Stable and unstable manifolds of diffeomorphisms with positive entropy (Singular phenomena of dynamical systems)

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Stable and unstable manifolds of
diffeomorphisms with positive entropy

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Abstract
We show that $C^2$-diffeomorphisms with positive entropy are chaotic in the sense of Li-Yorke. To do so we prove that these maps are $*$-chaotic on the closure of stable manifolds for some points. The notion of "$*$-chaos" was introduced by Kato and it is related to chaos in the sense of Li-Yorke.

1 Introduction
We study chaotic properties of diffeomorphisms with positive entropy. Notions of chaos have been given by Li and Yorke [9], Devaney [2] and others. It is known that if a continuous map of interval has positive entropy, then it is chaotic according to the definition of Li and Yorke (cf. [1]).

In [6] Katok proved the following :let $f$ be a $C^{1+\varepsilon}$-diffeomorphism of a closed surface. If the topological entropy of $f$ is positive, then there exists a hyperbolic set $\Gamma$ such that the restriction of $f$ into $\Gamma$ is topologically conjugate to a subshift of finite type with positive entropy. This implies that $f$ is chaotic in the sense of Li-Yorke.

However, Katok's theorem does not hold for the high dimensional case. Indeed, let $f$ be a surface diffeomorphism with positive entropy and let $r : S^1 \to S^1$ be an irrational rotation. Then a product map $f \times r$ has the same positive entropy, but it does not have $\Gamma$ as above because there are no periodic points of $f \times r$.

In this paper we show the following:

**Theorem A** Let $f$ be a $C^2$-diffeomorphism of a closed $C^\infty$-manifold. If the topological entropy of $f$ is positive, then $f$ is chaotic in the sense of Li-Yorke.

To my knowledge this theorem gives the most simplest sufficient condition for chaotic phenomena of high dimensional dynamical systems. It remains a question whether Theorem A is true for homeomorphisms. However this question is still unsolved.

Let $M$ be a closed $C^\infty$-manifold and let $d$ be the distance for $M$ induced by a Riemannian metric $\| \cdot \|$ on $M$. A subset $S$ of $M$ is a scrambled set of $f$ if there is a positive number $\tau > 0$ such that for any $x, y \in S$ with $x \neq y$,
1. \( \limsup_{n \to \infty} d(f^n(x), f^n(y)) > \tau, \)

2. \( \liminf_{n \to \infty} d(f^n(x), f^n(y)) = 0. \)

If there is an uncountable scrambled set \( S \) of \( f \), then we say that \( f \) is chaotic in the sense of Li-Yorke. Li and Yorke showed in [9] that if \( f: [0, 1] \to [0, 1] \) is a continuous map with a periodic point of period 3, then \( f \) is chaotic in this sense. In [9] there was the following one more condition: for any \( x \in S \) and any periodic point \( p \in M \), \( \limsup_{n \to \infty} d(f^n(x), f^n(p)) > 0. \) But this condition is unnecessary because a scrambled set contains at most one point which does not satisfy this condition. For the examples and the properties of scrambled sets, the readers may refer to [1], [5], [11], [12], [13], [22], [23] and [24].

Concerning the chaos in the sense of Li-Yorke, Kato introduced the notion of "*chaos" as follows: let \( F \) be a closed subset of \( M \). A map \( f : M \to M \) is *-chaotic on \( F \) (in the sense of Li-Yorke) if the following conditions are satisfied:

1. there is \( \tau > 0 \) such that if \( U \) and \( V \) are any nonempty open subsets of \( F \) with \( U \cap V = \emptyset \) and \( N \) is any natural number, there is a natural number \( n \geq N \) such that \( d(f^n(x), f^n(y)) > \tau \) for some \( x \in U, y \in V, \) and

2. for any nonempty open subsets \( U, V \) of \( F \) and any \( \varepsilon > 0 \) there is a natural number \( n \geq 0 \) such that \( d(f^n(x), f^n(y)) < \varepsilon \) for some \( x \in U, y \in V. \)

Such a set \( F \) is called a *-chaotic set. If \( S \) is a scrambled set, then the closure of \( S, \overline{S}, \) is a *-chaotic set. In [4] Kato showed that the converse is true.

**Lemma 1.1 ([4], Theorem 2.4)** If \( f : M \to M \) is continuous and is *-chaotic on \( F \), then there \( F_{r} \)-set \( S \subset F \) such that \( S \) is a scrambled set of \( f \) and \( \overline{S} = F. \) If \( F \) is perfect (i.e. \( F \) has no isolated points), we can choose \( S \) such that it is a countable union of Cantor sets.

To obtain Theorem A we need the following theorem.

**Theorem B** Let \( f \) be a \( C^2 \)-diffeomorphism of a closed \( C^\infty \)-manifold \( M \) and let \( \mu \) be an \( f \)-invariant ergodic Borel probability measure on \( M \). If the metric entropy of \( \mu \) is positive, then for \( \mu \)-almost all \( x \in M \) the following hold:

(a) \( \overline{W^s(x)} \) is a perfect *-chaotic set, and

(b) \( \overline{W^u(x)} \) contains a perfect *-chaotic set.

