MODULI OF SEXTICS AND ITS GEOMETRY

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1. INTRODUCTION

Let $\mathcal{M}$ be the moduli space of sextics with 6 cusps and 3 nodes. A sextic $C$ is called of (2,3)-torus type if its defining polynomial $f$ has the expression $f(x, y) = f_2(x, y)^3 + f_3(x, y)^2$ for some polynomials $f_2, f_3$ of degree 2, 3 respectively. Hereafter we simply say of torus type in the sense of (2,3)-torus type. We denote by $\mathcal{M}_{\text{torus}}$ the component of $\mathcal{M}$ which consists of curves of torus type and by $\mathcal{M}_{\text{gen}}$ the curves of non-torus type. We denote the dual curve of $C$ by $C^*$. In our previous paper [O2], we have shown that the dual curve operation $C \mapsto C^*$ gives an involution on $\mathcal{M}$ and it preserves the type of the curve in $\mathcal{M}$, i.e., $C^* \in \mathcal{M}_{\text{torus}}$ if and only if $C \in \mathcal{M}_{\text{torus}}$. Let $\mathcal{N}_3$ be the moduli space of sextics with 3 (3,4)-cusps as in [O2]. For brevity, we denote $\mathcal{N}_3$ by $\mathcal{N}$. We have shown that $\mathcal{N}$ is in the closure of $\mathcal{M}$ and the dual curve $C^*$ of a generic $C \in \mathcal{N}$ is a sextic with 6 cusps and three nodes i.e., $C^* \in \mathcal{M}$ ([O2]). Let $G := \text{PGL}(3, \mathbb{C})$. The quotient moduli spaces are by definition the quotient spaces of the moduli spaces by the action of $G$.

In §2, we will study the quotient moduli space $\mathcal{M}/G$ and we will show that there exists an involution $\iota$ on $\mathcal{M}/G$ such that $\iota$ is different from the dual curve operation and $\iota$ preserves the types of the sextics (Theorem 2.3).

In §3, we study the quotient moduli space $\mathcal{N}/G$. We will show that $\mathcal{N}/G$ is one dimensional and consists of two components $\mathcal{N}_{\text{torus}}/G$ and $\mathcal{N}_{\text{gen}}/G$ consisting of sextics of torus type and non-torus type respectively. Using their normal forms, we show that $\mathcal{N}_{\text{torus}}/G$ contains a unique sextic which is self dual (Theorem 3.9).

2. INVOLUTION ON THE QUOTIENT MODULI $\mathcal{M}/G$

Let $\mathcal{M}$ and $\mathcal{\widetilde{M}}$ be the moduli space of sextics with three nodes and 6 cusps and the moduli space of irreducible plane curves of degree 12 with 24 cusps and 24 nodes respectively. Note that the genus of a generic curve in $\mathcal{M}$ (respectively in $\mathcal{\widetilde{M}}$) is 1 (resp. 7). By the class formula ([N] or [O2]), it is easy to see that for a generic $C \in \mathcal{\widetilde{M}}$, the dual curve $C^*$ is also in $\mathcal{\widetilde{M}}$. We consider the mapping

$$\pi : \mathbb{P}^2 \to \mathbb{P}^2, \quad (X, Y, Z) \mapsto (X^2, Y^2, Z^2)$$

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which is a 4-fold covering branched along the coordinate axes \{X = 0\} ∪ \{Y = 0\} ∪ \{Z = 0\}. Take a generic curve \(C ∈ \mathcal{M}\) and let \(F(X, Y, Z)\) be the defining homogeneous polynomial of degree 6. As \(C^*\) has three nodes, \(C\) has three bi-tangent lines. We denote by \(\mathcal{M}^{nm}\) the subset of \(\mathcal{M}\) which consists of curves \(C ∈ \mathcal{M}\) whose three bitangent lines are \(X = 0\), \(Y = 0\) and \(Z = 0\). We define a mapping \(ψ : \mathcal{M}^{nm} → \tilde{\mathcal{M}}\) as follows. Let \(C ∈ \mathcal{M}^{nm}\) and let \(F(X, Y, Z)\) be the defining homogeneous polynomial. We define \(ψ(C) := \pi^{-1}(C)\). Note that \(ψ(C)\) is defined by \(\tilde{F}(X, Y, Z) := F(X^2, Y^2, Z^2)\). Each cusp of \(C\) produces 4 cusps on \(ψ(C)\). Thus \(ψ(C)\) has 24 cusps. Each node of \(C\) also gives 4 nodes on \(ψ(C)\), thus we get 12 nodes on \(ψ(C)\) which are mapped onto the nodes of \(C\). As the restriction of \(π\) to the affine chart \(\{Z ≠ 0\}\) is the composition of double coverings \((x, y) → (x, y^2)\) and \((x, y) → (x^2, y)\), each simple tangent on the coordinate axis \(X = 0\), \(Y = 0\) gives 2 nodes on \(ψ(C)\). This is the same for the simple tangents for \(Z = 0\). Thus there are 12 nodes on \(ψ(C)\) which are on the three coordinate axes and they are mapped to simple tangents on coordinate axis by \(π\). Thus \(ψ(C)\) has 24 nodes. Thus \(ψ(C) ∈ \tilde{\mathcal{M}}\).

