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THE INDEX OF A LOG-CANONICAL SINGULARITY

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Abstract. In this paper we study the index of an isolated strictly log-canonical singularity. As a result, we obtain the boundedness of indices of these singularities of dimension 3 and determine all possible indices.

1. Introduction

A log-canonical, non-log-terminal singularity is called strictly log-canonical. Let \((X, x)\) be an isolated strictly log-canonical singularity over \(\mathbb{C}\). If its dimension is 2, then the index is 1, 2, 3, 4, or 6. This is observed by checking the list of the weighted dual graphs of all strictly log-canonical singularities. This is also proved by Shokurov [14] by means of complements and by Okuma [13] by means of plurigenera. In the 3-dimensional case, the author heard that boundedness of indices of such singularities is proved by Shokurov in [15]. In this paper, we study the quotient of isolated strictly log-canonical singularities by finite group actions. First, in case that the group acts freely in codimension 1, we obtain the formula of the index of the quotient singularity (Lemma 3.3). By this it follows a different proof of above fact on indices for dimension 2. We then prove that the index of 3-dimensional strictly log-canonical singularity is less than or equal to 66. More precisely, a positive integer \(r\) is the index of such a singularity if and only if \(\varphi(r) \leq 20\) and \(r \neq 60\), where \(\varphi\) is the Euler function. This is related to finite automorphisms on \(K3\)-surfaces, Abelian surfaces and elliptic curves.

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2. Isolated strictly log-canonical singularities.

2.1. Isolated strictly log-canonical singularities are studied in [5]. In this section we summarize those results and add some basic facts on these singularities.

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Definition 2.2. Let \((X, x)\) be a germ of normal singularity. If there is an integer \(r\) such that \(\omega^{[r]}_X\) is invertible, the singularity is called a \(\mathbb{Q}\)-Gorenstein singularity. We call the minimum positive such number \(r\) the index of \((X, x)\) and denote by \(\text{Ind}(X, x)\).

Definition 2.3. A \(\mathbb{Q}\)-Gorenstein singularity \((X, x)\) is called a log-canonical singularity (resp. log-terminal singularity) if for a good resolution \(f : Y \rightarrow X\) the canonical divisor on \(Y\) has an expression in \(\text{Div}(Y) \otimes \mathbb{Q}\):

\[
K_Y = f^* K_X + \sum_i m_i E_i
\]

with \(m_i \geq -1\) (resp. \(m_i > -1\)) for every irreducible exceptional divisor \(E_i\) with 
\(x \in f(E_i)\). Here a good resolution means a resolution whose exceptional set is a 
normally crossing divisor with the non-singular irreducible components. We call \(m_i\) the discrepancy over \(X\) at \(E_i\) or the discrepancy for \(f\) at \(E_i\) for each irreducible component \(E_i\).

2.4. In case of index 1, a strictly log-canonical singularity is equivalent to a purely 
elliptic singularity ([5]). In this case we define the essential divisor in the exceptional 
divisor of a good resolution. It actually plays an essential role in the exceptional 
divisor (cf. Lemma 3.7 [5]).

Definition 2.5. Let \((X, x)\) be an isolated strictly log-canonical singularity of index 
1 and \(f : Y \rightarrow X\) a good resolution. Then one has a representation

\[
K_Y = f^* K_X + \sum_{i \in I} m_i E_i - \sum_{j \in J} E_j,
\]

with \(m_i \geq 0\), \(I \cap J = \emptyset\) and \(J \neq \emptyset\). The divisor \(E_J := \sum_{j \in J} E_j\) is called the essential 
divisor for a good resolution \(f\).

2.6. Let \((X, x)\) be an \(n\)-dimensional isolated strictly log-canonical singularity of index 
1 and \(f : Y \rightarrow X\) a good resolution with the essential divisor \(E_J\). Since \(E_J\) is a 
complete variety with normal crossings,

\[
H^{n-1}(E_J, \mathcal{O}_{E_J}) \simeq Gr^0_F H^{n-1}(E_J, \mathbb{C}) = \bigoplus_{i=0}^{n-1} H^{0,i}_{n-1}(E_J),
\]

where \(F\) is the Hodge filtration and \(H^{i,j}(\ast)\) is the \((i, j)\)-Hodge-component of \(H^m(\ast, \mathbb{C})\). 
As the left hand side is 1-dimensional \(\mathbb{C}\)-vector space (Lemma 3.7 [5]), it must coincide 
with one of \(H^{0,i}_{n-1}(E_J)\) \((i = 0, 1, 2, \ldots, n-1)\).