Here \( W^s(x) \) and \( W^u(x) \) are defined by

\[
W^s(x) = \{ y \in M : \limsup_{n \to \infty} \frac{1}{n} \log d(f^n(x), f^n(y)) < 0 \} \quad \text{and,}
\]

\[
W^u(x) = \{ y \in M : \limsup_{n \to \infty} \frac{1}{n} \log d(f^{-n}(x), f^{-n}(y)) < 0 \}
\]

respectively.
We notice that for \( \mu \)-almost all \( x \in M \), the above sets \( W^s(x) \) and \( W^u(x) \) are \( C^2 \) immersed manifolds under the assumptions of theorem B. Indeed, let \( f \) and \( \mu \) be as above. For \( \mu \)-almost all \( x \in M \), there exist a splitting of the tangent space \( T_xM = \oplus_{i=1}^{s(x)} E_i(x) \) and real numbers \( \lambda_1(x) < \lambda_2(x) < \cdots < \lambda_{s(x)}(x) \) such that

(a) the maps \( x \mapsto E_i(x) \), \( \lambda_i(x) \) and \( s(x) \) are Borel measurable, moreover \( E_i(f(x)) = D_xf(E_i(x)) \) and \( \lambda_i(x) \), \( s(x) \) are \( f \)-invariant \( (i = 1, \cdots, s(x)) \),

\[
\lim_{n \to \pm\infty} \frac{1}{n} \log \|D_x f^n(v)\| = \lambda_i(x) \quad (0 \neq v \in E_i(x), \ i = 1, \cdots, s(x)) \text{ and }
\]

(c) \( \lim_{n \to \pm\infty} \frac{1}{n} \log |\det(D_x f^n)| = \sum_{i=1}^{s(x)} \lambda_i(x) \dim E_i(x) \)

([14]). The numbers \( \lambda_1(x), \cdots, \lambda_{s(x)}(x) \) are called Lyapunov exponents of \( f \) at \( x \). Since \( \mu \) is ergodic, we can put \( s = s(x) \), \( \lambda_i = \lambda_i(x) \) and \( m_i = \dim E_i(x) \) \( (i = 1, \cdots, s) \) for \( \mu \)-almost all \( x \in M \).

Let \( h_\mu(f) \) denote the metric entropy of \( f \) (see [10] for definition). A well-known theorem of Margulis and Ruelle [21] says that entropy is always bounded above by the sum of positive Lyapunov exponents; i.e. \( h_\mu(f) \leq \sum_{\lambda_i > 0} \lambda_i m_i \).

Since \( f \) has positive entropy, we have

\[
0 < h_\mu(f) \leq \max \{ \lambda_i \} = \lambda_s.
\]

Therefore, by Pesin's stable manifold theorem ([3], [15], [17]), the set \( W^u(x) \) is the image of a \( C^2 \) injective immersion of an euclidean space such that \( T_x W^u(x) = \oplus_{\lambda_i > 0} E_i(x)(\neq \{0\}) \) for \( \mu \)-almost all \( x \in M \). \( W^u(x) \) is called an unstable manifold. Similarly, \( W^s(x) \) is a \( C^2 \) immersed manifold because \( W^s(x) \) is the unstable manifold of \( f^{-1} \), which has positive entropy \( h_\mu(f^{-1}) = h_\mu(f) > 0 \). \( W^s(x) \) is called a stable manifold.

Let us see how Theorem A follows from Theorem B. We denote as \( h(f) \) the topological entropy of \( f \) (see [10] for definition). Then we know that \( h(f) = \sup \{ h_\mu(f) : \mu \in M_e(f) \} \) where \( M_e(f) \) is the set of all \( f \)-invariant ergodic Borel probability measures (cf [20]). Thus, if \( h(f) > 0 \), then we can choose \( \mu \in M_e(f) \) with \( h_\mu(f) > 0 \). Therefore, by Theorem B and Lemma 1.1, \( f \) is chaotic in the sense of Li-Yorke.

Remark that (a) of Theorem B does not hold for unstable manifolds in general. For example, the Smale horseshoe has unstable manifolds which intersect a stable manifold of a fixed point (cf [18]). Since all points in the stable manifold converge to the fixed point, they do not satisfy the first condition of \( * \)-chaos.

Now we shall give a sufficient condition that \( f \) is \( * \)-chaotic on \( \overline{W^u(x)} \) as follows.

**Theorem C** If \( \mu \) is an ergodic SRB measure, then both \( \overline{W^s(x)} \) and \( \overline{W^u(x)} \) are perfect \( * \)-chaotic sets for \( \mu \)-almost all \( x \in M \).

If \( \xi \) is a measurable decomposition of \( M \), then a family \( \{ \mu_{\xi}^x | x \in M \} \) of Borel probability measures exists, and it satisfies the following conditions:
1. for \( x, y \in M \) if \( \xi(x) = \xi(y) \) then \( \mu^\xi_x = \mu^\xi_y \), here \( \xi(x) \) denotes a set containing \( x \) in \( \xi \),

2. \( \mu^\xi_x(\xi(x)) = 1 \) for \( \mu \)-almost all \( x \in M \),

3. for any Borel set \( A \) a function \( x \mapsto \mu^\xi_x(A) \) is measurable and \( \mu(A) = \int_M \mu^\xi_x(A) d\mu(x) \).

The family \( \{\mu^\xi_x | x \in M\} \) is called a canonical system of conditional measures for \( \mu \) and \( \xi \) (see [19] for more details).

An \( f \)-invariant Borel probability measure \( \mu \) is called a Sinai-Ruelle-Bowen measure (SRB measure for abbreviation) provided

(A) for \( \mu \)-almost all \( x \in M \), there exists a positive Lyapunov exponent of \( x \),

(B) \( \mu \) has a conditional measure that is absolutely continuous (with respect to the Lebesgue measure) on unstable manifolds, which is defined as follows:

From (A), the unstable manifold \( W^u(x) \) is a \( C^2 \) submanifold for \( \mu \)-almost all \( x \) in \( M \). Let \( m^u_x \) denote the Lebesgue measure of \( W^u(x) \). A measurable decomposition \( \xi \) of \( M \) is said to be subordinate to unstable manifolds if for \( \mu \)-almost all \( x \) in \( M \)

(C) \( \xi(x) \subset W^u(x) \),

(D) \( \xi(x) \) contains an open neighborhood of \( x \) in \( W^u(x) \).