Now for \(C ∈ \mathcal{M}\), we define \(ψ(C)\) as \(ψ(C^g)\) by choosing a \(g ∈ G\) such that \(C^g ∈ \mathcal{M}^{nm}\). The ambiguity for the choice of \(g ∈ G\) are in the stabilizer \(G_{\mathcal{M}^{nm}}\) of \(\mathcal{M}^{nm}\) which is a direct product of \(S_3\) (the permutations of coordinates) and \(C^* × C^* × C^*\) (scalar multiplications). Thus the polynomial \(\tilde{F}(X, Y, Z)\) is also unique up to a \(G_{\mathcal{M}^{nm}}\) action, and therefore \(\tilde{F}(X, Y, Z)\) is also unique up to a \(G_{\mathcal{M}^{nm}}\) action. Thus \(ψ : \mathcal{M}/G → \tilde{\mathcal{M}}/G\) is well-defined.

Recall that a polynomial \(F(X, Y, Z)\) is called even in \(X\) (respectively symmetric in \(X, Y\)) if \(F(−X, Y, Z) = F(X, Y, Z)\) (resp. \(F(Y, X, Z) = F(X, Y, Z)\)). Thus the polynomial \(F(X^2, Y^2, Z^2)\) is even in \(X, Y, Z\).

Assume that \(C ∈ \mathcal{M}\) is defined by \(F(X, Y, Z) = 0\). If \(F\) is a even polynomial in the variable \(X\) (respectively a symmetric polynomial in \(X, Y\)), then 6 cusps are stable by the involution \((X, Y, Z) → (−X, Y, Z)\) (respectively \((X, Y, Z) → (Y, X, Z)\)). Then there exists a homogeneous polynomial \(F_2(X, Y, Z)\) of degree 2 which is even in \(X\) (respectively symmetric in \(X, Y\)) such that the conic \(F_2(X, Y, Z) = 0\) passes through the 6 cusps of \(C\). By the criterion of Degtyarev [D], the sextic \(F(X, Y, Z) = 0\) is of torus type.

Now we take a generic \(C ∈ \mathcal{M}^{nm}\) and consider the dual curve \(ψ(C)^*\) and let \(\tilde{G}(X^*, Y^*, Z^*)\) be a defining homogeneous polynomial of degree 12, where \((X^*, Y^*, Z^*)\) is the dual coordinates of \((X, Y, Z)\). As \(\tilde{F}(X, Y, Z)\) is even in \(X, Y, Z\), so is \(\tilde{G}(X^*, Y^*, Z^*)\) in \(X, Y, Z\).