Definition 2.7. An \(n\)-dimensional isolated strictly log-canonical singularity \((X, x)\) 
of index 1 is called of type \((0, i)\), if \(H^{n-1}(E_J, \mathcal{O}_{E_J}) = H^{0,i}_{n-1}(E_J)\).

2.8. The type is independent of the choice of a good resolution (Proposition 4.2 in 
[5]).
Example 2.9. A 2-dimensional strictly log-canonical singularity \((X, x)\) of index 1 is of type \((0, 1)\) if and only if \((X, x)\) is a simple elliptic singularity and of type \((0, 0)\) if and only if it is a cusp singularity.

Proposition 2.10. Let \((X, x)\) be a 3-dimensional isolated strictly log-canonical singularity of index 1 and of type \((0, 2)\) and \(f : Y \to X\) the canonical model, i.e. \(Y\) has at worst canonical singularities and \(K_Y\) is \(f\)-ample. Let \(D\) be the exceptional divisor of \(f\) with the reduced structure. Then \(Y\) has at worst terminal singularities and \(D\) is isomorphic to either a normal \(K3\)-surface or an Abelian surface. Here a normal \(K3\)-surface is a normal surface whose minimal resolution is a \(K3\)-surface.

Proof. First note that \(E_J\) is irreducible by Lemma 6, [7]. Since the discrepancy for \(f\) at each exceptional component is negative (the proof of Lemma 3.7 [7]), \(D\) is irreducible. Let \(g : Y' \to Y\) be a proper birational morphism whose composite \(f \circ g : Y' \to X\) is a good resolution. One sees that \(Y\) has at worst terminal singularities. Indeed, if not, there exists an exceptional divisor \(E_0\) which is crepant for \(g\). Then the discrepancy at \(E_0\) for \(f \circ g\) is less than 0, so \(E_0\) becomes another component of the essential divisor, which is a contradiction. Now one can prove that \(Y\) is non-singular away from finite points. If \(D\) has 1-dimensional singular locus, then by the blowing-up at a 1-dimensional irreducible component of the singular locus one obtains a component \(E_1\) whose discrepancy for \(f \circ g\) is \(-m+1 < 0\), where \(m\) is the multiplicity of \(D\) at a general point on the curve. It implies that \(E_1\) is another component of the essential divisor, which is a contradiction. Therefore \(D\) is non-singular away from finite points. On the other hand, since \(\omega_Y \simeq O_Y(-D)\) is Cohen-Macaulay, so is \(D\). Hence by Serre's criterion \(D\) is normal. The condition \(\omega_Y \simeq O_Y(D)\) yields that \(\omega_D \simeq O_D\). A normal surface with this condition and \(H^2(E_J, O_{E_J}) = \mathbb{C}\), where \(E_J\) is a resolution of \(D\), is either a normal \(K3\)-surface or an Abelian surface ([16]). □

Proposition 2.11. ([6]) Let \((X, x)\) be a 3-dimensional isolated strictly log-canonical singularity of index 1 and of type \((0, 1)\) and a finite group \(G\) act on \((X, x)\). Then either:

(i) for every good resolution \(f : \tilde{X} \to X\), the essential divisor \(E_J\) is a cycle \(E_1 + E_2 + \ldots + E_s, (s \geq 2)\) of elliptic ruled surfaces, where \(E_i\) and \(E_{i+1}\) intersect at a section on each component for \(i = 1, \ldots, s\) \((E_{s+1} = E_1)\) or

(ii) there is a \(G\)-equivariant good resolution \(f : \tilde{X} \to X\) such that the essential divisor \(E_J\) contains a \(G\)-invariant chain \(E^{(0)} = E_1 + \ldots + E_s, (s \geq 1)\) of elliptic ruled surfaces, where \(E_i\) and \(E_{i+1}\) intersect at a section on each component for \(i = 1, \ldots, s - 1\). There are disjoint subdivisors \(E^{(-)}\) and \(E^{(+)}\) of \(E_J\) such that \(E_J = E^{(-)} + E^{(0)} + E^{(+)}\), where \(E^{(-)} \cap E^{(0)}\) is a section of \(E_1\) and \(E^{(+)} \cap E^{(0)}\) is a section of \(E_s\).
3. Finite groups which act freely in codimension 1.

**Definition 3.1.** Let $G$ be a group and $(X, x)$ a germ of a singularity. We say that $G$ acts on $(X, x)$ if $G$ acts on a neighbourhood of $x$ and fixes the point $x$. We say that $G$ acts on $(X, x)$ freely in codimension 1, if there exists a closed subset $S$ of codimension greater than or equal to 2 on a neighbourhood $X$ such that $G$ acts freely on $X \setminus S$.

