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# ベキ乗された平面グラフの彩色問題

## Coloring Powers of Planar Graphs

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**摘要:** We give nontrivial bounds for the chromatic number of power graphs $G^k$ of a planar graph $G$. In particular, we show that they can be colored with $O(\Delta^{\lfloor k/2 \rfloor})$ colors, which is best possible, and give 2-approximation for square graphs and $O(1)$-approximation for cubic graphs.

キーワード: graph coloring, distance-$k$ coloring, frequency allocation, power graphs, inductiveness.

## 1 Introduction

The $k$-th power $G^k$ of a graph $G$ is defined on the same set of vertices as $G$, and has an edge between any pair of vertices of distance at most $k$ in $G$. The topic of this paper is the coloring of power graphs, or equivalently coloring the underlying graphs so that vertices of distance at most $k$ receive different colors. We focus primarily on the planar case, long the center of attention for graph coloring. We upper-bound the chromatic number by the *inductiveness* of the graph, $\mathrm{ind}(G)$, defined to be $\max_{H \subseteq G}\{\min_v(d_H(v))\}$, where $H$ runs through all the subgraphs of $G$. Inductiveness leads to an ordering of the vertices, $\{v_1, \ldots, v_n\}$, such that the pre-order of any $v_i$, $d^+(v_i) = |\{v_j \in N_G(v_i) : j > i\}|$, is at most $\mathrm{ind}(G)$.

The problem of coloring squares of graphs has been studied recently for its applications to frequency allocation. Transceivers in a radio network communicate using channels at given radio frequencies. Graph coloring formalizes this problem well when the constraint is that nearby pairs of transceivers cannot use the same channel due to interference. However, if two transceivers are using the same channel and both are adjacent to a third station, a clashing of signals is experienced at that third node. This can be avoided by additionally requiring all neighbors of a node to be assigned different colors. That implies that vertices of distance at most two must receive different colors, which is equivalent to coloring the square of the underlying network. Another application of this problem, from a completely different direction, is that of approximating certain Hessian matrices, see [9].

Observe that neighbors form a clique in the square of the graph. Thus, the minimum number of colors needed to color any square graph is at least $\Delta + 1$, where $\Delta = \Delta(G)$ is the maximum degree of the original graph. As a result, the number of colors our algorithms use on power graphs will necessarily be a function of $\Delta$, and we are particularly interested in the asymptotic behavior.

The first reference on coloring squares of planar graphs is by Wegner [14], who gave bounds on the clique number of such graphs. In particular, he gave an instance for which the clique number is at least $3\Delta/2 + 1$ (which is largest possible), and conjectured this to be an upper bound on the chromatic number, for $\Delta$ large. Some work has been done on the case $\Delta = 3$, as listed in [5, Problem 2.18].

McCormick [9] showed that the problem of coloring the power of a graph is NP-complete, for any fixed power, and a later proof was given by Lin and Skiena [8]. McCormick gave a greedy algorithm that gives a $O(\sqrt{n})$-approximation for squares of general graphs. Heggernes and Telle [4] showed that determining if the square of a cubic graph can be colored with 4 colors or less is NP-complete, while it is easily determined if 3 colors suffice.

Ramanathan and Lloyd [13, 12] showed the problem of coloring squares of planar graphs to be NP-complete. They also gave an algorithm with a performance ratio of 9 and no better. This was the best previously result known for squares of planar graphs. More generally, they showed a constant approximation on graphs of constant inductiveness, and a $O(q)$-ratio for graphs of inductivensss $q$. Krumke, Marathe and Ravi [7] showed more precisely that the ratio is $2q-1$. They also gave a polynomial algorithm for graphs of bounded treewidth and bounded degree, and used that to give a 2-approximation for bounded-degree planar graphs.

This paper attempts to further the knowledge on the colorability and inductiveness of powers of planar and general graphs. We first show that for large values of $\Delta$, squares of planar graphs are $9\Delta/5+1$-inductive, implying a $9\Delta/5+2$-coloring. This is the tightest possible, since there are graphs attaining this bound. We combine this with previous results for bounded-degree graphs to obtain a 2-approximation for coloring that holds for all values of $\Delta$.

We next show that the power $G^k$ of a planar graph $G$ is $O(\Delta^{\lfloor k/2 \rfloor})$-inductive, for any $k \geq 1$. This gives an asymptotically tight algorithmic bound for the chromatic number of the power graph. In particular, this yields the first constant factor approximation for coloring cubes of planar graphs.

Finally, we consider the problem of approximately coloring powers $G^k$ of general graphs. We give a construction showing that the problems are hard to approximate within $\Omega(n^{1/2-\epsilon})$, for any $\epsilon > 0$ and any $k \geq 2$. This nearly matches the bound known for $k$ even. On the other hand, for $k$ odd, we give an $O(n^{1/2+1/(2k-4)})$-approximation. This implies that odd powers are a rare class of graphs where coloring is significantly easier than finding independent sets, since the latter problem is hard for this class within $\Omega(n^{1-\epsilon})$ factor, for any $\epsilon > 0$ [3].

Note the fine distinction between coloring the power graph $G^k$, and finding a *distance-$k$* coloring of $G$. The resulting coloring is naturally the same. However, in the latter case, the original graph is given. While it is easy to compute the power graph $G^k$ from $G$, Motwani and Sudan [10] showed that it is NP-hard to compute the $k$-th root $G$ of a graph $G^k$. All of the algorithm presented in this paper work without knowledge of the underlying root graph.

The rest of the paper is organized as follows. We bound the inductiveness of squares of planar graphs in Section 2, and general powers of planar graphs in Section 3. We consider the implications of these bounds to approximate colorings of powers of planar graphs in Section 4, and give bounds on the approximability of coloring powers of general graphs.