We say that \( \mu \) has an absolutely continuous conditional measure on unstable manifolds provided \( \mu^\xi_x \) is absolutely continuous with respect to \( m^u_x \) for \( \mu \)-almost all \( x \) in \( M \) if \( \xi \) is subordinate to unstable manifolds. It is known ([7], [8]) that \( \mu \) satisfies (B) if and only if the following equation holds:

\[
h_\mu(f) = \int \sum_{\lambda_i(x) > 0} \lambda_i(x) \dim E_i(x) d\mu(x).
\]

This is sometimes known as Pesin's formula. For the examples and the stochastic properties of diffeomorphisms with SRB measures, the readers may refer to [25]. In a similar way we can define a measurable partition subordinate to stable manifolds.

2 Preliminaries

In this section we introduce \( f \)-invariant measurable partitions each of whose elements is contained in the closure of (un)stable manifolds. Let \( f \) be a \( C^2 \)-diffeomorphism of a closed \( C^\infty \)-manifold \( M \) and let \( \mu \) be an \( f \)-invariant ergodic Borel probability measure on \( M \) with \( h_\mu(f) > 0 \). Denote as \( \mathcal{B} \) the family of Borel sets.
Lemma 2.1 (Proposition 3.1 [7]) Let $f$ and $\mu$ be as above. Then there exist measurable partitions $\xi^s$ and $\xi^u$ of $M$ such that

(a) $\xi^s \leq f\xi^s$ and $\xi^u \leq f^{-1}\xi^u$,

(b) $\xi^s$ and $\xi^u$ are subordinate to stable manifolds and unstable manifolds respectively,

(c) both $\bigvee_{n=0}^{\infty}f^n\xi^s$ and $\bigvee_{n=0}^{\infty}f^{-n}\xi^u$ are the partitions into points,

(d) for $\mu$-almost all $x \in M$,

$$\bigcup_{n=0}^{\infty}f^{-n}\xi^s(f^n(x)) = W^s(x) \quad \text{and} \quad \bigcup_{n=0}^{\infty}f^n\xi^u(f^{-n}(x)) = W^u(x).$$

Lemma 2.2 (Corollary 5.3 [8]) Let $f$ and $\mu$ be as above and let $\xi^\sigma (\sigma = s, u)$ be as in Lemma 2.1. Then,

$$h_\mu(f) = H_\mu(f\xi^s|\xi^s) = \int -\log \mu_x^\sigma (f\xi^s(x)) d\mu(x) = H_\mu(f^{-1}\xi^u|\xi^u) = \int -\log \mu_x^\sigma (f^{-1}\xi^u(x)) d\mu(x)$$

where the family $\{\mu_x^\sigma | x \in M\}$ is a canonical system of conditional measures for $\mu$ and $\xi^\sigma$.

Let us introduce two measurable partitions defined by

$$\eta^s = \bigwedge_{i=0}^{\infty}f^{-i}\xi^s \quad \text{and} \quad \eta^u = \bigwedge_{i=0}^{\infty}f^i\xi^u.$$ 

By Lemma 2.1(a) and (d) we can easily check that $f\eta^\sigma = \eta^\sigma$ and $\eta^\sigma(x) \subset \overline{W^\sigma(x)}$ for $\mu$-almost all $x (\sigma = s, u)$. Let $\{\mu_x^\sigma | x \in M\}$ be a canonical system of conditional measures for $\mu$ and $\eta^\sigma (\sigma = s, u)$. By Doob's theorem it follows that for $A \in \mathcal{B}$

$$\mu^s_x(A) = \lim_{n \to \infty} \mu_x^{f^{-n}\xi^s}(A) \quad \text{and} \quad \mu^u_x(A) = \lim_{n \to \infty} \mu_x^{f^n\xi^u}(A) \quad (\mu\text{-almost all } x).$$

Since $f\eta^\sigma = \eta^\sigma$ and $f$ preserves $\mu$, we have $\mu^\sigma_x(A) = \mu_x^\sigma(fA)$ (\(\mu\)-almost all $x$) for $A \in \mathcal{B}$ and $\sigma = s, u$.

Let $C(M)$ be the Banach space of continuous real-valued functions of $M$ with the sup norm $\| \cdot \|$, and let $\mathcal{M}(M)$ be a set of all Borel probability measures on $M$ with the weak topology. Since $C(M)$ is separable, there exists a countable set $\{\varphi_1, \varphi_2, \cdots\}$ which is dense in $C(M)$. For $\nu, \nu' \in \mathcal{M}(X)$ define

$$\rho(\nu, \nu') = \sum_{n=1}^{\infty} \frac{|\int \varphi_n d\nu - \int \varphi_n d\nu'|}{2^n \|\varphi_n\|}.$$ 

Then $\rho$ is a compatible metric for $\mathcal{M}(X)$ and $(\mathcal{M}(X), \rho)$ is compact (cf.[10]).
Lemma 2.3 Let \( f, \mu \) and \( \{\mu_x^\sigma | x \in M\} \) be as above. Then for \( \epsilon > 0 \) and \( \sigma = s, u \) there exists a closed set \( F_\epsilon^\sigma \) with \( \mu(F_\epsilon^\sigma) \geq 1 - \epsilon \) satisfying the map

\[
F_\epsilon^\sigma \ni x \mapsto \mu_x^\sigma \in \mathcal{M}(X)
\]
is continuous.