**Proposition 2.1.** \(ψ(C)^*\) has 4 nodes on each coordinate axis \(X^* = 0\), \(Y^* = 0\) or \(Z^* = 0\).

**Proof.** Let \(C = \{F(X, Y, Z) = 0\}\) and let us consider the discriminant polynomial \(Δ_Y F(X, Z)\). This is a homogeneous polynomial of degree 30 ([O1]). We assume that the singularities of the sextic \(F(X, Y, Z) = 0\) are not on the coordinate axis. Assume that \(P := (α, β, γ) ∈ C\) is a singular point of \(C\) with Milnor number \(μ\) and multiplicity \(m\). Then \(Δ_Y F(X, Z)\) has a linear term \((γX − αZ)^ρ\) with \(ρ ≥ μ + m − 1\) and the equality holds if the line \(γY − βZ = 0\) is generic with respect to \(C\) (see [O2]). Thus to each cusp (respectively to each node), there is an associated linear term with multiplicity \(3\) (resp. with multiplicity 2). The factor \(X = 0\) and \(Z = 0\) has also multiplicity 2 in \(Δ_Y F(X, Z) = 0\), as they are bi-tangent lines. Assume \(C\) is generic in \(\mathcal{M}\). Then the
sum of degrees is $18 + 6 + 4 = 28$ by the above consideration. Thus there exists two simple
tangent lines of the form $X - \eta_1 Z = 0$ and $X - \eta_2 Z = 0$ for some $\eta_1, \eta_2 \neq 0$. Then
four lines $X = \pm \sqrt{\eta_i} Z, i = 1, 2$ are bitangent lines for the curve $\psi(C)$. This implies that
$(1, 0, \pm \sqrt{\eta_i})$, $i = 1, 2$ are nodes of the dual curve $\psi(C)^*$. Thus the coordinate axis $Y^* = 0$
contains 4 nodes of $\psi(C)^*$. By the same argument, $X^* = 0$ and $Z^* = 0$ contains also 4
nodes respectively.

**Definition 2.2.** For $C \in \mathcal{M}_{nnml}$, we define a polynomial of degree 6 by $G(X^*, Y^*, Z^*) :=$
$\tilde{G}(\sqrt{X^*}, \sqrt{Y^*}, \sqrt{Z^*})$ and we define $\iota(C)$ by the sextics defined by $G(X^*, Y^*, Z^*) = 0$. For
$C \in \mathcal{M}$, take $g \in G$ so that $C^g \in \mathcal{M}_{nnml}$ and we define an involution $\iota : \mathcal{M}/G \to \mathcal{M}/G$
by $\iota(C) = \iota(C^g)$.

**Claim 1.** $\iota(C) \in \mathcal{M}$ for a generic $C \in \mathcal{M}$ and $\iota$ is an involution which preserves the type of
sextics, that is we have the commutative diagram:

\[
\begin{array}{ccc}
\mathcal{M}/G & \overset{\iota}{\to} & \mathcal{M}/G \\
\downarrow \psi & & \downarrow \psi \\
\mathcal{M}_{torus}/G & \overset{\iota}{\to} & \mathcal{M}_{torus}/G
\end{array}
\]

**Proof.** We may assume that $C \in \mathcal{M}_{nnml}$. By the above consideration, we have seen
that the dual curve $\psi(C)^*$ of $\psi(C)$ is defined by a polynomial $G(X^*, Y^*, Z^*)$ of degree 12
which is even in each of the three variables and it has 24 cusps and 12 nodes outside of
coordinate axis and 4 nodes on each coordinate axis. Thus $\iota(C)$ has 6 cusps and 3 nodes.
Note that nodes of $\psi(C)^*$ on the coordinate axes are mapped on simple tangents on the
perpendicular coordinate axes of $\iota(C)$. Thus the curve $\iota(C)$, defined by $g(\sqrt{x^*}, \sqrt{y^*}) = 0$,
belongs to $\mathcal{M}_{nnml}$. Finally we will show that $\iota$ keeps the type of the curve. As the
curves $\{\iota(C); C \in \mathcal{M}_{torus}/G\}$ are topologically equivalent, the image is contained in a
connected component. Thus it is enough to show that there exists a $C \in \mathcal{M}_{torus}/G$ such
that $\iota(C) \in \mathcal{M}_{torus}/G$. To see this, it is enough to take $C \in \mathcal{M}_{torus}^{nml}$ whose defining
polynomial $F(X, Y, Z)$ is symmetric in each of $X, Y$. Then $\tilde{F}(X, Y, Z)$ is also symmetric
in $X, Y$. This implies also that $\tilde{G}(X^*, Y^*, Z^*)$ and $G(X^*, Y^*, Z^*)$ symmetric in $X^*, Y^*$.
By the Degtyarev’s criterion, this implies that $\iota(C)$ is a sextic of torus type. The following example shows that $\iota(C) \neq C^*$ in general.

Thus we have proved the following:

**Theorem 2.3.** There exists an involution $\iota$ on the quotient moduli space $\mathcal{M}/G$ such that
$\iota$ is different from the dual curve operation and $\iota$ preserves the types of the sextics, that
is $\iota(C) \in \mathcal{M}_{torus}/G \iff C \in \mathcal{M}_{torus}/G$.