NOTATION: The degree of a vertex $v$ within a graph $G$ is denoted by $d_G(v)$ or simply by $d(v)$ when there is no danger of ambiguity. The maximum degree of $G$ is denoted by $\Delta = \Delta(G)$. For a vertex $V$ denote by $d_k(v)$ the degree of $v$ in $G^k$. Distance between two vertices $u$ and $v$ in a graph is the number of edges on the shortest path from $u$ to $v$, and is denoted by $d_G(u,v)$. Let $G[W]$ denote the subgraph of $G$ induced by vertex subset $W$.

## 2 Squares of planar graphs

We first take a look at the main technique we use to derive bounds on the inductiveness of a square graph (and more generally, power graphs). The argument used e.g. to show that planar graphs are 5-inductive is the following. By Euler's theorem, any planar graph contains a vertex of degree at most 5. Place one such node first in the inductive ordering, and remove it from the graph. Now the remaining graph is planar, so inductively we obtain a 5-inductive ordering.

The bound of 5 on the minimum degree of a planar graph also implies that squares of planar graphs are of minimum degree at most $5\Delta$. This would seem to imply a $5\Delta$-ordering. However, when a vertex is deleted from the graph, its incident edges are deleted as well, so that two remaining vertices that originally were of distance 2 apart, may not stay that way. Namely, the problem is that an induced subgraph does not preserve the paths of length two between vertices within the subgraph. Our solution is to replace the *deletion* of vertex by the *contraction* of an incident edge.

The *contraction* of an edge $uv$ in graph $G$ is the operation of collapsing the vertices $u$ and $v$ into a new vertex, giving the graph $G/uv$ defined by $V(G/uv) = V(G) \setminus \{v\}$ and $E(G) = \{ww \in E(G) | w, w' \neq v\} \cup \{uw | vw \in E(G)\}$. Observe that if $G$ is planar, then $G/uv$ is also planar. This is a property of various classes of graphs that are *closed under minor operations*. By the classic theorem of Kuratowski, planar graphs are precisely those graphs for which repeated contractions do not yield supergraphs of $K_5$ or $K_{3,3}$. Minor closedness holds for various other classes of graphs, e.g. partial-$k$ trees, but not $d$-inductive graphs in general.

Since our bounds on the inductiveness are functions of $\Delta$, it is imperative that the contraction operations do not increase the maximum degree. In summary, in order to show that a power graph $G^k$ is $q$-inductive, where $q$ is necessarily a function of the maximum degree $\Delta$, we show the existence of a vertex $v \in V(G^k) = V(G)$ such that

- $d_k(v) \leq q$, and
- $v$ has a neighbor $u$ such that $d(u) + d(v) - 2 \leq \Delta$. (1)

If such an edge $uv$ exists, then the contraction of $uv$ in $G$ yields yield a simple planar graph $G/uv$ whose distance function is dominated by the one on $G$ (i.e. distances in $G/uv$ are at most those in $G$). Further, by the second condition, the maximum degree of $G/uv$ stays at most $\Delta$.

We illustrate the technique first by a simple example. A theorem of Kotzin [6] states that a maximal planar graph contains an edge $uv$ such that $d(u) + d(v) \leq 13$. We first argue that this implies that any maximal planar graph $G$ with $\Delta(G) \geq 11$ is $5\Delta + 6$-inductive. We find an edge as guaranteed by Kotzin's theorem, select the vertex of lower degree, contract the edge, and inductively apply the argument on the resulting maximal planar graph. The degree of the lower degree vertex $u$ is at most 6, and that of $v$ at most $13 - d(u)$ (including the edge $uv$), thus the number of distance-2 neighbors of $u$ is at most $(d(u) - 1)\Delta + (13 - d(u) - 1) \leq 5\Delta + 6$. The degree of the new contracted edge is at most $(d(u) - 1) + (d(v) - 1) \leq 11$, hence maximum degree does not increase. The contracted graph is also maximally planar, hence this yields a $5\Delta + 6$-inductive ordering of $G^2$.

For a non-maximal planar graph $G$, we first form an arbitrary maximal supergraph $G'$, find an inductive ordering as above, and use that to color $G^2$. Consider a vertex $u$ and let $G'_v$ be the contracted subgraph when $v$ was selected. $u$ had at most 6 neighbors in $G'_u$ (including $v$ of degree at most $13 - d_{G'_v}(u)$). Each neighbor $w$ was either a contracted node of degree at most 11, or a node that had not received any new neighbors. In the latter case, the degree of $w$ in $G$ is at most $\Delta(G)$; the other neighbors of $w$ do not count as neighbors of $u$ in $G^2$, unless it is through some other path. Hence, we have a $5\Delta + 6$-inductive ordering of the square of any planar graph with $\Delta \geq 11$. We can use that to improve the $9\Delta$ inductiveness bound of [13] for every value of $\Delta$. For smaller values of $\Delta$, we know that any graph is trivially $\Delta^2$-inductive, and the above also gives us an upper bound of 61. In particular, we have that the square of any planar graph is $8\Delta$-inductive.

We now turn to the main result of this section, which is that when $G$ is planar and $\Delta$ large enough, then $G^2$ is $\lfloor \frac{9\Delta(G)}{5} \rfloor + 1$-inductive. The following lemma is the key to this result.