Proof. Let \( \{\varphi_1, \varphi_2, \cdots\} \) be as above. By the definition of conditional measures the map

\[
M \ni x \mapsto \int \varphi_n d\mu_x^\sigma
\]
is Borel measurable for \( n \geq 1 \). Thus, by Lusin’s theorem, for \( n \geq 1 \) there exists a closed set \( F_n^\sigma \) with \( \mu(F_n^\sigma) \geq 1 - \epsilon/2^n \) satisfying

\[
F_n^\sigma \ni x \mapsto \int \varphi_n d\mu_x^\sigma : \text{continuous}.
\]
Then \( F_\epsilon^\sigma = \bigcap_{n=1}^\infty F_n^\sigma \) has the desired property.

\( \square \)

Lemma 2.4 Let \( f, \mu \) and \( \{\mu_x^\sigma | x \in M\} \) be as above. Then for \( \mu \)-almost all \( x \in M \) and \( \sigma = s, u \), \( \text{supp}(\mu_x^u) \) has no isolated points.

Proof. We will give the proof for \( \sigma = u \) and so we here omit for \( \sigma = s \) since the technique of the proof is similar.

If this lemma is false, \( \text{supp}(\mu_x^u) \) has an isolated point for \( x \) belonging to some Borel set with positive measure. Let \( \xi^u \) be as in Lemma 2.1. Since \( \text{diam}(f^{-k}\xi^u(x)) \to 0 \) \( (k \to \infty) \) by Lemma 2.1 (c),

\[
P_{-k} = \{ x \in M : \mu_x^{f^{-k}\xi^u} \text{ is a point measure} \}
\]
is positive \( \mu \)-measure for \( k \) large enough. Since \( \mu \) is \( f \)-invariant, we have \( f^n \mu_x^{f^{-k}\xi^u} = \mu_x^{f^{-k}n\xi^u} \) for \( \mu \)-almost all \( x \) and \( n \in \mathbb{Z} \). Then

\[
f^n(P_{-k}) = \{ f^n(x) \in M : \mu_x^{f^{-k}n\xi^u} \text{ is a point measure} \}
\]

\[
= \{ x \in M : f^n \mu_x^{f^{-k}n\xi^u} \text{ is a point measure} \}
\]

\[
= \{ x \in M : \mu_x^{f^{-k}n\xi^u} \text{ is a point measure} \}
\]

\[
= P_{n-k} \quad (n \in \mathbb{Z}).
\]

Put

\[
P = \cap_{j \geq 1} \cup_{n \geq j} P_{n-k} = \cap_{j \geq 1} \cup_{n \geq j} f^n P_{-k}.
\]

Since \( P \) is \( f \)-invariant and \( \mu \) is ergodic, we have \( \mu(P) = 1 \).

For \( x \in P \) there exists an increasing sequence \( \{ n_i \}_{i \geq 0} \) such that \( x \in P_{n_i} \) for \( i \geq 0 \). Since \( \mu_x^u = \lim_{n \to \infty} \mu_x^{f^n\xi^u} = \lim_{i \to \infty} \mu_x^{f^{ni}\xi^u} \) and \( \mu_x^{f^{ni}\xi^u} \) is a point measure for \( i \), so is \( \mu_x^u \). Therefore \( \mu_x^u \) is a point measure for \( \mu \)-almost all \( x \in M \).
Since $\xi^u$ is finer than $\eta^u$ and $\mu^u_x$ is a point measure for $\mu$-almost all $x \in M$, so is $\mu^u_x$. Thus $\mu^u_x(f^{-1}\xi^u(x)) = 1$ for $\mu$-almost all $x$. Therefore

$$h_\mu(f) = \int - \log \mu^u_x(f^{-1}\xi^u(x)) d\mu(x) = 0$$

by Lemma 2.2. This is a contradiction.

Let $B(x, r)$ and $U(x, r)$ denote the closed and open balls in $M$ with center $x \in M$ and radius $r > 0$ respectively.

**Lemma 2.5** Let $f$, $\mu$ and $\{\mu_x^s| x \in M\}$ be as above. Then for $\mu$-almost all $x \in M$

$$\overline{W^s(y)} = \overline{W^s(x)} \quad (\mu_x^s \text{-almost all } y \in M).$$

**Proof.** Let $\xi^s$ be as in Lemma 2.1. Then we have that for $\mu$-almost all $x \in M$

$$\xi^s(y) \subset \overline{W^s(x)} \quad (\mu_x^s \text{-almost all } y).$$

Indeed, let $d_x^s$ denote the distance induced by the Riemannian metric on $W^s(x)$. Then there exist an increasing family $\{\Lambda_\ell\}_{\ell \geq 1}$ of closed sets of $M$ with $\mu(\bigcup_{\ell \geq 1}\Lambda_\ell) = 1$ and a sequence $\{A_\ell\}_{\ell \geq 1}$ of positive numbers satisfying that

(e) for each $x \in \Lambda_\ell$ there exists $\epsilon = \epsilon(x) > 0$, such that

$$B(x, \epsilon) \cap \Lambda_\ell \ni y \mapsto W_{A_\ell}^s(y) = \{z \in W^s(y) : d_x^s(z, y) \leq A_\ell\}$$

is continuous with respect to the Hausdorff metric $d_H$: i.e.

