数理解析研究所講究録
1121 巻 2000 年 221-228

A Statistical Model of Vortex Filaments

Zensho YOSHIDA
University of Tokyo, Graduate School of Frontier Sciences

吉田 善章
東京大学 大学院 新領域創成科学研究科

1 Introduction

The Beltrami condition describes the alignment of the vector field and its own vorticity. In many different convective nonlinear systems, such as fluids and plasmas, the Beltrami condition plays an important role in characterizing self-organized structures.

A plasma flow induces mixing of magnetic flux. The length scale cascades toward a small scale, resulting in amplification of the magnetic field. If the enhanced Lorentz force becomes to dominate the dynamics, its back-reaction must be taken into account. The Beltrami condition, which reads as the magnetic force-free condition, must apply to slow motion of a strongly magnetized plasma, i.e., the magnetic field $\boldsymbol{B}$ must satisfy $\nabla\times\boldsymbol{B}=\lambda\boldsymbol{B}$. This relation characterizes the local structure of stable vorteses. In this paper, we derive a Boltzmann distribution of the size of the filaments that maximizes the entropy for an ensemble defined by the total current. An interesting assertion is that the time series produced by a random-motion model of such filaments generates a power-law spectra which agrees well with experimantal observation [1].

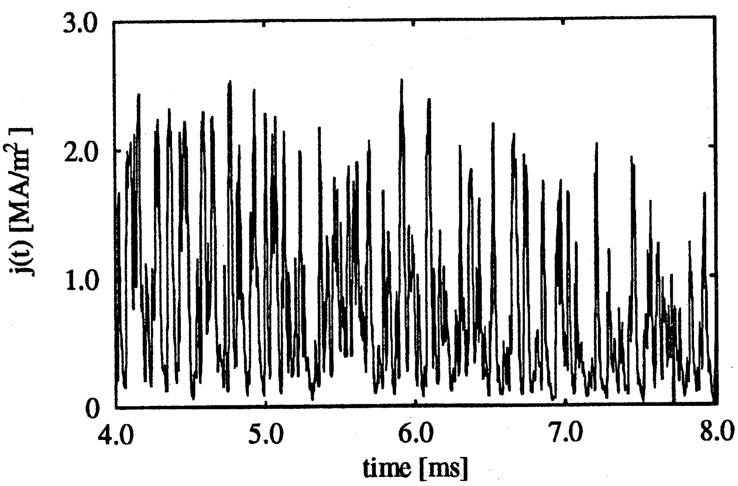
2 Intermittency

Intermittency is one of the common issues in various complex systems. The underlying mechanism may be either low-dimensional chaotic dynamics or high-dimensional statistical dynamics. A typical example of the latter category is the fluid turbulence [2]. When a turbulent flow has filamentary structures of vortex tubes, local measurements of the flow, or other related physical quantities, detect intermittent fluctuations. Three-dimensional shear flows stretch vortex tubes and produce thinner tubes carrying stronger vorticities (Kelvin's circulation law), and hence, the vorticity ($\boldsymbol{\omega}=\nabla\times\boldsymbol{v}$; $\boldsymbol{v}$ is the flow) tends to concentrate into filamentary tubes.

In plasmas, the induction effect and the corresponding Lorentz back reaction bring about coupling between the flow $\boldsymbol{v}$ and the magnetic field $\boldsymbol{B}$, resulting in a considerable complication of the dynamics. The stretching effect applies also to the magnetic flux, and $\boldsymbol{B}$ tends to take irregular distributions. There are many different examples where various physical quantities exhibit intermittent behavior and filamentary structures. In the solar photosphere, magnetic field strengths vary in the range of 0 to 0.2 T. The characteristic length scale of the magnetic field distribution is of the order of 100km [3]. Satellite measurements of the magnetotail of the earth detect intermittent fluctuations in ion bulk velocities and magnetic fields [4].

We consider the filamentation of the electric current $\boldsymbol{j} = \nabla \times \boldsymbol{B}/\mu_0$ (μ_0; vacuum permeability), which is the vorticity of the magnetic field in a plasma. The reduction of the length scale of the magnetic field has been discussed by many authors in its relation to the turbulent cascade process and self-organizations [5, 6]. The induction effect due to turbulent plasma flow brings about stretching of flux tubes and generates locally-stable flux filaments [6]. Such filamentary structures of magnetic fields resemble those of vortices created in three-dimensional turbulent flows [7]. While each filamentary flux tube, such as a Beltrami field [6], can be locally stable, their movement and mutual interactions are very complicated [8], and hence, they invoke a statistical mechanical treatment.