**Lemma 2.1** *Let $G$ be a simple planar graph of maximum degree $\Delta \geq 26$. Then there exists a vertex $v \in V(G)$ satisfying one of the following conditions:*

1. *$d(v) \leq 25$ and at most one neighbor of $v$ has degree $\geq 26$.*
2. *$d_2(v) \leq \lfloor \frac{9}{5}\Delta \rfloor + 1$ and only two neighbors of $v$ in $G$ have degree $\geq 26$.*

*Proof.* We assume that we have a fixed planar embedding of $G$, and hence $G$ is a plane graph. Let $V_h = \{v \in V(G) : d(v) \geq 26\}$ and $V_l = V(G) \setminus V_h$. If there is a vertex in $V_l$ with at most one neighbor in $V_h$, then we are done, so assume the contrary.

Call a cycle of four vertices in $G$ *forbidden*, if exactly two opposite vertices of the cycle are in $V_h$ and the enclosed region formed by the cycle in the plane properly contains at least one vertex in $V_h$. If $G$ contains a forbidden 4-cycle then let $G'$ be the subgraph of $G$ induced by the region bounded by a minimal such 4-cycle. (Here, minimal means that no other 4-cycle is inside.) If $G$ contains no such cycle then let $G'$ be $G$.

Consider now the multigraph $H$ with vertex set $V_h \cap V(G')$ and with colored edges defined as follows. For each edge $uw$ in $E(G')$ with both $u, w \in V_h$ connect $u$ and $w$ with a red edge. For each vertex $v \in V_l$ adjacent to $u$ and $w \in V_h$ in $G'$ and to no other vertex in $V_h$, connect $u$ and $w$ in $H$ with a green edge. Finally for $v \in V_l$ adjacent to $u_1, u_2, \ldots, u_k \in V_h$ in $G'$ in a clockwise order for $k \geq 3$, connect $u_1$ to $u_2$, $u_2$ to $u_3, \ldots, u_{k-1}$ to $u_k$ and $u_k$ to $u_1$ with blue edges in $H$.

Since $G$ is planar we note that $H$ is also a planar multigraph, and hence, we can assume we have a drawing of $H$ in the plane such that

1. The vertices of $H$ have the same configuration as they have in the plane graph $G$.
2. For every pair $\{u, w\}$ of vertices of $H$ connected by green or blue edges, their order is the same as the order of the corresponding vertices of $V_l$.

By our assumption there is no vertex in $V_l$ with at most one neighbor in $V_h$ in $G$ and hence in $G'$. Therefore, the degree of a vertex in $H$ is at least that in $G'$.

Using Euler's formula for planar graphs, it is easy to show that there are at least three vertices of $V(H) = V_h \cap V(G')$ with at most 5 neighbors in $H$, and hence there is such a vertex $v \in V(H) \subseteq V(G')$ that is not on the 4-cycle defining $G'$ (if $G'$ was so defined.)

Consider now a neighbor $u$ of this $v \in V(H)$. Let $m_{uv}$ be the multiplicity of the edge $uv$ in $H$. By our definition of $G'$ there are at most two blue edges connecting $u$ and $v$ since the third one would imply a forbidden 4-cycle within $G'$. Also, there is only one red edge connecting $u$ and $v$. Hence, if $m_{uv} \geq 4$ there are at least $m_{uv} - 3 \geq 1$ green edges connecting $u$ and $v$ in $H$. We note that all the blue and green edges connecting $u$ and $v$ in $H$ correspond to different vertices of $V_l$ in $G'$.

Let $c_{uv}$ be the number of common neighbors of $u$ and $v$ in $G'$ (if $u$ and $v$ are connected in $G'$, then both $u$ and $v$ are counted as well.) The combined closed neighborhood of $u$ and $v$ in $G'$ has precisely $(d_{G'}(u)+1)+(d_{G'}(v)+1)-c_{uv}$ vertices. Since $m_{uv} \leq c_{uv}$ (in fact, $m_{uv}+1 \leq c_{uv}$, if $u$ and $v$ are connected in $G'$), we have that this closed neighborhood of $u$ and $v$ in $G'$ is bounded above by $(d_{G'}(u)+1)+(d_{G'}(v)+1)-m_{uv}$ vertices.

Letting $w$ run through all the neighbors of $v$ in $H$, we note that $\sum_w m_{vw} = d_H(v) \geq d_{G'}(v)$. Since $v$ has at most 5 neighbors in $G'$, there must be a neighbor $u$ of $v$ such that $m_{uv} \geq \lceil d_{G'}(v)/5 \rceil$ and hence the combined neighborhood of $u$ and $v$ is at most

$$
\begin{aligned}
& d_{G'}(v) + d_{G'}(u) + 2 - \left\lceil \frac{d_{G'}(v)}{5} \right\rceil \\
= & \left\lfloor \frac{4d_{G'}(v)}{5} \right\rfloor + d_{G'}(u) + 2 \\
\leq & \left\lfloor \frac{9\Delta(G')}{5} \right\rfloor + 2 \\
\leq & \left\lfloor \frac{9\Delta(G)}{5} \right\rfloor + 2.
\end{aligned}
$$

Since $v \in V(H) \subseteq V_h$ is in the interior of the 4-cycle defining $G'$, we have $d_{G'}(v) = d_G(v) \geq 26$

and hence $m_{uv} \geq \lceil d_{G'}(v)/5 \rceil \geq 6$. Hence, $u$ and $v$ are connected by at least 5 nonred edges. Choose 5 consecutive nonred edges between $u$ and $v$, and let $z_1, z_2, \ldots, z_5$ be the neighbors of $u$ and $v$ in $G'$, in a clockwise order, corresponding to these chosen nonred edges. The edges corresponding to $z_2$, $z_3$ and $z_4$ are green, since otherwise we would have a forbidden 4-cycle within $G'$.