$$\lim_{\Lambda_\ell \ni y \to x} d_H(W_{A_\ell}^s(y), W_{A_\ell}^s(x)) = 0,$$

(f) for each $x \in \Lambda_\ell$, $\xi^s(x) \subset W_{A_\ell}^s(x)$

(cf.[7], [15], [16]). Take arbitrary $y \in \text{supp}(\mu_x^s|\Lambda_\ell) \ (\ell \geq 1)$. Let $\epsilon = \epsilon(y) > 0$ be as in (e) and let $0 < r < \epsilon$. Recall that for $\mu$-almost all $x \in M$

$$\mu_x^s(\bigcup_{\ell \geq 1}\Lambda_\ell) = 1 \quad \text{and} \quad \mu_x^s|\Lambda_\ell = \lim_{n \to \infty} \mu_x^{f^{-n}\xi^s}|\Lambda_\ell \quad (\ell \geq 1).$$

Since $U(y, r)$ is open, we have $\mu_x^{f^{-n}\xi^s}(U(y, r) \cap \Lambda_\ell) > 0$ for $n$ large enough. So we can take $y' \in U(y, r) \cap \Lambda_\ell \cap f^{-n}\xi^s(x)$. Since $y' \in f^{-n}\xi^s(x) \subset W^s(x)$, we have $W_{A_\ell}^s(y') \subset W^s(x)$. Since $y' \in U(y, r) \cap \Lambda_\ell$ and $r$ is arbitrary, it follows that

$$\lim_{r \to 0} d_H(W_{A_\ell}^s(y'), W_{A_\ell}^s(y)) = 0.$$ 

Therefore $\xi^s(y) \subset W_{A_\ell}^s(y) \subset \overline{W^s(x)}$.

From this fact it follows that for $n \geq 0$ and $\mu$-almost all $x \in M$

$$\xi^s(f^n(y)) \subset \overline{W^s(f^n(x))} \quad (\mu_x^s \text{-almost all } y).$$
Thus
\[ W^s(y) = \bigcup_{n \geq 0} f^{-n} \xi^s(f^n y) \subset \bigcup_{n \geq 0} f^{-n} (W^s(f^n(x))) \subset W^s(x) \]
for \( \mu^s_x \)-almost all \( y \). On the other hand, by the definition of conditional measures, \( \mu^s_y = \mu^s_x \) for \( y \in \eta^s(x) \). This implies that \( W^s(y) = W^s(x) \) for \( \mu^s_x \)-almost all \( y \).

\section{Proof of Theorem B(a)}

The purpose of this section is to show Theorem B(a). Let \( f \) be a \( C^2 \)-diffeomorphism of a closed \( C^\infty \)-manifold \( M \) and let \( \mu \) be an \( f \)-invariant ergodic Borel probability measure on \( M \) with positive entropy. As described in §1 the stable manifold \( W^s(x) \) is a \( C^2 \) immersed manifold for \( \mu \)-almost all \( x \in M \) and so the closure of \( W^s(x) \), \( \overline{W^s(x)} \), is perfect.

Let \( \eta^s \) and \( \{ \mu^s_x \mid x \in M \} \) be as in §2. By Lemma 2.4, \( \text{supp}(\mu^s_x) \) has no isolated points for \( \mu \)-almost all \( x \in M \). Therefore, to obtain the conclusion it suffices to show the following.

\textbf{Proposition 1} If \( \mu^s_x \) is not a point measure for \( \mu \)-almost all \( x \in M \), then \( \overline{W^s(x)} \) is a \(*\)-chaotic set for \( \mu \)-almost all \( x \in M \).

\textbf{Proof.} Fix \( 0 < \epsilon < 1 \) and let \( F^s_\epsilon \) be as in Lemma 2.3. By assumption we can take and fix \( x_0 \in \text{supp}(\mu|F^s_\epsilon) \) such that \( \mu^s_{x_0} \) is not a point measure. Since \( \text{supp}(\mu^s_{x_0}) \) is not one point, there are disjoint open sets \( O_1 \) and \( O_2 \) of \( M \) satisfying that
\[ d(O_1, O_2) = \inf\{d(x, y) : x \in O_1, \ y \in O_2\} > \delta \text{ and } \mu^s_{x_0}(O_i) > \delta \quad (i = 1, 2) \]
for some \( \delta > 0 \). By Lemma 2.3 we can choose \( \epsilon' > 0 \) such that
\[ \mu^s_x(O_i) > \delta \quad (i = 1, 2) \]
for \( x \in U(x_0, \epsilon') \cap F^s_\epsilon \). Put \( K = \bigcap_{n=0}^{\infty} \bigcup_{k \geq n} f^{-k}(U(x_0, \epsilon') \cap F^s_\epsilon) \). Since \( \mu(U(x_0, \epsilon') \cap F^s_\epsilon) > 0 \), by ergodicity of \( \mu \) we have \( \mu(K) = 1 \).