Some experimental observations also give us a motivation to develop a model of strongly irregular current in a turbulent plasma. Measurements of local currents in turbulent plasmas show intermittent fluctuations. In a reversed-field-pinch plasma, internal currents were measured by inserting a small electrostatic energy analyzer [9]. The data exhibit strong fluctuations, suggesting that the current density has strongly inhomogeneous distributions [10]. We observe similar behavior on REPUTE-1 [11] (Fig. 1).



T. Suzuki and Z. Yoshida, Fig. 1

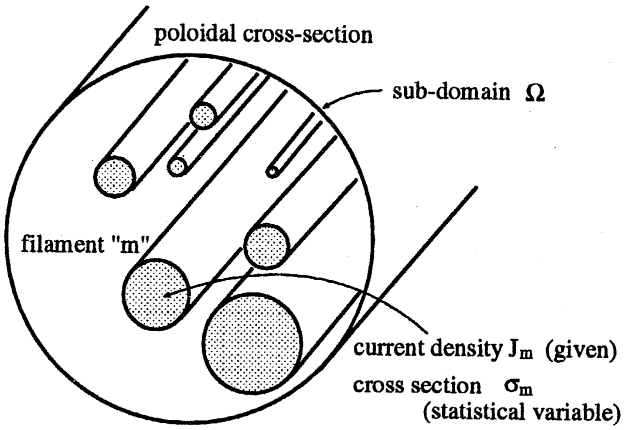
Figure 1: The waveform of the local current density (toroidal component) measured inside the plasma. The distance of the detector from the center is 75% of the radius of the plasma column. The detector is directed parallel to the local magnetic field during the flat top phase of the discharge.

Fluctuations in the local current density are more than 100% of the mean current density, while the system is in a macroscopic quasi-steady state. Such a system is insuperable from the viewpoint of conventional theories such as the diffusion model of the current density or the magnetohydrodynamic relaxation model [12].

3 Statistical Model of Filaments

We apply a statistical mechanical method to characterize the spectral structure of the current-density fluctuations. The principal macroscopic parameter that characterizes the system is the total current, which is a controllable stationary parameter. We note that we do not invoke "energy" to characterize the system. We are not discussing the dynamical process of generating filaments or the dynamics of many filaments, and hence, the energy arguments do not apply. Because of the resistive dissipation, the energy is not in equilibrium. The filamentary structure of the current yields an enhanced dissipation. In such a "dissipative structure", fluctuations creating filaments yield large dissipation, so that the principle of the minimum energy dissipation is invalidated.

The total current is the integral (sum) of local currents (filament currents). We study the most probable statistical distribution of filament currents for a given total current. Theoretical predictions will be compared with experimental observations.



T. Suzuki and Z. Yoshida, Fig. 2

Figure 2: A model of current filaments in a plasma column.

Let us consider a system of current filaments. Each filament is denoted by an index m ($m = 1, \cdots, N$; N is the total number of filaments in the ensemble). We specify the current density $\boldsymbol{J}_m$ on each filament. The toroidal component is given by $J_m = \boldsymbol{J}_m \cdot \boldsymbol{n}$, where $\boldsymbol{n}$ is the unit vector in the toroidal direction. We can invoke the analogy of the standard statistical mechanics of particles, where J_m parallels the energy level of the eigenstate m. The cross section σ_m of the filament m is the statistical variable of the present model (σ_m may be regarded as the number of particles allocated to the eigenstate m). Each filament has the current of $I_m = J_m \sigma_m$ (see Fig. 2).

We consider a sub-domain Ω of the total cross section of the plasma column as to be a canonical ensemble that contains N filaments. This N may change, as a grand canonical ensemble; however, it is appropriate to assume that the "chemical potential" is zero. The total current in Ω is given by

$$I = \sum_{m=1}^{N} I_m = \sum_{m=1}^{N} J_m \sigma_m. \tag{1}$$

A micro-state is characterized by specifying $\ell \equiv \{\sigma_1, \sigma_2, \cdots, \sigma_N\}$. The probability of a micro-state ℓ is denoted by $p(\ell) \equiv p(\sigma_1, \sigma_2, \cdots, \sigma_N)$. We have to normalize the probability by imposing

$$P = \sum_{\ell} p(\ell) \equiv 1, \tag{2}$$

where the summation is taken over all possible micro-states. The expectation value of the current is given by

$$\langle I \rangle = \sum_{\ell} p(\ell) I(\ell), \tag{3}$$

where $I(\ell)$ is the current of the micro-state ℓ;

$$I(\ell) \equiv I(\sigma_1, \sigma_2, \cdots, \sigma_N) = \sum_{m} J_m \sigma_m. \tag{4}$$