Now, if $z_i$, $i \in \{2,3,4\}$, is adjacent to a vertex in $V_l$ that does not represent a green nor blue edge between $u$ and $v$, then by our assumption that every vertex in $V_l$ has at least two neighbors in $V_h$ in the graph $G'$, one of these neighbors in $V_h$ must be distinct from $u$ and $v$ and therefore contained in the region formed by the 4-cycle $(u, z_{i-1}, v, z_{i+1})$. Again this would imply a forbidden 4-cycle and contradict our definition of $G'$.

Therefore, the only vertices of $V_l$ that $z_i$ can possibly be adjacent to in $G'$ are $z_{i-1}$ and $z_{i+1}$. In particular, the neighbors of $z_3$ in $G'$ are among $\{u, v, z_2, z_4\}$, and the neighbors of $z_2$ and $z_4$ are among $\{u, v, z_1, z_3\}$ and $\{u, v, z_3, z_5\}$ respectively. In any case, the combined neighborhood of $z_2$ and $z_4$ is contained in the closed combined neighborhood of $u$ and $v$. Hence the number of vertices of distance at most 2 from $z_3$ are at most $\lfloor \frac{9\Delta(G)}{5} \rfloor + 2$ (including $z_3$ itself). □

**Theorem 2.2** *If $G$ is a planar graph with maximum degree $\Delta \geq 749$, then $G^2$ is $\lfloor \frac{9}{5}\Delta \rfloor + 1$-inductive.*

*Proof.* Assume that $\Delta \geq 25 + 25 - 2$ and that we have a vertex $v$ of $G$ which satisfies the first condition of Lemma 2.1. If $v$ has a neighbor $u$ of degree 25 or less, then $d_2(v) \leq 600 + \Delta$, and moreover $d(v)+d(u)-2 \leq \Delta$. If $v$ has no neighbor of degree 25 or less, then it has only one neighbor $u$. In this case $d_2(v) \leq \Delta$ and $d(v)+d(u)-2 \leq \Delta$.

In the proof of Lemma 2.1 we assumed that there is no vertex in $V_l$ with at most one neighbor of $V_h$. In that case there is a vertex of $G$, called $z_3$ in the last paragraph of the proof, with $d_2(z_3) \leq \lfloor \frac{9}{5}\Delta \rfloor + 1$. Also, $z_3$ has at most two neighbors $z_2$ and $z_4$ of $V_l$. If $z_3$ has no neighbors of $V_l$ (that is, is connected to neither $z_2$ nor $z_4$), then since the only neighbors of $z_3$ in $V_h$ are $u$ and $v$, we have $d(z_3) + d(v) - 2 = d(z_3) + d(u) - 2 \leq \Delta$. If $z_3$ has a neighbor $z_1$ or $z_2$ of $V_l$, say $z_1$, then, $d(z_3) + d(z_1) - 2 \leq \Delta$.

In any case, we see that we can always find a vertex $w$ of $G$ with $d_2(w) \leq \max\{600+\Delta, \lfloor \frac{9}{5}\Delta \rfloor + 1\}$, and such that $w$ has a neighbor $w'$ with $d(w)+d(w') - 2 \leq \Delta$. □

It turns out that $\frac{9}{5}\Delta+1$ is a *sharp* upper bound.

**Observation 2.3** *For any $\Delta$, there exists a planar graph $G$ of maximum degree $\Delta$ such that $G^2$ is of minimum degree $\frac{9}{5}\Delta + 1$.*

*Proof.* Take a 5-regular planar graph (e.g. the graph corresponding to the regular icosahedron), and add to each edge $k$ parallel paths of length 2. Then, $\Delta = 5(k+1)$. The two vertices adjacent to a given degree-2 vertex $v$ have a common neighborhood of size $k$, and the union of their closed neighborhoods is thus of size $(\Delta+1)+(\Delta+1)-(k+1) = \frac{9}{5}\Delta + 2$ (including $v$ itself.) □

## 3 General powers of planar graphs

In this section we prove the following theorem.

We summarize our explorations in the follow theorem.

**Theorem 3.1** *Let $G$ be a planar graph with maximum degree $\Delta$. For any $k \geq 1$, $G^k$ is $O(\Delta^{\lfloor k/2 \rfloor})$-colorable. Also, there is a family of graphs that attains this bound. This bound is also asymptotically tight for the clique number, inductiveness, arboricity, and minimum degree of $G^k$.*

Let us first give a construction that matches the bound of the theorem. Given $k, \Delta \geq 1$, consider the tree $T$ of height $\lfloor k/2 \rfloor$ where internal vertices have degree $\Delta$. The number of vertices in $T$ is

$$
\begin{aligned}
D_{\Delta,k} &= 1 + \Delta + \Delta(\Delta-1) + \cdots + \Delta(\Delta-1)^{\lfloor k/2 \rfloor - 1} \\
&= \frac{\Delta(\Delta-1)^{\lfloor k/2 \rfloor} - 2}{\Delta - 2}.
\end{aligned}
$$

Observe that $T^k$ is a complete graph, thus thus $\chi(T^k) = D_{\Delta,k}$.

We now turn to proving the upper bound of the theorem. The rest of this section is divided up into several subsections, each of which deals with necessary tools to complete the proof of our main Theorem 3.1 below. First let us set forth some useful terminology.