**3.2.** We denote the set of non-singular points of $X$ by $X_{\text{reg}}$. Let $(X, x)$ be a $\mathbb{Q}$-Gorenstein singularity of index $m$ and a group $G$ act on $(X, x)$. We denote the germ $(X/G, x')$ by $(X, x)/G$, where $x' \in X/G$ is the image of $x$. Denote the maximal ideal of $x$ by $m_x$. Then it induces a canonical representation

$$\rho : G \to GL(\omega_{X/m}^{[m]}/m_x\omega_{X}^{[m]}) \simeq \mathbb{C}^*.$$ 

because $G$ fixes the point $x$.

**Lemma 3.3.** Let $(X, x)$ be a $\mathbb{Q}$-Gorenstein normal singularity of index $m$. Let $G$ be a finite group which acts on $(X, x)$ freely in codimension 1 and $\rho : G \to GL(\omega_{X/m}^{[m]}/m_x\omega_{X}^{[m]}) \simeq \mathbb{C}^*$ the canonical representation. Then

$$\text{Ind}((X, x)/G) = m |\text{Ind}\rho|.$$ 

In particular,

$$\text{Ind}((X, x)/G) \leq m |G|.$$ 

**Proof.** Denote the order of $G$ by $d$, $|\text{Ind}\rho|$ by $r$ and $\text{Ind}((X, x)/G)$ by $I$. Let $g$ be a generator of $\text{Ind}\rho$ and $e$ the primitive $r$-th root of 1 which corresponds to $g$. Let $\omega$ be a generator of $\omega_{X/m}^{[m]}$. By the pull-back of a generator of $\omega_{X/G}^{[m]}$, one has a $G$-invariant $I$-ple $n$-form $\theta$ which is holomorphic and does not vanish on $X_{\text{reg}}$. Therefore $I = mm'$ for some $m' \in \mathbb{N}$ and $\theta = h\omega^{\otimes m'}$, where $h$ is a nowhere vanishing holomorphic function on $X$. Since $\theta^g = \theta$ as an element of $\omega_{X/m}^{[m]}/m_x\omega_{X}^{[m]}$, one obtains that $\epsilon^{m'} h(x) \omega^{\otimes m'} = h(x) \omega^{\otimes m'}$. Hence $\epsilon^{m'} = 1$. This shows $I \geq mr$. Next, to prove $I \leq mr$, we construct a $G$-invariant $mr$-ple $n$-form which is holomorphic and does not vanish on $X_{\text{reg}}$. Denote an element of $G$ which corresponds to $g \in \text{Ind}\rho$ by the same symbol $g$. Let $\theta$ be an $mr$-ple $n$-form $\omega \otimes \omega^g \ldots \otimes \omega^{\sigma r-1}$ and $\tilde{\theta}$ be $(1/d) \sum_{\sigma \in G} \theta^\sigma$. Then $\tilde{\theta}$ is an invariant $mr$-ple $n$-form. Let $\rho(\sigma) = g^i$ for $\sigma \in G$. Then in $\omega_{X/mr}^{[m]}/m_x\omega_{X}^{[mr]}$, $\theta^\sigma = \epsilon^{ri+1+2+\ldots+r-1} \omega^{\otimes r}$ which is $\omega^{\otimes r}$ if $r$ is odd and $-\omega^{\otimes r}$ if $r$ is even. Therefore $\tilde{\theta} = \pm \omega^{\otimes r} + \lambda$, where $\lambda \in m_x\omega_{X}^{[mr]}$. Since $\tilde{\theta} \notin m_x\omega_{X}^{[mr]}$, $\tilde{\theta}$ does not vanish on $X_{\text{reg}}$, which shows that $\tilde{\theta}$ is a required form. \[ \square \]

**Corollary 3.4.** Let $(X, x)$ be an isolated strictly log-canonical singularity of index 1 on which a finite group $G$ acts. Let $f : \tilde{X} \to X$ be a $G$-equivariant resolution of the singularities and $\rho : G \to GL(\omega_{X}/f_*\omega_{X}) \simeq \mathbb{C}$ the induced representation. Then $\text{Ind}((X, x)/G) = |\text{Ind}\rho|$.
**Proof.** For an isolated strictly log-canonical singularity of index 1, it follows that $m_x \omega_X = f_* \omega_X$.

**Corollary 3.5.** Let $(X, x)$ be an $n$-dimensional isolated strictly log-canonical singularity of index 1 on which a finite group $G$ acts. Assume there exists the canonical model $\varphi : X' \to X$ and let $E$ be the reduced exceptional divisor. Then the action induces a representation $\rho : G \to GL(H^{n-1}(E, \mathcal{O}_E))$ and $\text{Ind}(X, x)/G = |\text{Im}\rho|$.

