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On approximation algorithms for intersection graphs of rectangles

矩形によるインターセクショングラフに関する近似アルゴリズムについて

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Abstract: In this paper we show some graph theoretical properties of intersection graphs on rectangles and that the minimum coloring problem can be approximated within ratio $O((\log|V(G)|)^2)$ for the intersection graphs represented by sets of rectangles on the plane.

Keywords: Intersection graphs, rectangles, approximation algorithms, maximaum weight independent set problem, minimum coloring problem;

1 Definitions and notation

Let $G = (V, E)$ be a graph. We denote the subgraph of G induced by $V' \subseteq V$ by $G[V']$, the degree of vertex u in G by $d_G(u)$, the maximum degree of vertex in G by Δ_G, the neighborhoods of v by $N_G(v)$, and $\{v\} \cup N_G(v)$ by $N_G^+(v)$. If G is understood, then we often omit the inscription G in $d_G(u)$, Δ_G, $N_G(v)$, and $N_G^+(v)$.

Let $\mathcal{F} = \{S_1, \ldots, S_k\}$ be a family of nonempty subset of a set S. We will call the pair $(\mathcal{F}, S)$ *representation*. We will refer to the set S as the *host* and the subsets S_i as *objects*. A graph G is an *intersection graph represented by* a representation $(\mathcal{F}, S)$ if $G = (\mathcal{F}, E)$ such that $\forall S_i, S_j \in \mathcal{F}$ $(i \neq j)$, $\{S_i, S_j\} \in E$ iff $S_i \cap S_j \neq \emptyset$. Let $\mathcal{R} = (\mathcal{F} = \{S_1, \ldots, S_k\}, S)$ be a representation. We will say that $\mathcal{R}$ is *unit* if the objects of $\mathcal{F}$ have the same shape. $\mathcal{R}$ is *injective* if $S_i = S_j$ implies $i = j$ (i.e. the subsets are distinct). Objects we consider in the paper are open.

A *closed (open) rectangle* on the plane is a set R_i of points such that $\exists(x_1, y_1), (x_2, y_2)$ $(x_1 \leq x_2, y_1 \leq y_2)$ for which $R_i = \{(x, y) \mid x_1 \leq x \leq x_2,\ y_1 \leq y \leq y_2\}$ $(\{(x, y) \mid x_1 < x < x_2,\ y_1 < y < y_2\}$ respectively). We denote the projection of a rectangle R_i on the x-axis (y-axis) $I_x(R_i)$ ($I_y(R_i)$ respectively). Let R be a sets of rectangles on the plane. R is *x-axis (y-axis) non-proper* if $\forall R_i, R_j \in R$ $I_x(R_i) \subsetneq I_x(R_j)$ and $I_x(R_j) \subsetneq I_x(R_i)$ $((I_y(R_i) \subsetneq I_y(R_j)$ and $I_y(R_j) \subsetneq I_y(R_i))$ respectively). R is *strongly non-proper* if R is x and y-axis non-proper.

Let I be a sets of intervals on the real line. A graph G is an *interval graph represented by I* if G is an intersection graph represented by I (so the host is the real line in this case). Let $MI = \{A_1, \ldots, A_k\}$ be a set such that A_i $(\forall 1 \leq i \leq k)$ is a union of intervals on the real line. A graph G is a *multiple interval graph represented by MI* if G is an intersection graph represented by MI (so the host is the real line in this case).

2 Known techniques

In the section, we will review several techniques (and properties) which are useful in designing approximation algorithms for the problems.

2.1 Claw-free property

A graph is k *claw-free* if the graph does not have $K_{1,k}$ as induced subgraph (See [18]). A set of graphs is *claw-free* if there is a positive integer k such that all graphs in the set are k claw-free. For example, the following types of intersection graphs have claw-free property.

Unit intersection graphs

Most of intersection graphs with unit representations have the claw-free property. For example, a graph represented by unit iso-oriented rectangles on the plane is a 5 claw-free graph. A graph represented by unit disks on the plane is a 7 claw-free graph (in our definition all objects we consider are open) (See [15]).

Representations with objects of bounded area

Let G be a graph represented by a set objects $\mathcal{F}$ on the plane with following two properties; there is a positive integer k such that the area of each object in $\mathcal{F}$ is at most k, and any two intersecting objects in $\mathcal{F}$ share a region with an area of at least one. Then it is easy to see that all graph represented by $\mathcal{F}$ on the plane are $k+1$ claw-free graphs.

The claw-free property plays an important role in the two (or more) dimensional packing problem (See [2, 15]), because packing problem can be thought as a maximum independent set problem, and it is known that the independent number of a 3 claw-free weighted graph can be computed in polynomial time [16] and also that the independent number of a k claw-free graph can be approximated within ratio of $(k+1)/2$ for unweighted graphs [10] and k for weighted graphs [11].

2.2 The most left object strategy

Let G be an intersection graph of strongly non-proper rectangles on the plane, and let $v \in V(G)$ be the vertex corresponding to the most left object in a representation of G. Then since $G[N_G^+(v)]$ is a 3 claw-free graph, we have $\alpha(G[N_G^+(v)]) \leq 2$. Similarly, for an intersection graph G of unit disks on the plane, we have $\alpha(G[N_G^+(v)]) \leq 3$ (note that in our definition all objects we consider are open) [15]. Clearly if G is an intersection graph of strongly non-proper rectangles (and/or unit desk) on the plane then so is an induced subgraph of G. Hence the intersection graphs of strongly non-proper rectangles on the plane (and/or of unit disks on the plane [15]) have the following properties: Let $\mathcal{I}$ be a set of intersection graphs.