**Notation** A *$k$-path* is a path of length exactly $k$. A *$(k, \leq)$-path* is a path of length $k$ or less. If $u$ and $v$ are two given vertices then an *$(k; u, v)$-path* is a path between $u$ and $v$ of length exactly $k$, and finally a *$(k, \leq; u, v)$-path* is a path between $u$ and $v$ of length $k$ or less. A vertex $w$ is called a *$(k, \leq; u, v)$-link* if $w$ is on every $(k, \leq; u, v)$-path. $N(v)$ will denote the set of the neighbors of $v$ in $G$, and $N[v]$ the closed neighborhood of $v$, that is $N(v) \cup \{v\}$.

**Definition 3.2** *For a simple planar graph $G$, an integer $k \geq 1$ and a subset $U \subseteq V(G)$, denote by $\mathcal{P}_k(G; U)$ the set of all $W$ with $U \subseteq W \subseteq V(G)$ and such that any two vertices in $U$ connected by a $(k, \leq)$-path in $G$, are also connected by a $(k, \leq)$-path in $G[W]$.*

We will derive the following bound on the size of each minimal element of $\mathcal{P}_k(G; U)$, that is linear in $|U|$, for any fixed $k$.

**Theorem 3.3** *There exists an integer sequence $(d_k)_{k\geq 1}$ with $d_k \leq 10^{k-1}$, such that for every connected simple planar graph $G$, every integer $k \geq 1$ and every $U \subseteq V(G)$, each minimal element of $\mathcal{P}_k(G; U)$ has at most $d_k|U|$ vertices.*

Let us get a better grasp of this by examining the first two cases $k = 1, 2$. Clearly $U$ itself is the only minimal element in $\mathcal{P}_1(G; U)$, for any $U$ and $G$, thus $d_1 = 1$.

For the case $k = 2$, let $W$ be a minimal element in $\mathcal{P}_2(G; U)$, for a given $U$. We form a graph $G'$ on vertex set $U$ as follows. For each $w \in W \setminus U$, select a pair $u_1, u_2$ in $U$ for which $w$ is a 2-link, and add an edge $u_1u_2$ to $G'$. Note that $G'$ is a simple graph, since each $w$ was the only path $G[W]$ between the endpoints of the corresponding edge in $G'$, and it is planar since it is an edge contraction of a subgraph of $G[W]$ (where all vertices in $W \setminus U$ are of degree 2). By Euler's formula, $|E(G')| \leq 3|U| - 6$. Since each edge of $G'$ corresponds to a distinct vertex of $W \setminus U$, we have that $|W| \leq 4|U| - 6$. Thus, $d_2 \leq 4$.

Before proving the general case of Theorem 3.3, let us continue and derive our conclusions.

**Arboricity** For a graph $G$, define its *arboricity* as $\mathrm{arb}(G) = \max_{H \subseteq G} \left\lceil \frac{|E(H)|}{|V(H)|-1} \right\rceil$. By the Nash-Williams theorem [11] there are $\mathrm{arb}(G)$ edge-disjoint subforests of $G$ that cover all the edges of $G$.

Arboricity is closely related to inductiveness.

**Lemma 3.4** *For any graph $G$, we have $arb(G) \leq ind(G) < 2\, arb(G)$.*

*Proof.* Assume first $\mathrm{ind}(G) = q$. We will show that $E(G)$ can be partitioned into $q$ forests. Given a linear arrangement of the vertices, such that the pre-order is at most $q$, we arbitrarily color the $q$ edges from a vertex $v_i$ to later vertices with $q$ colors. In this way, each color class is acyclic – since two edges of the same color cannot have the same first-labeled endpoint – and thus a forest. Therefore $\mathrm{arb}(G) \leq q$, proving the first inequality.

For the other inequality, let $\mathrm{ind}(G) = q$. Let $H$ be a subgraph of $G$ such that $\min_v(d_H(v)) = q$. Since $2|E(H)| = \sum_v d_H(v) \geq q|V(H)|$, we have $\mathrm{arb}(G) > |E(H)|/|V(H)| \geq q/2$, which completes our lemma. □

From Lemma 3.4 we have in particular from [13] that $\mathrm{arb}(G^2) \leq 9\Delta$.

Consider now the power graph $G^k$ of $G$. For a vertex set $U \subseteq V(G)$, let $E^k(U)$ be the edgeset of the subgraph of $G^k$ induced by $U$. Then, the arboricity of $G^k$ is

$$\mathrm{arb}(G^k) = \max_{U \subseteq V(G)} \left\lceil \frac{|E^k(U)|}{|U| - 1} \right\rceil .$$

Note that every edge in $E^k(U)$ is represented by at least one $(k, \leq)$-path between vertices of $U$. Let $W_U \in \mathcal{P}_k(G; U)$ be a minimal element. By Theorem 3.3, $|W_U| \leq 10^{k-1}|U|$ and we have that $|E^k(U)|$ is less than the number of $(k, \leq)$-paths in $G[W_U]$. We note that all $(k, \leq)$-paths in $G[W_U]$

connecting two vertices of $U$, except the $(2, \leq)$-paths, are represented by an edge $uv$ in $G[W_U]^{k-2}$ together with edges $e$ and $e'$ in $G$, with one endpoint $u$ and $v$ respectively. Hence,

$$|E^k(U)| \leq |E^2(U)| + \sum_{uv \in G[W_U]^{k-2}} d(u)d(v). \quad (2)$$

**Degree products over edges** The following lemma will be used in our inductive argument.

**Lemma 3.5** *If $G$ is a simple graph of maximum degree $\Delta$ and $F$ is a forest with $V(F) \subseteq V(G)$, then*

$$\sum_{uv \in E(F)} d_G(u)d_G(v) \leq 2\Delta |E(F)|.$$

*Proof.* For any graph $H$, with $V(H) \subseteq V(G)$ let

$$S(H) = \sum_{uv \in E(H)} d_G(u)d_G(v).$$

For each tree $T$ of $F$, direct its edge away from an arbitrarily chosen root. Thus, $T$ becomes a directed tree $T^d$ in which every vertex but the root has indegree one. For each arc $\vec{uv}$ in $T^d$ bound the summand $d_G(u)d_G(v)$ from above by $\Delta d_G(v)$. Then,

$$\begin{aligned} S(T) &\leq \sum_{\vec{uv} \in E(T^d)} \Delta d_G(v) = \Delta \left( \sum_{v \in V(T) \setminus \{r\}} d_G(v) \right) \\ &\leq \Delta \left( \sum_{v \in V(T)} d_G(v) \right). \end{aligned}$$