**Proof.** Take a $G$-equivariant resolution $f : \tilde{X} \to X$. Then $\bigoplus_{m \geq 0} f_* \omega_{X}^{\otimes m}$ admits the action of $G$. So the canonical model admits the equivariant action of $G$, therefore the exceptional divisor $E$ also does. Since $\omega_{X'} \simeq \mathcal{O}_{X'}(-E)$ (proof of Lemma 7 of [7]) and $X'$ is Gorenstein in codimension 2, $E$ is Cohen-Macaulay and $\omega_E \simeq \mathcal{O}_E$. These yield that $H^{n-1}(E, \mathcal{O}_E) = \mathbb{C}$. As $R^{n-1} \varphi_* \mathcal{O}_{X'} \simeq R^{n-1} f_* \mathcal{O}_X \simeq \mathbb{C}$, the surjection $R^{n-1} \varphi_* \mathcal{O}_{X'} \to H^{n-1}(E, \mathcal{O}_E)$ is an isomorphism. On the other hand $R^{n-1} f_* \mathcal{O}_X$ is dual to $\omega_X/f_* \omega_{\tilde{X}}$, on which one can apply Corollary 3.4.

**Corollary 3.6.** Let $(X, x)$ be an $n$-dimensional isolated strictly log-canonical singularity of index 1 on which a finite group $G$ acts. Let $f : Y \to X$ be a $G$-equivariant good resolution and $E_J$ the essential divisor. Then the action induces a representation $\rho : G \to GL(H^{n-1}(E_J, \mathcal{O}_{E_J}))$ and $\text{Ind}(X, x)/G = |\text{Im}\rho|$.

**Proof.** It is clear that $G$ acts on $E_J$. Since $E_J$ is the essential divisor, $R^{n-1} f_* \mathcal{O}_{X'} \simeq H^{n-1}(E_J, \mathcal{O}_{E_J})$ by Lemma 3.7 [5]. On the other hand $R^{n-1} f_* \mathcal{O}_X$ is dual to $\omega_X/f_* \omega_{\tilde{X}}$, on which one can apply Corollary 3.4.

## 4. Index of isolated strictly log-canonical singularities

### 4.1. In this section, one proves that the indices of isolated strictly log-canonical singularities of dimension 2 and 3 are determined. Here one should note that the boundedness of indices does not hold for log-terminal singularities and non-log-canonical singularities even for 2-dimensional case.

**Example 4.2.** (1) Let $(Z_m, z_m)$ be the cyclic quotient singularity $\mathbb{C}^2/G$, where $G$ is generated by

\[
\begin{pmatrix}
\epsilon & 0 \\
0 & \epsilon
\end{pmatrix}
\]

Here $\epsilon$ is a primitive $m$-th root of unity. Then the exceptional curve on the minimal resolution is $\mathbb{P}^1$ and its self-intersection number is $-m$. Therefore the index of $(Z_m, z_m)$ is $m$ if $m$ is odd and $m/2$ if $m$ is even. This shows that the indices of log-terminal singularities are not bounded.
(2) Let \((X, x) \subset (\mathbb{C}^3, 0)\) be a hypersurface singularity defined by \(x^4 + y^4 + z^4 = 0\) and \((Z_m, z_m)\) is its quotient by the cyclic group generated by
\[
\begin{pmatrix}
\epsilon & 0 & 0 \\
0 & \epsilon & 0 \\
0 & 0 & \epsilon
\end{pmatrix},
\]
where \(\epsilon\) is a primitive \(m\)-th root of unity. Then the index of \((Z_m, z_m)\) is \(m\). This shows that the indices of non-log-canonical singularities are not bounded.

4.3. Let \(\pi : (X, x) \to (Z, z)\) be a finite morphism étale in codimension 1. Then \((X, x)\) is strictly log-canonical if and only if \((Z, z)\) is (see for example Proposition 1.7, [?]). Hence by the canonical cover, an arbitrary strictly log-canonical singularity is regarded as the quotient of such a singularity of index 1 by a finite group which acts on the singularity freely in codimension 1.

**Definition 4.4.** An isolated strictly log-canonical singularity is called of type \((0, i)\), if its canonical cover is of type \((0, i)\).

**Theorem 4.5.** An arbitrary dimensional isolated strictly log-canonical singularity of type \((0, 0)\) has index either 1 or 2.