**Example 2.4.** Let $C \in \mathcal{M}_{torus}^{nml}$ be the sextic defined by the symmetric polynomial:

\[
f := -684(x^3 + xy^3) - 1055(x^3 + y^3) + 2235(x^2 + y^2) - 2178(x + y) + \frac{189}{16}(x^5 y + y^5 x) + \frac{1767}{8}(x^4 y^2 + x^2 y^4) + \frac{881}{8} x^3 y^3 + \frac{405}{16} (x^6 + y^6) - \frac{372}{8} (x^5 y^2 + y^5 x^2) + \frac{2001}{4} (x^6 + y^6) - \frac{91}{8} (x^4 y + y^4 x) - \frac{6947}{2} y^2 x^2 + 2268 + 1038(x^2 y + xy^2) - 4883 y x - \frac{378}{2} (x^2 y^3 + x^3 y^2)
\]

Then $\psi(C)$ is defined by $f(x^2, y^2)$ and $\psi(C)^*$ is defined by $g(x^2, y^2) = 0$ and $\iota(C)$ is
the sextic defined by the symmetric polynomial

\[
g(x^*, y^*) := 908294 x^2 y^2 - 354000(x^* y^2 + x^2 y^*) + 302745(y^4 + x^4) + 529284(x^4 y^2 + y^4 x^2) - 396458(x^* y^4 + y^4 x^*) - 722148(x^3 y^2 + y^3 x^2) + 11340(y^6 + x^6) - 109170(x^5 +
\]

$y^5 + 86296x^5y^* + 482724(x^3y^* + y^*x^3) - 158508(y^*x^5 + y^5x^*) + 103096y^*x^3 - 22230(x^* + y^*) - 203920(y^*x^3 + x^*y^3) + 90570(y^*x^2 + x^*y^2) + 2025$

The dual curve $C^*$ of $C$ is defined by the following symmetric polynomial and we can easily check that $\tau(C) \neq C^*$.

$h(x^*, y^*) := 3(x^*^4 + y^*^4) + 14(x^*^3 + y^*^3) + 3(x^*^2 + y^*^2) + 4(y^*x^* + y^*y^*) + 36(y^*x^*y^* + x^*y^*y^*) + 6(y^*x^* + x^*y^*) - 2y^*x^* + 12(y^*x^2 + x^*y^2) + 84(y^*x^3 + x^*y^3) + 14y^*x^4 + 88y^*x^3 + 4y^*x^2 + 3$.

3. Normal forms of the moduli $\mathcal{N}$

We consider the submoduli $\mathcal{N}^{(1)}$ of the sextics whose cusps are at $O := (0, 0)$, $A := (1, 1)$ and $B := (1, -1)$. Under the action of $G$, every sextic in $\mathcal{N}$ can be represented by a curve in $\mathcal{N}^{(1)}$. Consider the stabilizer group $G^{(1)} := \{g \in G; g(\mathcal{N}^{(1)}) = \mathcal{N}^{(1)}\}$. By an easy computation, we see that $G^{(1)}$ is the semi-direct product of the group $G_0^{(1)}$ and a finite group $\mathcal{K}$ where $\mathcal{K}$ is a finite linear subgroup of $G$, isomorphic to the permutation group $S_3$, and $G_0^{(1)}$ is defined by

$G_0^{(1)} := \{M \in G; a_3(a_1^2 - a_2^2) \neq 0\}$

which fix singular points pointwise. Note that $G_0^{(1)}$ is normal in $G^{(1)}$. The isomorphism $\mathcal{K} \cong S_3$ is given by identifying $g \in \mathcal{K}$ as the permutation of three singular locus $O, A, B$. We will study the normal forms of the quotient moduli $\mathcal{N}/G \cong \mathcal{N}^{(1)}/G^{(1)}$.

Lemma 3.1. For a given line $L := \{y = bx\}$ with $b^2 - 1 \neq 0$, there exists $M \in G_0^{(1)}$ such that $L^M$ is given by $x = 0$.

Proof. By an easy computation, the image of $L$ by the action of $M^{-1}$, where $M$ is as above, is defined by $(a_1 - ba_2)y + (a_2 - ba_1)x = 0$. Thus we take $a_1 = ba_2$. Then $a_1^2 - a_2^2 = a_2^2(b^2 - 1) \neq 0$ by the assumption. \qed

Lemma 3.2. The tangent cone at $O$ is not $y = x = 0$ for $C \in \mathcal{N}^{(1)}$.