Take arbitrary \( \delta' \) with \( 0 < \sqrt{\delta'} < \min\{\mu(U(x_0, \epsilon') \cap F^s_\epsilon), \delta\} \). Let \( \xi^s \) be as in Lemma 2.1 and put
\[ A^s_m(n) = \left\{ x \in M \left| \begin{array}{l} d_H(f^{-\lfloor k/2 \rfloor} \xi^s(f^{\lfloor k/2 \rfloor} x), W^s(x)) \leq 1/m, \\ \text{diam}(f^{-\lfloor k/2 \rfloor} \xi^s(f^{\lfloor k/2 \rfloor} x)) \leq 1/m \quad (k \geq n) \end{array} \right\} \right. \]
for \( n, m \geq 1 \). Then \( A^s_m(n) \subset A^s_m(n + 1) \) and \( \mu(\bigcup_{n=0}^{\infty} A^s_m(n)) = 1 \) by Lemma 2.1 (c) and (d). Thus there exists an increasing sequence \( \{n_m\} \) such that \( \mu(A^s_m(n_m)) \geq 1 - \delta'/2^m \) \((m \geq 1) \). Since
\[ \int \mu^s_x(\cap_{n=1}^{\infty} A^s_m(n_m)) d\mu = \mu(\cap_{n=1}^{\infty} A^s_m(n_m)) \geq 1 - \sum_{m=1}^{\infty} \delta'/2^m = 1 - \delta', \]
we can find a Borel set $C^s_\delta \subset M$ with $\mu(C^s_\delta) \geq 1 - \sqrt{\delta'}$ satisfying
\[
\mu_x^s(\cap_{m=1}^\infty A^s_m(n_m)) \geq 1 - \sqrt{\delta'} \tag{4}
\]
for $x \in C^s_\delta$. To obtain the conclusion it suffices to show that $\overline{W^s(x)}$ is a *-chaotic set for $x \in K \cap C^s_\delta$ because $\delta'$ is arbitrary.

For $x \in K \cap C^s_\delta$, by the definition of $K$, there exists a sequence of positive integers $\{k_m\}_m$ with $k_m > n_m$ such that $f^{k_m}(x) \in U(x_0, \varepsilon') \cap F^s_\varepsilon$. Then by (4) and (2)
\[
\begin{align*}
\mu_x^s(A^s_m(k_m)) &\geq \mu_x^s(\cap_{m=1}^\infty A^s_m(k_m)) \geq \mu_x^s(\cap_{m=1}^\infty A^s_m(n_m)) \geq 1 - \sqrt{\delta'}, \\
\mu_x^s(f^{-k_m}(O_i)) &= \mu_x^s(f^{-k_m}(x))(O_i) > \delta > \sqrt{\delta'} \quad (i = 1, 2, m \geq 1).
\end{align*}
\]

Thus we have $\mu_x^s(A^s_m(k_m)) \cap f^{-k_m}(O_i) > 0$ for $i = 1, 2$ and $m \geq 1$. From Lemma 2.5 we may assume that for $m \geq 1$ and $i = 1, 2$ there exists a point $y_i = y_i(m) \in A^s_m(k_m) \cap f^{-k_m}(O_i)$ such that
\[
\overline{W^s(y_i)} = \overline{W^s(x)}.
\]

By (3) we have
\[
\begin{align*}
d_H(f^{-[k_m/2]}\xi^s(f^{[k_m/2]}y_i), \overline{W^s(x)}) &= d_H(f^{-[k_m/2]}\xi^s(f^{[k_m/2]}y_i), \overline{W^s(y_i)}) \leq 1/m, \\
\text{diam}(f^{k_m-[k_m/2]}\xi^s(f^{[k_m/2]}y_i)) &\leq 1/m \quad (i = 1, 2, m \geq 1).
\end{align*}
\]

To show that $\overline{W^s(x)}$ is a *-chaotic set, suppose that nonempty open sets $U_1$ and $U_2$ satisfy
\[
U_1 \cap U_2 \neq \emptyset, \quad U_j \cap \overline{W^s(x)} \neq \emptyset \quad (j = 1, 2).
\]

By (5) we may assume that
\[
\begin{align*}
y_i &\in f^{-[k_m/2]}\xi^s(f^{[k_m/2]}y_i) \cap U_j \neq \emptyset, \\
f^{k_m}(y_i) &\in f^{k_m-[k_m/2]}\xi^s(f^{[k_m/2]}y_i) \subset O_i \quad (1 \leq i, j \leq 2)
\end{align*}
\]

if $m$ is sufficiently large. Take
\[
a_{i,j} = a_{i,j}(m) \in f^{-[k_m/2]}\xi^s(f^{[k_m/2]}y_i) \cap U_j
\]
for $1 \leq i, j \leq 2$. Then we have that for $1 \leq i, j \leq 2$
\[
a_{i,j} \in U_j, \quad d(f^{k_m}(a_{1,1}), f^{k_m}(a_{2,2})) > \tau \quad \text{and} \quad d(f^{k_m}(a_{1,1}), f^{k_m}(a_{1,2})) < 1/m
\]
by (1), (5) and (6). Since $m$ is arbitrary, $\overline{W^s(x)}$ is a *-chaotic set for $x \in K \cap C^s_\delta$. 
\[\square\]
4 Proof of Theorem B(b)

In this section we will prove Theorem B (b). Let $f$, $\mu$, $\eta^u$ and $\{\mu^u_x | x \in M\}$ be as in §2. By Lemma 2.4, $\text{supp}(\mu^u_x)$ has no isolated points for $\mu$-almost all $x \in M$. Therefore, to obtain the conclusion it suffices to show the following.

**Proposition 2** If $\mu^u_x$ is not a point measure for $\mu$-almost all $x \in M$, then $\text{supp}(\mu^u_x) \subset \overline{W(u)(x)}$ is a $*$-chaotic set for $\mu$-almost all $x \in M$.