As $F$ is a disjoint union of trees $T_i$, we have that $S(F) = \sum_{i=1}^{k} S(T_i) \leq 2\Delta|E(F)|$. □

**Arboricity of power graphs.** We now want to show inductively that there is a sequence $(\alpha_k)_{k=1}^{\infty}$ such that for every planar $G$ with maximum degree $\Delta$ we have

$$\text{arb}(G^k) \leq \alpha_k \Delta^{\lfloor k/2 \rfloor}. \quad (3)$$

We know at this point that $\alpha_1 = 3$ and $\alpha_2 = 9$ satisfy (3). We proceed by induction and consider general $k \geq 3$. By (2) we get

$$|E^k(U)| \leq 9\Delta(|U| - 1) + \sum_{uv \in G[W_U]^{k-2}} d(u)d(v),$$

where now $W_U$ is our minimal element of $\mathcal{P}_k(G; U)$. By the induction hypothesis we have that $\text{arb}(G[W_U]^{k-2}) \leq \alpha_{k-2}\Delta^{\lfloor \frac{k-2}{2} \rfloor} = a_{k-2}$ and hence by the Nash-Williams theorem [11] there are $a_{k-2}$ edge-disjoint forests $F_1, F_2, \ldots, F_{a_{k-2}}$ covering all the edges of $G[W_U]^{k-2}$. By Lemma 3.5, and Theorem 3.3,

$$\begin{aligned} \sum_{uv \in G[W_U]^{k-2}} d(u)d(v) &= \sum_{i=1}^{a_{k-2}} \sum_{uv \in E(F_i)} d(u)d(v) \\ &\leq \sum_{i=1}^{a_{k-2}} 2\Delta|E(F_i)| \\ &\leq a_{k-2}(2\Delta(|W_U| - 1)) \\ &< 2a_{k-2}\Delta|W_U| \\ &\leq 2 \cdot 10^{k-1}\alpha_{k-2}\Delta^{\lfloor k/2 \rfloor}|U|. \end{aligned}$$

Since $k \geq 3$ and $\alpha_k \geq 1$, we can assume $|U| \geq 3$. Thus,

$$\begin{aligned} |E^k(U)| &\leq 9\Delta(|U| - 1) + 2 \cdot 10^{k-1}\alpha_{k-2}\Delta^{\lfloor k/2 \rfloor}|U| \\ &\leq 4 \cdot 10^{k-1}\alpha_{k-2}\Delta^{\lfloor k/2 \rfloor}(|U| - 1). \end{aligned}$$

Thus, $\text{arb}(G^k) \leq \alpha_k \Delta^{\lfloor k/2 \rfloor}$, where $\alpha_1 = 3, \alpha_2 = 9$ and $\alpha_k = 4 \cdot 10^{k-1}\alpha_{k-2}$. By an easy induction, we obtain the following lemma.

**Lemma 3.6** *If $G$ is a planar graph with a maximum degree $\Delta$, and $k \geq 1$ is an integer, then we have arb$(G^k) \leq \alpha_k \Delta^{\lfloor k/2 \rfloor}$, where $\alpha_k = 4^{k-1}10^{k^2/4}$.* □

Letting $\alpha_k$ be as in the previous lemma, we get by Lemma 3.4 the following corollary.

**Corollary 3.7** *For a simple planar graph $G$ and an integer $k \geq 1$, we have that $G^k$ is $2\alpha_k \Delta^{\lfloor k/2 \rfloor}$-inductive.* □

**Proof of Theorem 3.2** We have already proved the theorem in the case where $k \in \{1,2\}$. When considering the general case of $(k, \leq)$-paths, we proceed by induction on $k$ and assume $k \geq 3$. Let $U \subseteq V(G)$ be given. Let $W \in \mathcal{P}_k(G; U)$ be a minimal element. Note that every vertex $w \in W \setminus U$ is *nonremovable*, in that there is a pair of vertices $\{u_{w1}, u_{w2}\}$ in $U$ such that $w$ is a $(k, \leq; u_{w1}, u_{w2})$-link in $G[W]$.

Let $U' \subseteq W \setminus U$ be the set of vertices of $W$ that are connected to some vertex in $U$ by an edge. We want to show that there is a constant $c$ such that $|U'| \leq c|U|$. We can partition $U'$ as $U' = U'_1 \cup U'_2 \cup U'_3$, where

$$U'_1 = \{v \in U' : |N(v) \cap U| = 1\},$$
$$U'_2 = \{v \in U' : |N(v) \cap U| = 2\},$$
$$U'_3 = \{v \in U' : |N(v) \cap U| \geq 3\}.$$

When estimating the sizes of $U'_1$, $U'_2$ and $U'_3$, the easiest case to deal with is $U'_3$. By the following Lemma 3.8 we have that $|U'_3| \leq 2|U| - 4$.