**Proof.** This is proved in Theorem 3.10, [?]. One can also prove it by using 3.6. Let \(\pi : (X, x) \to (Z, z)\) be the canonical cover of an \(n\)-dimensional isolated strictly log-canonical singularity \((Z, z)\) and \(G = \langle g \rangle\) the associated cyclic group. Let \(f : \bar{X} \to X\) be a \(G\)-equivariant good resolution of \((X, x)\) such that \(\pi \circ f\) factors through a good resolution \(g : \bar{Z} \to Z\) of \((Z, z)\). Denote the essential divisor for \(f\) by \(E_J\) and its dual complex by \(\Gamma\). Then \(g\) induces an automorphism \(g^*\) on \(H_{n-1}(\Gamma, \mathbb{Z})\). Since \((X, x)\) is of type \((0, 0)\), \(C \simeq H_{n-0}^0(E_J)\) and this is isomorphic to \(H_{n-1}^0(\Gamma, \mathbb{C})\) by 2.5, [?]. Therefore \(H_{n-1}^0(\Gamma, \mathbb{Z})\) is of rank 1. Let \(\lambda\) be a free generator of \(H_{n-1}^0(\Gamma, \mathbb{Z})\). Then \(g^*(\lambda) = \pm \lambda + (\text{torsion})\) in \(H_{n-1}^0(\Gamma, \mathbb{Z})\). Therefore \(g^*(\lambda) = \pm \lambda\) in \(H_{n-1}(\Gamma, \mathbb{C})\). Hence the order of the action of \(G\) on \(H_{n-1}(E_J, O_{E_J})\) is 1 or 2. Now apply 3.6. \(\square\)

4.6. A non-singular projective variety \(X\) is called a Calabi-Yau variety, if it satisfies that \(\omega_X \simeq O_X\). It is well known that a 1-dimensional Calabi-Yau variety is an elliptic curve and 2-dimensional one is either a \(K3\)-surface or an Abelian surface. An automorphism \(g\) on \(X\) induces a linear automorphism \(g^*\) on \(\Gamma(X, \omega_X) = \mathbb{C}\) which is dual to \(H^n(X, O_X)\), where \(n = \dim X\). Now let us introduce a conjecture on finite automorphisms on Calabi-Yau varieties, which is essential to our problem.

**Conjecture 4.7.** For \(n \in \mathbb{N}\), there is a number \(B_n\) such that \(n\)-dimensional Calabi-Yau variety \(X\) and a finite automorphism \(g\) on \(X\), the order of the induced automorphism \(g^*\) on \(H^n(X, O_X) = \mathbb{C}\) is bounded by \(B_n\).

For \(n = 1, 2\), the conjecture holds true.
Proposition 4.8. For an arbitrary elliptic curve $X$, denote the order $|\text{Imp}|$ by $r$, where $\rho : \text{Aut}(X) \to GL(H^1(X, \mathcal{O}_X)) = \mathbb{C}^*$ is the induced representation. Then $\varphi(r) \leq 2$, which means $r = 1, 2, 3, 4$ or 6.

Proof. This is a classical result and proved in various ways. For example, note that an automorphism of $X$ is the composite of a group homomorphism and a translation. Since the translation has no effect on $H^1(X, \mathcal{O}_X) = \mathbb{C}$, $\text{Imp}$ is $\rho(\text{Aut}(X, 0))$, where $\text{Aut}(X, 0)$ is the group of automorphisms. Since $\text{Aut}(X, 0)$ fixes the zero element of the group, it is a finite group of order 1, 2, 4 or 6 (see, for example, IV, 4.7, [4]).

Proposition 4.9. (i) (10.1.2, [11]) For an arbitrary K3-surface $X$, denote the order $|\text{Imp}|$ by $r$, where $\rho : \text{Aut}(X) \to GL(H^2(X, \mathcal{O}_X)) = \mathbb{C}^*$ is the induced representation. Then $\varphi(r) \leq 20$, in particular $r \leq 66$. Here $\varphi$ is the Euler function.

(ii) (3.2, [3]) For an arbitrary Abelian surface $X$, the order $r$ of a finite automorphism on $X$ satisfies $\varphi(r) \leq 4$, which means that $r = 1, 2, 3, 4, 5, 6, 8, 10, 12$.

Now one obtains a new proof of the following result.

Theorem 4.10. A 2-dimensional strictly log-canonical singularity has index 1, 2, 3, 4 or 6.

Proof. Let $\pi : (X, x) \to (Z, z)$ be the canonical cover of the strictly log-canonical singularity $(Z, z)$ and $G$ be the associated cyclic group. By 4.5, it is sufficient to prove for the case that $(X, x)$ is of type $(0, 1)$. Let $f : Y \to X$ be the minimal resolution and $E$ the exceptional curve. Then $f$ is a $G$-equivariant good resolution with the essential divisor $E$ which is an elliptic curve. By 4.8, $|\text{Imp}| = 1, 2, 3, 4$, or 6, where $\rho : G \to GL(H^1(E, \mathcal{O}_E)) = \mathbb{C}^*$ is the induced representation. Now apply 3.6.