We derive the probability distribution by maximizing the Shannon entropy

$$S = -\sum_{\ell} p(\ell) \ln p(\ell). \tag{5}$$

The use of the Shannon entropy is appropriate when we can assume that the whole system of filaments are statistically homogeneous (in a more restricted system, we use the Rényi entropy). Apparently, some filaments are connected through turns around the torus. However, we neglect the long-range correlations through toroidal turns, assuming that the current channels are strongly chaotic. For the above-mentioned canonical ensemble, we maximize

$$H = S - \alpha P - \beta \langle I \rangle, \tag{6}$$

where α and β are Lagrange multipliers. The reciprocal of β represents the "temperature" of fluctuations. The variation principle $\delta H = 0$ yields the canonical distribution

$$p(\ell) = \exp(-\alpha - \beta I(\ell)). \tag{7}$$

The normalization condition (2) determines α;

$$Z \equiv \exp(\alpha) = \sum_{\ell} \exp(-\beta I(\ell)). \tag{8}$$

This Z is the partition function of the canonical distribution.

The probability (7) now reads

$$p(\ell) \equiv p(\sigma_1, \sigma_2, \cdots, \sigma_N) = \frac{\exp(-\beta \sum_m J_m \sigma_m)}{Z}. \tag{9}$$

We assume that the probability for a filament "m" to have a cross-section σ_m (denoted by $p_m(\sigma_m)$) is independent to the all other filaments. Then, $p(\sigma_1, \sigma_2, \cdots, \sigma_N)$ can be broken down into the simple product of $p_m(\sigma_m)$ $(m = 1, \cdots, N)$, and we obtain the "Boltzmann distribution"

$$p_m(\sigma_m) = \frac{\exp(-\beta J_m \sigma_m)}{Z_m}, \tag{10}$$

where

$$Z_m = \sum_{\sigma_m} \exp(-\beta J_m \sigma_m). \tag{11}$$

The inverse temperature β characterizes the intensity of fluctuations. For a certain σ, we observe

$$\begin{aligned} \frac{p_m(\sigma)}{\sum_k p_k(\sigma)} &= \frac{Z_m^{-1}\exp(-\beta J_m \sigma)}{\sum_k Z_k^{-1}\exp(-\beta J_k \sigma)} \\ &= \frac{1}{1 + \sum_{k \neq m}(Z_m/Z_k)\exp(-\beta\sigma(J_k - J_m))} \\ &\equiv \frac{1}{1 + A_m}. \end{aligned} \tag{12}$$

In the limit of zero temperature ($\beta \to \infty$), $A_m = 0$ for the filament m that satisfies $J_m = \min_k(J_k)$, and $A_m = \infty$ for other filaments. This implies that the whole current condensates on the "grand state", that is the filament which has the minimum current density $J_{\min}$. This $J_{\min}$ must be the average current density over the cross section Ω, and then, σ is the total area of Ω. Therefore, the current distributes homogeneously on Ω.

By calculating the total ohmic dissipation produced by a local resistivity η, we easily find that the effective resistivity for the mean current is given by

$$\eta^* = \eta \frac{\langle J^2 \rangle}{\langle J \rangle^2}. \tag{13}$$

In the limit of $\beta \to \infty$, $\eta^* = \eta$, while a finite β yields an enhanced effective resistivity.

4 Time Seriese Generated by Ramdom Motion of Filaments

Using the probability distribution (10), we generate the time series of the current density $j(t)$ that is measured at a fixed position in Ω. Note that the probability of measuring the filament m differs from $p_m(\sigma_m)$ (the probability of the filament m to have a cross section σ_m). We denote by q_m the probability of measuring a filament m at a fixed point, which is proportional to σ_m. Here, we assume that the filaments cover Ω, and set

$$q_m = \frac{\sigma_m}{|\Omega|}, \tag{14}$$

where $|\Omega|$ is the area of Ω.