**Lemma 3.8** *For a simple planar graph with vertex set $U \cup V$ such that every vertex in $V$ is connected to at least three vertices of $U$, we have that $|V| \leq 2|U| - 4$.*

*Proof.* The bipartite subgraph on $(U, V, E)$ has at most $2(|U| + |V|) - 4$ edges by Euler's formula, but at least $3|V|$ edges by the degree bound on $V$. □

The proof of the following bound on $U'_2$ is omitted for reasons of space.

**Lemma 3.9** $|U'_2| \leq 9|U|$.

We now derive the the final step towards completion of the proof of Theorem 3.3.

**Lemma 3.10** *For a minimal element $W$ of $\mathcal{P}_k(G; U)$, $|W| \leq 84 d_{k-2}|U|$.*

*Proof.* Let $U_1 \subseteq U$ be the set of vertices that have neighbors in $U'_1$. We now have the following partition

$$U'_1 = \bigcup_{u \in U_1} N_{U'_1}(u)$$

where $N_{U'_1}(u) = \{v \in U'_1 : uv \in E(G[W])\}$. Consider the planar graph $C[W]$ we get from $G[W]$ by contracting $N_{U'_1}[u]$ to a single vertex $u^*$, for each $u \in U_1$. Let $U^* = \{u^* : u \in U_1\} \cup (U \setminus U_1)$. If we let $U'' = W \setminus (U \cup U')$, then clearly $W$ is a disjoint union of $U, U'_1, U'_2, U'_3$ and $U''$. In view of this, $C[W]$ will become a graph whose vertices are a disjoint union of $U^*, U'_2, U'_3$ and $U''$. For convenience define a map $c : W \to U^* \cup U'_2 \cup U'_3 \cup U''$ by

$$c(w) = \begin{cases} u^* & \text{if } w \in N_{U'_1}[u], \text{ for } u \in U_1, \\ w & \text{otherwise.} \end{cases}$$

Note that every $(k, \leq)$-path between a pair of vertices of $U$ in $G[W]$ gives a $(k-2, \leq)$-path between a pair of vertices of $U^* \cup U'_2 \cup U'_3$.

Let us now show that every vertex of $U''$ is nonremovable in $C[W]$ when considering $(k-2, \leq)$-paths between pairs of vertices of $U^* \cup U'_2 \cup U'_3$. Let $u'' \in U''$. Since $u''$ is nonremovable in $G[W]$ there is a pair $u, u'$ of vertices of $U$ such that $u''$ is a $(k, \leq; u, u')$-link. Pick a fixed $(k, \leq; u, u')$-path $\gamma$ and let $v$ and $v'$ be the endpoints of $\gamma \setminus \{u, u'\}$. Now $u''$ is a $(k-2, \leq; v, v')$-link in $G[W]$, since otherwise $u''$ would not be a $(k, \leq; u, u')$-link in $G[W]$. That $u''$ is a $(k-2, \leq; c(v), c(v'))$-link in $C[W]$ can be seen as follows. If $u''$ is not such a link, then there is a $(k-2, \leq; c(v), c(v'))$-path $\gamma'$ not including the vertex $u''$ in $C[W]$. It then gives a $(k, \leq; u, u')$-path $\gamma''$ in $G[W]$ not including $u''$, which is a contradiction.

By induction hypothesis on $k$, we now have that the number of vertices of $C[W]$ are bounded, that is $|U^* \cup U'_2 \cup U'_3 \cup U''| \leq d_{k-2}|U^* \cup U'_2 \cup U'_3|$. By previous arguments and the fact that $|U^*| = |U|$, we have

$$|U^* \cup U'_2 \cup U'_3 \cup U''| \leq d_{k-2}(|U| + 9|U| + 2|U|) = 12 d_{k-2}|U|.$$

The only thing left to conclude our inductive argument is to show that $|W \setminus (U \cup U'_2 \cup U'_3 \cup U'')| = |U'_1| \leq c|U|$ for some constant $c$.

Let $u \in U$ be fixed. For each neighbor $v$ of $u$ in $G[W]$, let $p_u(v) \in U$ be a vertex such that $v$ is a $(k, \leq; u, p_u(v))$-link. We assume further that for a fixed $u$ and distinct $v$, all the $p_u(v)$ are distinct.

**Claim 3.11** *With the notation from above, for each neighbor $v$ of $u$ in $G[W]$, let $\gamma_v$ be a $(k, \leq; u, p_u(v))$-path. Except for the vertex $u$, all these paths are vertex-disjoint.*

*Proof.* Assume $\gamma_{v_1}$ and $\gamma_{v_2}$ have a common vertex $x$ other than $u$. Hence, for $i = 1, 2$, $\gamma_{v_i} =$

$\gamma_{ix}\beta_i$, where $\gamma_{ix}$ is the path from $u$ to $x$ along $\gamma_{v_i}$ and $\beta_i$ the path from $x$ to $p_u(v_i)$ along $\gamma_{v_i}$. If now $l(\gamma_{1x}) \leq l(\gamma_{2x})$, then $\gamma' = \gamma_{1x}\beta_2$ is a $(k, \leq; u, p_u(v_2))$-path not including the vertex $v_2$, contradicting the definition of $p_u(v_2)$. □

By Claim 3.11, all the $(k, \leq; u, p_u(v))$-paths from $u$ to each of the $p_u(v)$, are vertex disjoint. Therefore the number of vertices of $N_{U_1'}(u)$ is less than the number of edges going out of $u^*$ in the contracted graph $C[W]$. Since each edge in $C[W]$ connects to at most two vertices of $U^*$, we have that $|U_1'| \leq 2|E(C[W])|$. Since $|E(C[W])| \leq 3|V(C[W])| - 6$ and $V(C[W]) = U^* \cup U_2' \cup U_3' \cup U''$, we have

$$\begin{aligned} |W| &= |U^* \cup U_2' \cup U_3' \cup U''| + |U_1'| \\ &\leq 12d_{k-2}|U| + 2|E(C[W])| \\ &\leq 12d_{k-2}|U| + 6|U^* \cup U_2' \cup U_3' \cup U''| \\ &\leq 84d_{k-2}|U|. \end{aligned}$$