Theorem 4.11. An isolated 3-dimensional strictly log-canonical singularity of type (0, 2) has index $r$, where $\varphi(r) \leq 20$.

Proof. Let $\pi : (X, x) \to (Z, z)$ be the canonical cover of a 3-dimensional strictly log-canonical singularity $(Z, z)$ and $G$ the associated cyclic group. Let $E$ be the exceptional divisor on the canonical model of $X$. Then by 2.10 $E$ is either a normal K3-surface or an Abelian surface. Note that the action of $G$ on $E$ is lifted onto the minimal resolution $\tilde{E}$ of $E$. Since the singularities on $E$ are at worst rational double, one obtains that $\Gamma(E, \omega_E) = \Gamma(\tilde{E}, \omega_{\tilde{E}})$. By the Serre duality, the action of $G$ on $H^2(E, \mathcal{O}_E)$ is the same as the one on $H^2(\tilde{E}, \mathcal{O}_{\tilde{E}})$. Therefore by 3.5 and 4.9 $r = \text{Ind}(Z, z)$ satisfies $\varphi(r) \leq 20$.

Theorem 4.12. An isolated 3-dimensional strictly log-canonical singularity of type (0, 1) has index 1, 2, 3, 4 or 6.
4.13. For the proof of Theorem 4.12 one needs the discussion on the following divisor: Let $E_J$ be a simple normal crossing divisor on a non-singular 3-fold. Assume $E_J = E_1 + E_2 + \ldots + E_s$ is a cycle of elliptic ruled surfaces $E_i$ and every intersection curve is a section on the ruled surfaces. Decompose $E_J$ into two connected chains $E^{(i)}_J$ ($i = 1, 2$) with no common components. Let $C_1$ and $C_2$ be the irreducible curves of $E^{(1)}_J \cap E^{(2)}_J$. Let $p : E^{(1)}_J \to C$ and $q : E^{(2)}_J \to C$ be the rulings and $p_i : C_i \to C$ be the restriction of $p$ on $C_i$. Then one obtains the Mayer-Vietoris exact sequence:

$$H^1(E^{(1)}_J, \mathcal{O}) \oplus H^1(E^{(2)}_J, \mathcal{O}) \overset{\Phi}{\longrightarrow} H^1(C_1, \mathcal{O}) \oplus H^1(C_2, \mathcal{O}) \overset{\Psi}{\longrightarrow} H^2(E_J, \mathcal{O}) \rightarrow 0,$$

which is an exact sequence of mixed Hodge structure. By taking $Gr^F_0$, where $F$ is the Hodge filtration, one obtains the following:

$$H^1(E^{(1)}_J, \mathcal{O}) \oplus H^1(E^{(2)}_J, \mathcal{O}) \overset{\Phi}{\longrightarrow} H^1(C_1, \mathcal{O}) \oplus H^1(C_2, \mathcal{O}) \overset{\Psi}{\longrightarrow} H^2(E_J, \mathcal{O}) \rightarrow 0.$$

**Lemma 4.14.** Assume that $H^2(E_J, \mathcal{O}) = \mathbb{C}$. Let $\Phi|_{H^1(E^{(i)}_J, \mathcal{O})} = \varphi_i$ and $\Psi|_{H^1(C_i, \mathcal{O})} = \psi_i$. Then the following hold:

(i) $\text{Im} \varphi_1 = \text{Im} \varphi_2 = \text{Im} \Phi$;

(ii) $\psi_i$ is an isomorphism for $i = 1, 2$ and $\text{Ker} \Psi \circ (p_1^{*} \oplus p_2^{*}) = \Delta$, where $\Delta$ is the diagonal subspace of $H^1(C, \mathcal{O}) \oplus H^1(C, \mathcal{O})$;

(iii) fix $C_1$, then the isomorphism $\psi_1$ is independent of the choice of the decomposition of $E_J$ as in 4.13.

**Proof.** If (i) does not hold, then $\text{Im} \Phi \neq \text{Im} \varphi_1$, where $\text{Im} \varphi_1$ is of dimension 1, because $\varphi_1$ is a non-zero map from 1-dimensional vector space. Therefore $\Phi$ becomes surjective, a contradiction to $H^2(E_J, \mathcal{O}_E) \neq 0$. For (ii), consider the composite:

$$H^1(E^{(i)}_J, \mathcal{O}_E) \overset{\varphi_i}{\longrightarrow} H^1(C_1, \mathcal{O}_C) \oplus H^1(C_2, \mathcal{O}_C)$$