- $\exists$ a small integer k such that $\forall G \in \mathcal{I}$, $\exists v \in V(G)$ for which $\alpha(G[N_G^+(v)]) \leq k$,
- $\forall G \in \mathcal{I}$ and $\forall V' \subseteq V(G)$, $G[V'] \in \mathcal{I}$.

Using this properties, Marathe et al. showed better approximation algorithms for minimum coloring problem and maximum independent set problem for unit disk graphs [15]. The method in [15] leads the the following proposition (See concluding remarks in [15]). The proofs (for minimum coloring problem) is quite similar to the unit disk case presented in [15], hence are omitted.

Proposition 2.1 *Let $\mathcal{I}$ be a set of graphs with properties that (1) $\exists$ a small integer k such that $\forall G \in \mathcal{I}$, $\exists v \in V(G)$ for which $\alpha(G[N_G^+(v)]) \leq k$, and (2) $\forall G \in \mathcal{I}$ and $\forall V' \subseteq V(G)$, $G[V'] \in \mathcal{I}$. Then, minimum coloring problem and (unweighted) maximum independent set problem for $\mathcal{I}$ can be approximated within ratio of k.*

Corollary 2.2 *Let R be a strongly non-proper set of rectangles on the plane. Then minimum coloring problem and (unweighted) maximum independent set problem for intersection graphs represented by R can be approximated within ratio of 2.*

2.3 Shifting strategy

Hochbaum and Maass introduced a method, called *shifting strategy*, which applies to covering and packing problems in the plane in order to yield a polynomial time approximation scheme [8, 9].

2.4 Decomposition strategy

In [14], S.Khanna et al. introduced the following simple and useful technique to partition a graph G represented by rectangles on the plane into $O((\log|V(G)|)^2)$ 9 claw-free induced subgraphs of G: Partition the set of given rectangles into $\lceil\log|V(G)|\rceil^2$ classes $(i,j), 1 \leq i \leq \lceil\log|V(G)|\rceil$ and $1 \leq j \leq \lceil\log|V(G)|\rceil$. The class (i,j) comprises all rectangles with width $\in [2^{i-1}+1, 2^i]$, and height $\in [2^{j-1}+1, 2^j]$. Then it is easy to see that each intersection graph represented by rectangles in class (i,j) (on the plane) is a 9 claw-free graph. We will refer to the technique as *decomposition strategy.*

Decomposition strategy is very simple but useful. For example, we can give much more simple proof than one in chapter 6 in [4] for the following theorem by using decomposition strategy.

Theorem 2.3 *$\tau(n) \geq n/\lceil\log_2 n\rceil$ for all $n \geq 3$, where $\tau(n) = \max\{k \mid$ every interval graph of size n has a 3 claw-free induced subgraph of size $k\}$.*

3 Results

3.1 Graph thoretical properties of rectangle graphs

Forbidden induced subgraphs

Lemma 3.1 *Let R be a set of rectangles on the plane. And let G be the intersection graph represented by R. Then, G does not have an octahedron as an induced subgraph.*

Chromatic number and clique number

Let R be a set of rectangles on the plane. And let G be the intersection graph represented by R. In [1], Asplund and Grünbaum showed that $4\omega(G)^2 > \chi(G)$. If R is strongly non-proper, then we have $4\omega(G)+1 \geq \chi(G)$, because $\omega(G) \geq \lceil\Delta(G)/4\rceil$ and $\Delta(G)+1 \geq \chi(G)$. By using the most left object strategy, we can show the following slightly better upper bound.

Proposition 3.2 *Let G be an intersection graph represented by a strongly non-proper set of rectangles on the plane. Then the chromatic number of G is at most two times the clique number of G plus one.*

Proof. Let $\mathcal{G}$ be the set of intersection graphs represented by a strongly non-proper set of rectangles on the plane. Any $G \in \mathcal{G}$ has a vertex v such that $d_G(v)$ is at most 2ω. For any induced subgraph G' of G, G' is also in $\mathcal{G}$, and $\omega(G') \leq \omega(G)$. Thus, $\chi(G) \leq 2\omega(G)+1$. □

3.2 An approximation algorithm for minimum coloring problem

Theorem 3.3 *The minimum coloring problem can be approximated within ratio $O((\log|V(G)|)^2)$ for the intersection graphs represented by sets of rectangles on the plane.*

Proof. By using decompositon strategy, we have at most $O((\log|V(G)|)^2)$ 9 claw-free subgraphs G_{ij} of G $(1 \leq i,j \leq \log|V(G)|)$. Obviously for each subgraph G_{ij}, $\chi(G_{ij}) \leq \chi(G)$. From propositon 2.2, the problem for each subgraph G_{ij} can be approximated within ratio 7. This means that $\sum_{ij}(7 \times \chi(G_{ij})) \leq \sum_{ij}(7 \times \chi(G))$ is $O((\log|V(G)|)^2) \times \chi(G)$, thus the proof is complete. □

4 Summary

Maximum independent set problem

object	unweighted	weighted
unit disk	PTAS [12], 3 [15]	——
unit rectangle	PTAS [8, 9], 2 *1	——
SNP rectangles	2 *1	3.25 [2]
rectangles	——	$O(\log n)$ [14]

Minimum coloring problem

object	injective	no restriction
unit disk	——	3 [15]
unit rectangle	——	2 *1
SNP rectangles	——	2 *1
rectangles	——	$O((\log n)^2)$ *2

*1: From corollary 2.2.

*2: From proposition 2.1 and decomposition strategy.

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