We see from the above display that $d_k = 84d_{k-2}$ is sufficient in the case for general $k$ provided that $d_{k-2}$ is known. Therefore the sequence $(d_k)_{k=1}^{\infty}$ defined inductively by $d_1 = 1$, $d_2 = 4$ and $d_k = 84d_{k-2}$ will give us the desired constants. A straightforward induction implies that $d_k \leq 10^{k-1}$, and hence we have Theorem 3.3. □

## 4 Approximating the chromatic number of power graphs

**Planar graphs** We can improve the best approximation factor known for coloring squares of planar graphs. Recall that since neighbors in $G$ must be colored differently in $G^2$, $\chi(G^2) \geq \Delta + 1$. Thus, for $\Delta \geq 748$, Corollary 3.7 yields a 1.8-approximation.

For constant values of $\Delta$, we can use a result of Krumke, Marathe and Ravi [7]. They stated a 3-approximation, but actually a 2-approximation easily follows from their approach. Hence, combined we obtain a 2-approximation for any value of $\Delta$.

Theorem 3.1 also immediately gives a $O(1)$-approximation to coloring cubes of planar graphs.

**Theorem 4.1** *The problem of coloring squares of planar graph has a 2-approximation, and coloring cubes of planar graphs has a constant approximation.*

**General graphs** We give upper and lower bounds on the approximability of coloring power graphs, for the case of general (not necessarily planar) graphs.

It is easy to see that coloring square graphs $G^2$ is $O(\sqrt{n})$-approximable. Namely, $G^2$ is $\min(\Delta^2, n-1)$-inductive, for a performance ratio of $\min(\Delta, n/\Delta) \leq \sqrt{n}$. Note that $G^{2t}$ are squares of $G^t$, and positive results for square graphs translate to all even powers. We can show that to be essentially tight.

We say that a problem is *hard to approximate* within a given factor, if the claim holds under the assumption that $NP \neq ZPP$, the class of problems with polynomial-time zero-error randomized algorithms.

**Theorem 4.2** *Let $\epsilon > 0$ and $d \geq 1$. Coloring power graphs $G^d$ is hard to approximate within $O(n^{1/2-\epsilon})$, even if $G$ is known.*

*Proof.* Consider first $d = 2$. Given a graph $G$ on $N$ vertices, we construct a graph $H$ on $n = N + N^2$ vertices, where $V(H) = \{v_i, u_{i,j} : 1 \leq i, j \leq n\}$, $E(H) = \{(v_i, u_{j,l}) : (v_i, v_j) \in E(G)\}$. Observe that if $G$ is $k$-colorable, then $H$ has a distance-2 coloring with $(k+1)N$ colors, which can be constructed by $k$-coloring each set $\{u_{1,j}, u_{2,j}, \ldots, u_{N,j}\}$ and coloring the $v$-vertices with additional $N$ colors. On the other hand, if $H$ has a distance-2 $qN$-coloring, then $G$ is $q \log n$-colorable, obtained by greedily using the largest distance-2 color class of $H$ as the first color in $G$ and recursing on the remaining graph. In fact, when $q = N^{\Omega(1)}$, then $G$ is $O(q)$-colorable.

Since it is hard to determine whether a given graph on $N$ vertices is $N^{\epsilon}$-colorable or $\Omega(N^{1-\epsilon})$-chromatic [2], it is also hard to determine whether the optimal distance-2 coloring of a given graph on $n$ vertices uses at most $O(N^{1+\epsilon}) = O(n^{1/2+\epsilon/2}$ or at least $\Omega(N^{2-\epsilon}) = \Omega(n^{1-\epsilon/2})$ colors.

A similar argument holds for other $d > 1$. For the case of $d = 2t$, with $t > 1$, we construct the following graph $H$ on $tN + N^2$ vertices, when given a graph $G$ on $N$ vertices. The graph consists of $G$, a path of $t-1$ vertices attached to each vertex of $G$, and a set of $N$ vertices attached to the end node of each path.

For the case of $d = 2t$, we use a construction similar to the theorem above, except a path of $t-1$ vertices lies between each $v_i$ vertex and the corresponding $u_{i,x}$ vertices. □

We note that NP-hardness reduction of Lin and Skiena [8] yields nearly the same result, or a $(n/d)^{1/2-\epsilon}$-hardness.

On the positive side, we can obtain nontrivial approximation for coloring all power graphs. In contrast with the Independent Set problem [3], the coloring problem becomes easier in odd powers.

**Theorem 4.3** *Coloring $G^{2t-1}$ is $O(n^{1/2+1/(4t-2)})$-approximable.*

*Proof.* Let $d = 2t - 1$ and $D$ be the maximum over all vertices $v$ of the number of vertices within distance $t-1$ from $v$. Then, the clique size of $G^d$ is at least $D$. By averaging, there exists a vertex in of degree at most $D^{1/(t-1)}$ in $G$. Hence, it is of degree at most $D^{2+1/(t-1)}$ in $G^d$, and by induction, $G^d$ is $D^{2+1/(t-1)}$-inductive. Thus, the performance ratio of an inductive coloring algorithm is at most $\min(D^{1+1/(t-1)}, n/D) \leq n^{(t-1)/(2t-1)}$. □

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