One obtains that $\text{Im}(p_1^{*\perp} \oplus p_2^{*\perp}) \circ \varphi_i = \Delta$. Therefore $\psi_i$ is not a zero map. For (iii), take another $C_2'$ and $E^{(i)'}_J$ ($i = 1, 2$) such that $E^{(1)'}_J \cap E^{(2)'}_J = C_1 \amalg C_2'$. One may assume that $E_2' \subset E^{(1)}_J$ and $E_1' \subset E^{(2)}_J$. Let $E^{(3)}_J$ be a subchain of $E_J$ such that $E^{(1)'}_J \cap E^{(2)'}_J = C_1 \amalg E^{(3)}_J$. Then $C_2, C_2' \subset E^{(3)}_J$. By these inclusions, it follows the commutative diagram:

$$\begin{array}{c}
H^1(E^{(1)}_J) \oplus H^1(E^{(2)}_J) \overset{\Psi}{\longrightarrow} H^2(E_J) \rightarrow 0 \\
\| \quad \uparrow \psi \\
H^1(E^{(1)}_J) \oplus H^1(E^{(2)'}_J) \rightarrow H^1(C_1) \oplus H^1(C_2) \rightarrow H^2(E_J) \rightarrow 0 \\
\| \quad \uparrow \psi' \\\nH^1(E^{(1)'}_J) \oplus H^1(E^{(2)'}_J) \rightarrow H^1(C_1) \oplus H^1(C_2') \rightarrow H^2(E_J) \rightarrow 0.
\end{array}$$

So the restrictions of $\Psi$ and $\Psi'$ on $H^1(C_1, \mathcal{O})$ are the same. \(\square\)
Proof of Theorem 4.12. Let \((Z, z)\) be an isolated strictly log-canonical singularity of type \((0, 1)\), \(\pi : (X, x) \to (Z, z)\) the canonical cover and \(G\) the associated cyclic group. Let \(f : Y \to X\) be a \(G\)-equivariant good resolution and \(E_J\) the essential divisor. Then \(E_J\) is either as in (i) or (ii) of Proposition 2.11.

Case 1. The case that \(E_J\) is as in (ii) of Proposition 2.11.

Let \(E_J = E^{(-)} + E^{(0)} + E^{(+)}\) be the decomposition as in (ii). Then there is a ruling \(p : E^{(0)} \to C\) over an elliptic curve \(C\). Since each fiber of \(p\) is mapped to a fiber of \(p\) by the action of \(G\), \(C\) admits the action of \(G\) and \(p\) becomes a \(G\)-equivariant morphism. Now by Mayer-Vietoris exact sequence:

\[
H^1(E^{(-)} + E^{(0)}, \mathcal{O}) \oplus H^1(E^{(0)} + E^{(+)}, \mathcal{O}) \to H^1(E^{(0)}, \mathcal{O})
\]

\[
\to H^2(E_J, \mathcal{O}) \to H^2(E^{(-)} + E^{(0)}, \mathcal{O}) \oplus H^2(E^{(0)} + E^{(+)}, \mathcal{O}) = 0,
\]

one obtains a \(G\)-equivariant isomorphism \(H^1(E^{(0)}, \mathcal{O}) \simeq H^2(E_J, \mathcal{O})\). On the other hand there is a \(G\)-equivariant isomorphism \(p^* : H^1(C, \mathcal{O}) \to H^1(E^{(0)}, \mathcal{O})\). Since the action of \(G\) on \(H^1(C, \mathcal{O})\) is induced from that on \(C\), the order of the action on \(G\) on \(H^1(C, \mathcal{O})\) is 1, 2, 3, 4, 6 by Proposition 4.8.

Case 2. The case that \(E_J\) is as in (i) of Proposition 2.11.

If the intersection curves are all fixed under the action of \(G\), the generator \(g\) of \(G\) induces an automorphism of each intersection curve. Take \(C_i\) and \(E^{(i)}\) \((i = 1, 2)\) as in 4.13. Then one obtains the commutative diagram of isomorphisms:

\[
\begin{array}{ccc}
H^1(C_1) & \xrightarrow{\psi_1} & H^2(E_J) \\
g|_{C_1} & \downarrow & g^* \\
H^1(C_1) & \xrightarrow{\psi_1} & H^2(E_J).
\end{array}
\]

Since \(g|_{C_1}\) is of order 1, 2, 3, 4, 6 by Proposition 4.8, so is \(g^*\).

