i) and (ii) we can check easily. It remains to show (iii). We may assume that $Q_{\overline{p}}(X)$ is a diagonal matrix. Let $A(-\overline{p}) = (a_{ij})$ and let $\lambda_1, \lambda_2, \ldots, \lambda_N$ be eigenvalues of $Q_{\overline{p}}(X)$ with $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_N$. Then we see

$$-\operatorname{trace}\{A(-\overline{p})Q_{\overline{p}}(X)\} = -\sum_{i=1}^{N} \lambda_i a_{ii}.$$

From the assumption $\lambda_N = \lambda_{\max}(Q_{\overline{p}}(X)) \le \lambda_0$ and $\lambda_1 = \lambda_{\min}(Q_{\overline{p}}(X)) \le -N_0$ we observe that

$$-\sum_{i=1}^{N} \lambda_i a_{ii} \ge -\lambda_1 a_{11} - \sum_{i=2}^{N} \lambda_0 a_{ii}.$$

If $|\lambda_1|$ is sufficiently large then the condition (iii) holds. A similar remark applies Lemma 3.2. (Geometricity is not invoked for Lemmas 3.1 and 3.2.) To extend Theorem 2.5 for (3.9) we notice that properties (i)–(iii) are invariant under translation in space independent variables and order-preserving change of the dependent variable of (3.9); (i)-(iii) are invariant under multiplication with -1 to the dependent variable by taking $\tilde{F}(p, X) = F(-p, -X)$.

This extended theory applies level set equations of anisotropic mean curvature flow equations (see e.g. [Gur]) provided that the Frank diagram of interfacial energy is strictly convex in the sense that its all (inward) principal curvatures are positive.

Remark 3.4. Recently, a strong maximum principle for degenerate elliptic equations in viscosity sense was established by Bardi and Da Lio. Although they study fully nonlinear partial differential equation of the form

$$F(x, u, \nabla u, \nabla^2 u) = 0,$$

here we only explain thier results on the strong maximum principle for the equation (3.9). Let $F : (\mathbf{R}^N \backslash \{0\}) \times \mathbf{S}^N \to \mathbf{R}$ be continuous and be lower semicontinuous on

$\mathbf{R}^N \times \mathbf{S}^N$. Assume that F is degenerete elliptic, i.e.,

$$F(p, X) \leq F(p, Y) \quad \text{if} \quad X \geq Y \quad \text{and for all} \quad p \neq 0.$$

Moreover, they assume two properties on F. One is *the nondegeneracy property*, that is, there exist $\gamma_0 > 0$ such that

$$F(\nu, I - \gamma\nu \otimes \nu) > 0 \quad \text{for all} \quad \gamma > \gamma_0, \ \nu \neq 0. \tag{3.10}$$

The other is *the scaling property*, that is, there exist a function $\varphi > 0$ such that

$$F(\xi s, \xi X) \geq \varphi(\xi) F(s, X) \quad \text{for all} \quad \xi > 0, \ s \in [-1, 0]. \tag{3.11}$$

There are many equations satisfying the above conditions. For example, the minus p-Laplacian, the minus ∞-Laplacian and the graph minimal surface equation. However, the level set minimal surface equation does not satisfy the conditon (3.10). Generally, geometric equations does not fulfill it. Thier proof of the strong maximum principle reflects the proof of the classical strong maximum principle in [PW], [GT] as same as our Lemma 3.1.

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