**Proof.** Fix $0 < \epsilon < 1$ and let $F^u_\epsilon$ be as in Lemma 2.3. By assumption we can take and fix $x_0 \in \text{supp}(\mu|F^u_\epsilon)$ such that $\mu^u_{x_0}$ is not a point measure. Choose two distinct points $y_1, y_2 \in \text{supp}(\mu^u_{x_0})$ and put $\tau = d(y_1, y_2)/2(>0)$. Take arbitrarily $0 < r < \tau/2$ and choose $\delta = \delta(r) > 0$ such that

$$\mu^u_{x_0}(U(y_i, r)) > \delta \quad (i = 1, 2).$$

(7)

Remark that

$$d(U(y_1, r), U(y_2, r)) = \inf\{d(x, y) : d(x, y_1) < r, d(y, y_2) < r\} > \tau.$$  

(8)

Since $U(y_i, r) (i = 1, 2)$ are open, by (7) there exists a large integer $M = M(r) > 0$ such that

$$\nu(U(y_i, r)) > \delta = \delta(r) \quad (i = 1, 2)$$

(9)

for $\nu \in \mathcal{M}(M)$ with $\rho(\nu, \mu^u_{x_0}) < 1/M$. We can find $\epsilon' = \epsilon'(r) > 0$ such that

$$\rho(\mu^u_x, \mu^u_{x_0}) < 1/2M = 1/2M(r) \quad (x \in U(x_0, \epsilon') \cap F^u_\epsilon).$$

(10)

Note that $\epsilon'$ depends on $r$.

Let $\xi^u$ be as in Lemma 2.1 and put

$$B^u_m(n) = \left\{ x \in M \left| \rho(\mu^u_x \xi^u, \mu^u_{x_0}) \leq 1/m, \right. \right.$$

$$\left. \text{diam}(\rho^{-k-[k/2]}(f^k \xi^u(x))) \leq 1/m \quad (k \geq n) \right\}$$

(11)

for $n, m \geq 1$. Then $B^u_m(n) \subset B^u_m(n+1)$ and $\mu(\bigcup_{n=0}^{\infty} B^u_m(n)) = 1$, by Lemma 2.1 (c) and Doob's theorem. Thus there exists an increasing sequence $\{n_m\}$ such that $\mu(B^u_m(n_m)) \geq 1 - 1/2^{m+1} (m \geq 1)$. Since $\mu(\bigcap_{k=m}^{\infty} B^u_k(n_k)) \geq 1 - 1/2^m$ for $m \geq 1$, we can find a Borel set $D^u_m$ with $\mu(D^u_m) \geq 1 - 2^{-m/2}$ satisfying

$$\mu^u_x(\bigcap_{k=m}^{\infty} B^u_k(n_k)) \geq 1 - 2^{-m/2} \quad (x \in D^u_m).$$

(12)

Put

$$K_r = \bigcap_{k=1}^{\infty} \bigcup_{m=k}^{\infty} \left( \bigcup_{n=0}^{\infty} \bigcup_{\ell \geq n} f^{-\ell}(U(x_0, \epsilon'(r)) \cap F^u_\epsilon \cap D^u_m) \right) (0 < r < \tau/2).$$

Then $\mu(K_r) = 1 (0 < r < \tau/2)$ by ergodicity of $\mu$. To obtain the conclusion it suffices to show that $\text{supp}(\mu^u_x)$ is a $*$-chaotic set for $x \in K = \cap_{n \geq 1} K_{1/n}$. 

To do this fix \( x \in K_r \) (\( r = 1/n, n \geq 1 \)) and suppose that nonempty open sets \( U_1 \) and \( U_2 \) satisfy
\[
U_1 \cap U_2 \neq \emptyset, \quad U_j \cap \text{supp}(\mu^u_x) \neq \emptyset \quad (j = 1, 2).
\]
Choose \( m_0 > 0 \) with
\[
0 < 2^{-m_0/2} < \min\{\mu^u_x(U_j) : j = 1, 2\} \quad \text{and} \quad m_0 \geq 2M.
\]
Since \( x \in K_r \), by the definition of \( K_r \), there exist \( m_1 > m_0 \) and a sequence of positive integers \( \ell_k \) with \( \ell_k > n_k \) such that
\[
f^{\ell_k}(x) \in U(x_0, \varepsilon'(r)) \cap F^u_{\varepsilon} \cap D^u_{m_1} \quad (k \geq 1).
\]
(13)
Since
\[
\mu^u_x(f^{-\ell_k}(B^u_k(n_k))) \geq \mu^u_x(f^{-\ell_k}(\cap_{k=m_1}^\infty B^u_k(n_k))) = \mu^u_x(f^{-\ell_k}(\cap_{k=m_1}^\infty B^u_k(n_k))) \geq 1 - 2^{-m_1/2} \geq 1 - 2^{-m_0/2} \quad (k \geq m_1)
\]
by (12), we have
\[
\mu^u_x(U_j \cap f^{-\ell_k}(B^u_k(n_k))) \geq \mu^u_x(U_j) - 2^{-m_0/2} > 0 \quad (k \geq m_1).
\]
Then, by the definition of \( \{\mu^u_x| x \in M\} \), we can choose
\[
z_j = z_j(k) \in U_j \cap f^{-\ell_k}B^u_k(n_k)
\]
with \( \eta^u(x) = \eta^u(z_j) \) for \( j = 1, 2 \) and \( k \geq m_1 \). Thus we have \( \eta^u(f^{\ell_k}(x)) = \eta^u(f^{\ell_k}(z_j)) \) and so \( \mu^u_x(f^{\ell_k}(z_j)) = \mu^u_x(f^{\ell_k}(z_j)) \). Since \( f^{\ell_k}(z_j) \in B^u_k(n_k) \subset B^u_k(\ell_k) \), by (11) we have
\[
\rho(\mu^u_x(f^{\ell_k}(z_j)), \mu^u_x(f^{\ell_k}(z_j))) = \rho(\mu^u_x(f^{\ell_k}(z_j)), \mu^u_x(f^{\ell_k}(z_j))) \leq 1/k \leq 1/m_0 \leq 1/2M,
\]
(14)
and
\[
diam(f^{-\ell_k+[\ell_k/2]}\xi^u(f^{\ell_k-[\ell_k/2]}(z_j))) \leq 1/k
\]
(15)
for \( j = 1, 2 \) and \( k \geq m_1 \). Thus, by (9), (10), (13) and (14),
\[
\mu^u_x(f^{-\ell_k}U(y_i, r)) = \mu^u_x(f^{\ell_k}(z_j)) (U(y_i, r)) > \delta.
\]
Since \( \text{supp}(\mu^u_x) \subset \xi^u(z_j) \), we have
\[
\xi^u(z_j) \cap f^{-\ell_k}U(y_i, r) \neq \emptyset
\]
for \( 1 \leq i, j \leq 2 \) and \( k \geq m_1 \). For \( k \) large enough, by (15) we may assume
\[
z_j \in f^{-\ell_k+[\ell_k/2]}\xi^u(f^{\ell_k-[\ell_k/2]}(z_j)) \subset U_j.
\]
Therefore
\[
\Delta U_j \cap f^{-\ell_k}U(y_i, r) \supset f^{-\ell_k+[\ell_k/2]}\xi^u(f^{\ell_k-[\ell_k/2]}(z_j)) \cap f^{-\ell_k}U(y_i, r)
\]
\[
\supset \xi^u(z_j) \cap f^{-\ell_k}U(y_i, r) \neq \emptyset
\]
for $1 \leq i, j \leq 2$ and $k$ large enough. Take