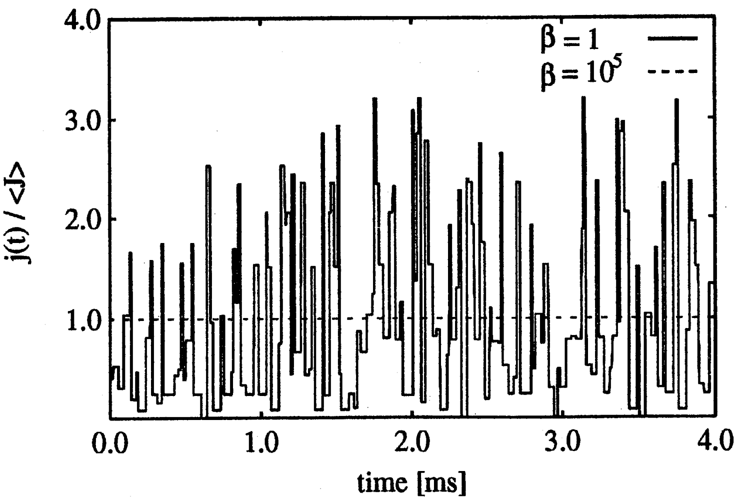
To evaluate the period of a certain filament being measured at the observation point, we specify the speed v, which is perpendicular to the toroidal direction, of the filament. We assume that v is common for all filaments. Let ρ_m be the radius of the filament m;

$$\rho_m = \sqrt{\frac{\sigma_m}{\pi}}. \tag{15}$$

The duration of the filament m appearing in the time series is given by

$$\Delta t_m = \frac{2\rho_m}{v}. \tag{16}$$

When we choose a filament m, we have the pulse height J_m and the pulse width Δt_m in the graph of $j(t)$. The probability of choosing the filament m must be proportional to the "cross section" $\rho_m \propto \sigma_m^{1/2}$. The long-term average of the total duration of the filament m appearing in $j(t)$ is given by $\rho_m \Delta t_m \propto \sigma_m \propto q_m$, which is consistent with (14).



T. Suzuki and Z. Yoshida, Fig. 3

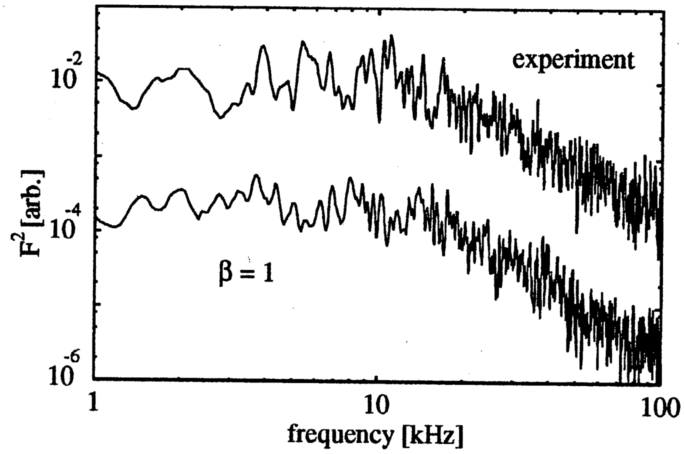
Figure 3: Time series $j(t)$ produced by the statistical model ($\beta = 1$ and $\beta = 10^5$).

Controllable parameters in the present statistical model are the velocity v, the number of filaments N, the "energy level" $\{J_1, J_2, \cdots, J_N\}$, and the inverse temperature β. By choosing an appropriate set of parameters, we can reproduce the spectral structure of the experimentally observed fluctuations of the current density. In Fig. 3, we show an example of the time series $j(t)$ generated by the model with assuming $v = 2\text{km/s}$, $N = 100$ and $\beta = 1$. We also show the non-fluctuating time series given by $\beta = 10^5$. Here, $\{J_m\}$ distributes uniformly, with respect to m, over the interval $(0, J_{\max}]$. We normalize $J_{\max}$ by imposing that $\langle J \rangle = \sum_m J_m q_m$ gives the mean current density. The typical value of σ_m is about 1/200 of the cross section $|\Omega|$. To compare with the experiment (Fig. 1), we take the radius of $|\Omega|$ to be 0.22m. The average radius of a filament (defined by $\rho_c = \sum_m \rho_m J_m / \sum_m J_m$) is 1.67×10^{-2}m.