Proof. Assume for example that $y - x = 0$ is the tangent cone of $C$ at $O$. The intersection multiplicity of the line $L_1 := \{y - x = 0\}$ and $C$ at $O$ is 4 and thus $L_1 \cdot C \geq 7$, an obvious contradiction to Bezout theorem. \qed

Let $\mathcal{N}^{(2)}$ be the subspace of $\mathcal{N}^{(1)}$ consisting of curves whose tangent cone at $O$ is given by $x = 0$. Let $G^{(2)}$ be the stabilizer of $\mathcal{N}^{(2)}$. By Lemma 3.1 and Lemma 3.2, we have the isomorphism:

Corollary 3.3. $\mathcal{N}^{(1)}/G^{(1)} \cong \mathcal{N}^{(2)}/G^{(2)}$. 

It is easy to see that $G^{(2)}$ is generated by the group $G^{(2)} = G^{(2)} \cap G^{(1)}$ and an element $\tau$ of order two defined by $\tau(x, y) = (x, -y)$. Note that

$$G^{(2)}_0 = \{ M = \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_1 & 0 \\ 1-a_3 & 0 & a_3 \end{pmatrix} \in G^{(1)}_0; \ a_1a_3 \neq 0 \}$$

For $C \in \mathcal{N}^{(2)}$, we associate complex numbers $b(C), c(C) \in \mathbb{C}$ which are the directions of the tangent cones of $C$ at $A, B$ respectively. This implies that the lines $y - 1 = b(C)(x - 1)$ and $y + 1 = c(C)(x - 1)$ are the tangent cones of $C$ at $A$ and $B$ respectively. We have shown that $C \in \mathcal{N}^{(2)}$ if and only if $b(C) + c(C) = 0$ and $C$ is not of torus type if and only if $c(C)^2 + 3c(C) - b(C)c(C) + 3 - 3b(C) + b(C)^2 = 0$ ($\S 4, [O2]$).

We consider the subspaces:

$$\mathcal{N}^{(3)}_{torus} := \{ C \in \mathcal{N}^{(2)}_{{torus}}; b(C) = 1 \}, \quad \mathcal{N}^{(3)}_{gen} := \{ C \in \mathcal{N}^{(2)}_{{gen}}; b(C) = c(C) = \sqrt{-3} \}$$

and we put $\mathcal{N}^{(3)} := \mathcal{N}^{(3)}_{torus} \cup \mathcal{N}^{(3)}_{gen}$.

**Remark.** The common solution of the both equations: $b + c = c^2 + 3c - bc + 3 - 3b + b^2 = 0$ is $(b, c) = (1, -1)$ and in this case, $C$ degenerates into two non-reduced lines $(y^2 - x^2)^2 = 0$ and a conic.

**Lemma 3.4.** Assume that $C \in \mathcal{N}^{(2)}$. Then there exists $C' \in \mathcal{N}^{(3)}$ and an element $g \in G^{(2)}$ such that $C^g = C'$ and such a $C'$ is unique. This implies that

$$\mathcal{N}^{(3)}_{torus}/G \cong \mathcal{N}^{(3)}_{torus}/G^{(2)} \cong \mathcal{N}^{(3)}_{torus}, \quad \mathcal{N}^{(3)}_{gen}/G \cong \mathcal{N}^{(3)}_{gen}/G^{(2)} \cong \mathcal{N}^{(3)}_{torus}$$

**Proof.** Assume that $C \in \mathcal{N}^{(1)}_{{torus}}, b + c = 0$. Consider an element $g \in G^{(1)}_0$,

$$g^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1-a_3 & 0 & a_3 \end{pmatrix}$$

The image $L^g_A$ is given by $y - x + xa_3 - a_3 + bxa_3 + ba_3 = 0$. Thus we can solve the equation $a_3^2(1 - b) - 1 = 0$ in $a_3$ uniquely as $a_3 = 1/(1 - b)$ as $b \neq 1$. Thus $g \in G^{(1)}_0$ is unique if it fixes the singular points pointwise and thus $C'$ is also unique. It is easy to see that the stabilizer of $\mathcal{N}^{(3)}_{torus}$ is the cyclic group of order two generated by $\tau$, as $C'$ is even in $y$ (see the normal form below) and $C'^{\tau} = C'$ for any $C' \in \mathcal{N}^{(3)}_{torus}$. Thus we have $\mathcal{N}^{(2)}_{torus}/G \cong \mathcal{N}^{(3)}_{torus}$.