$$b_{i,j} = b_{i,j}(k) \in f^{-k}(U(y_i, r)) \cap U_j$$

for $1 \leq i, j \leq 2$. Then we have that for $1 \leq i, j \leq 2$

$$b_{i,j} \in U_j, \quad d(f^k(b_{1,1}), f^k(b_{2,2})) > \tau \quad \text{and} \quad d(f^k(b_{1,1}), f^k(b_{1,2})) < r = 1/n$$

by (8). This implies that $\text{supp}(\mu^u_x)$ is a $*$-chaotic set for $x \in K = \bigcap_{n \geq 1} K_{1/n}$.

\[\square\]

5 Proof of Theorem C

The purpose of this section is to show Theorem C. Let $f$ be a $C^2$-diffeomorphism of a closed $C^\infty$-manifold $M$ and let $\mu$ be an ergodic SRB measure. As described in §1 the Pesin's formula holds: i.e. $h_\mu(f) = \sum_{\lambda_i > 0} \lambda_i m_i$. Thus we have $h_\mu(f) = \max \{\lambda_i\} > 0$ because $\mu$ satisfies the condition (A) mentioned in §1. Therefore, by Theorem B, $W^u(x)$ is a $*$-chaotic set for $\mu$-almost all $x \in M$.

To show that $W^u(x)$ is a $*$-chaotic set we need the following lemma.

Lemma 5.1 ([8], Corollary 6.1.4) Let $\mu$ be an ergodic measure satisfying Pesin's formula, let $\xi^u$ be as in Lemma 2.1 and let $\psi$ be the density of $\mu^u_x$ with respect to $m^u_x$. Then at $\mu$-almost all $x$, $\psi$ is a strictly positive function on $\xi(x)$ and $\log \psi$ is Lipschitz along $W^u$-leaves.

Let $\eta^u$ and $\{\mu^u_x | x \in M\}$ be as in §2. Then, by Proposition 2, $\text{supp}(\mu^u_x)(\subset W^u(x))$ is a $*$-chaotic set for $\mu$-almost all $x \in M$. Therefore, to obtain the conclusion it suffices to show the following.

Proposition 3 If $\mu$ is an SRB measure, then $\text{supp}(\mu^u_x) = \overline{W^u(x)}$ for $\mu$-almost all $x \in M$.

Proof. We first show that $\xi^u(x) \subset \text{supp}(\mu^u_x)$ for $\mu$-almost all $x \in M$. Since $\xi^u$ is finer than $\eta^u$, for $\mu$-almost all $z \in M$

$$\int \mu^u_x(\text{supp}(\mu^u_x))d\mu^u_z(x) = \mu^u_z(\text{supp}(\mu^u_x)) = 1.$$ 

Then $\mu^u_z(\text{supp}(\mu^u_x)) = 1$ for $\mu_z$-almost all $x$. Since $\text{supp}(\mu^u_z)$ is closed, by Lemma 5.1 we have that

$$\xi^u(x) \subset \text{supp}(\mu^u_z) = \text{supp}(\mu^u_x)$$

for $\mu^u_x$-almost all $x$. Therefore $\xi^u(x) \subset \text{supp}(\mu^u_x)$ for $\mu$-almost all $x$.

Since $f_*\mu^u_x = \mu^u_{fx}$ for $\mu$-almost all $x$, we have $f(\text{supp}(\mu^u_x)) = \text{supp}(\mu^u_{fx})$ for $\mu$-almost all $x$. By Lemma 2.1 (d)

$$W^u(x) = \bigcup_{n=0}^\infty f^n\xi^u(f^{-n}(x))$$

$c \bigcup_{n=0}^\infty f^n(\text{supp}(\mu^u_{f^{-n}(x)}))$

$c \text{supp}(\mu^u_x)$

for $\mu$-almost all $x \in M$. Therefore $\text{supp}(\mu^u_x) = \overline{W^u(x)}$ for $\mu$-almost all $x \in M$.

$\square$
References