Figure 4 shows the frequency spectrum of $j(t)$ in Fig. 3, in comparison with the experimental spectrum corresponding to Fig. 1. We observe that the theoretical spectrum agrees well with the experimental one. The spectrum of $j(t)$ has two distinct frequency ranges; the spectrum in the low frequency range ($f < f_c \approx 20\text{kHz}$) is "white", while in the high frequency range ($f > f_c$), the spectrum is the $1/f$ spectrum. In the theory, the critical frequency f_c is given by

$$f_c = \frac{1}{\pi \Delta t_c} = \frac{v}{2\pi \rho_c}, \tag{17}$$

where ρ_c is the average radius and Δt_c is the average duration of a filament; cf. (16). In the above example, $\rho_c = 1.67 \times 10^{-2}$m and $v = 2$km/s yield $f_c \simeq 20$kHz.



T. Suzuki and Z. Yoshida, Fig. 4

Figure 4: Power spectrum of $j(t)$ produced by the statistical model (Fig. 3), in comparison with the experimental spectrum corresponding to Fig. 1. We apply different normalizations to both spectra to plot them at different vertical positions. A triangle filter (weighted by 3 point for both sides) is applied.

The two different frequency ranges of the spectrum can be explained as follows. In a long time scale ($|t_1 - t_2| > \Delta t_c$), $j(t_1)$ and $j(t_2)$ come from different randomly-selected filaments. Therefore, we obtain a white spectrum. Agreement of our theoretical result with the experimental spectrum in this frequency range justifies our assumption of random movement of filaments.

In the short time scale range, the shape of each pulse (height $= J_m$ and width $= \Delta t_m$) dominates the spectral structure. The Fourier transform of a rectangular pulse yields

$$F_m(f) \propto \left| \frac{J_m}{f} \sin(\pi \Delta t_m f) \right| . \tag{18}$$

Summing up (18) for various pulses "m", we obtain the $1/f$ spectrum for the frequency range of $f > f_c$. Therefore, the $1/f$ spectrum in the high frequency range implies that the current density is approximately uniform in each filament, which justifies the model of filaments introduced in the present theory.

5 Summary

We have derived a statistical model of filament currents in a plasma, which reproduces the characteristic structure of the frequency spectrum of intermittent current fluctuations. The theory gives the Boltzmann distribution of the size σ_m of filaments under the constraint of the total current. An interesting assertion is that the time series produced by the model has non-Gaussian power-law spectra, although the statistical distribution of σ_m is Gaussian. This is because the spectrum of the time series is primarily characterized by the probability $q_m \propto \sigma_m$ (see (14)) where $p_m(\sigma_m) \propto \exp(-\beta J_m \sigma_m)$.

The authors are grateful to the REPUTE experiment group for their discussions. This work was supported by Grant-in-Aid for Scientific Research from the Japanese Ministry of Education, Science and Culture No. 09308011.

References

[1] T. Suzuki, *Intermittent Fluctuations of internal Current in a Turbulent Plasma*, dissertation, University of Tokyo (1999).

[2] Y. Couder, S. Douady and O. Cadot, *Turbulence, A Tentative Dictionary*, edited by P. Tabeling and O. Cardoso (Plenum, New York, 1995), p.131.

[3] J. O. Stenflo, Sol. Phys. **32**, 41 (1973).

[4] V. Angelopoulos *et al.*, J. Geophys. Res. **97**, 4027 (1992).

[5] D. Biskamp and H. Walter, Phys. Fluids B **1**, 1964 (1989); Phys. Fluids B **2**, 1787 (1990).

[6] Z. Yoshida, Phys. Rev. Lett. **77**, 2722 (1996).

[7] E. D. Siggia, J. Fluid Mech. **107**, 375 (1981).

[8] The dynamics of a system of current filaments was studied by Y. Yatsuyanagi, T. Hatori and T. Kato, J. Phys. Soc. Jpn. **65**, 745 (1996); see also D. W. Moore and P. G. Saffman, Philos. Trans. R. Soc. Lond. Ser. A **272**, 403 (1972); R. Kinney, T. Tajima, J. C. McWilliams and N. Petviashvili, Phys. Plasmas **1**, 260 (1994).

[9] J. C. Ingraham *et al.*, Phys. Fluids B **2**, 143 (1990).

[10] A plasma with imperfect magnetic surfaces must have a microscopically filamented current structure; see J. B. Taylor, Phys. Fluids B **5**, 4378 (1993).

[11] Z. Yoshida *et al.*, J. Plasma Phys. **59**, 103 (1998).

[12] J. B. Taylor, Phys. Rev. Lett. **33**, 1139 (1974).