If \(g(C_1) = C_2\) for \(C_1 \neq C_2\), then under the notation in 4.13 let \(h : C \to C\) be an automorphism \(p_2 \circ g|_{C_1} \circ p_1^{-1}\). By the definition of \(h\), it follows the commutative diagram of isomorphisms:

\[
\begin{array}{ccc}
H^1(C) & \xrightarrow{p_2^*} & H^1(C_2) \\
\downarrow h^* & g|_{C_1} & \downarrow g^* \\
H^1(C) & \xrightarrow{p_1^*} & H^1(C_1) \\
\psi_1 & \xrightarrow{\psi_2} & H^2(E_J).
\end{array}
\]

where \(\psi_2\) is induced from \(\psi_1\) through \(g\). Here, note that \(H^2(E_J, \mathcal{O}) = \mathbb{C}\) by the assumption of the singularity. So one can apply Lemma 4.14, (iii) and obtains that \(\psi_2 = \psi_2\). On the other hand, as \(\text{Ker}\Psi \circ (p_1^* \oplus p_2^*) = \Delta\) by Lemma 4.14, (ii), it follows that \(\psi_1 \circ p_1^* = -\psi_2 \circ p_2^*\). Hence, by the diagram above, the order of \(g^*\) is 1, 2, 3, 4, 6 since that of \(h^*\) is 1, 2, 3, 4, 6 by 4.8. \(\square\)
Theorem 4.15. For a positive integer $r$ the following are equivalent:
(i) $r$ is the index of a 3-dimensional strictly log-canonical singularity;
(ii) $\varphi(r) \leq 20$ and $r \neq 60$, where $\varphi$ is the Euler function.

Proof. First assume (i), then by theorems 4.5, 4.11 and 4.12, it follows that $\varphi(r) \leq 20$. If there exists a 3-dimensional strictly log-canonical singularity $(Z, z)$ of index 60. Then by 4.5 and 4.12, $(Z, z)$ must be of type $(0, 2)$. Let $E$ be the exceptional divisor on the canonical model of the canonical cover $(X, x)$, then $E$ is normal $K3$-surface. Let $G$ be the corresponding group of the canonical cover, then $G$ acts on $E$ whose induced action on $H^2(E, \mathcal{O}_E)$ is of order 60. Since this action is lifted to the minimal resolution $\tilde{E}$ of $E$, one obtains a $K3$-surface $\tilde{E}$ which admits an automorphism whose action on $H^2(\tilde{E}, \mathcal{O}_{\tilde{E}})$ is of order 60. However, it is proved by Machida-Oguiso [10] that there is no $K3$-surface with such an automorphism.

Next assume (ii), then by [8] and [12], there is a $K3$-surface $E$ with an automorphism $g : E \to E$ whose order and the order of induced automorphism on $H^2(E, \mathcal{O}_E)$ are both $r$. Let $G = \langle g \rangle$, $\pi : E \to E'/G$ the quotient map and $\mathcal{L}$ an ample invertible sheaf on $E'$. Let $Y'$ and $Y$ be the line bundles $\text{Spec} \left( \bigoplus_{m \geq 0} \mathcal{L}^\otimes m \right)$ and $\text{Spec} \left( \bigoplus_{m \geq 0} \pi^* \mathcal{L}^\otimes m \right)$ on $E'$ and on $E$, respectively. Then $Y \to E$ has the zero section $E_0$ whose normal bundle is $\pi^* \mathcal{L}^{-1}$, so there is a contraction $f : (Y, E_0) \to (X, x)$ of $E_0$. Since the exceptional divisor $E_0$ is $K3$-surface, the singularity $(X, x)$ is strictly log-canonical of index 1 and of type $(0, 2)$ by [7]. One defines an action of $G$ on $(X, x)$ in the following way: Let $\sigma$ be the action of $G$ on $E$. On the other hand there is also an action $\tau$ of $G$ on $Y'$ which is trivial on $E'$, because $Y'$ admits a canonical action of $\mathbb{C}^*$ and $G$ is considered as a subgroup of $\mathbb{C}^*$. Since $Y$ is the fiber product $E \times_{E'} Y'$, one obtains the action of $G$ on $Y$ which is compatible with $\sigma$ and $\tau$. It is clear that this action is free on $Y \setminus E_0$ and $E_0$ is $G$-invariant. Therefore one can introduce the action of $G$ on $(X, x)$. The quotient $(Z, z) = (X, x)/G$ is strictly log-canonical of index $r$ by Corollary 3.6. □

4.16. Boundedness of indices of higher dimensional strictly log-canonical singularities is also expected to follow from Conjecture 4.7. On the contrary, if indices of $n$-dimensional strictly log-canonical singularities are bounded, then Conjecture 4.7 holds for $(n - 1)$-dimensional Calabi-Yau varieties. Indeed, as in the proof of Theorem 4.15, for every Calabi-Yau $(n - 1)$-fold $E$ and a finite order automorphism $g$, one can construct a strictly log-canonical singularity of index $r$, where $r$ is the order of the induced automorphism $g^*$. Hence the boundedness of indices implies Conjecture 4.7.

References

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