Consider the case $C \in \mathcal{N}^{(2)}_{{gen}}$. Then the images of the tangent cones at $A, B$ by the action of $g$ are given by $y - x + xa_3 - a_3 - bxa_3 + ba_3 = 0$ and $y + x - xa_3 + a_3 + cxa_3 + ca_3$ respectively. Assume that $b(C^g) = c(C^g)$. Then we need to have $a_3(1 - b) - 1 = a_3(-1 - c) + 1$, which has a unique solution in $a_3$, if $(*) b - c - 2 \neq 0$. Assume that $c^2 + 3c - bc + 3 - 3b + b^2 = 0$ and $b - c - 2 = 0$. Then we get $(b, c) = (1, -1)$ which is excluded as it corresponds to non-reduced sextic. Thus the condition $(*)$ is always satisfied. Put $(b', c') := (b(C^g), c(C^g))$. They satisfy the equality $c'^2 + 3c' - b'c' + 3 - 3b' + b'^2 = 0$ and $b' = c'$. Thus we have either $b' = c' = \sqrt{-3}$ or $b' = c' = -\sqrt{-3}$. However in the second case, we can take the automorphism $(x, y) \rightarrow (x, -y)$ to change into the first case. Thus $b' = c' = \sqrt{-3}$ and $C^g \in \mathcal{N}^{(3)}_{{gen}}$ as desired. \[\square\]
3.1. Normal forms of curves of torus type. In [O2], we have shown that a curve in $\mathcal{N}_{\text{torus}}^{(1)}$ is defined by a polynomial $f(x, y)$ which is defined by a sum $f_2(x, y)^3 + s f_3(x, y)^2$ where $f_2(x, y)$ is a smooth conic passing through $O, A, B, f_3(x, y) = (y^2 - x^2)(x - 1)$ and $s \in C^*$.

Proposition 3.5. The direction of the tangent cones at $O, A$ and $B$ are the same with the tangent line of the conic $f_2(x, y) = 0$ at those points.

This is immediate as the multiplicity of $f_2(x, y)^3$ at $O, A, B$ are 4. See also Lemma 23 of [O2]. Assume that $C \in \mathcal{N}_{\text{torus}}^{(3)}$, that is, the tangent cones of $C$ at $O, A$ and $B$ are given by $x = 0, y - 1 = 0$ and $y + 1 = 0$ respectively. Thus the conic $f_2(x, y) = 0$ is also uniquely determined as $f_2(x, y) = y^2 + x^2 - 2x$. This is the circle with radius 1, centered at $(1,0)$. Therefore $\mathcal{N}_{\text{torus}}^{(3)}$ is one-dimensional and it has the representaion

$$
C_s : \quad f_{\text{torus}}(x, y, s) := f_2(x, y)^3 + s f_3(x, y)^2 = 0
$$

For $s \neq 0, 27, C_s$ is a sextic with three $(3,4)$ cusps, while $C_{27}$ obtains a node. As is easy to see, if $g \in G^{(2)}$ fixes the tangent lines $y \pm 1 = 0$, then $g = e$ or $\tau$ and $C^*_{\tau} = C_s$. Thus $C_s \neq C_t$ if $s \neq t$.

3.2. Normal form of sextics of non-torus type. For the moduli of non-torus type sextic $\mathcal{N}_{\text{gen}}$, we start from the expression given in §4.1, [O2]. We may assume $b = c = \sqrt{-3}$. Then the parametrization is given by

$$
f_{\text{gen}}(x, y, s) := f_0(x, y) + s f_3(x, y)^2, \quad f_3(x, y) = (y^2 - x^2)(x - 1)
$$

where $s$ is equal to $a_{06}$ in [O2] and $f_0$ is the sextic given by

$$
f_0(x, y) := y^6 + y^5(6\sqrt{-3} - 6\sqrt{-3}x) + y^4(35 - 76x + 38x^2)

+ y^3(-24\sqrt{-3}x + 36\sqrt{-3}x^2 - 12\sqrt{-3}x^3) + y^2(-94x^2 + 200x^3 - 103x^4)

+ y(24\sqrt{-3}x - 42\sqrt{-3}x^4 + 18\sqrt{-3}x^5) + 64x^3 - 133x^4 + 68x^5
$$

Let $D_s := \{f_{\text{gen}}(x, y, s) = 0\}$ for each $s \in C$. Observe that $D_0 = \{f_0(x, y) = 0\}$ is a sextic with three $(3,4)$-cusps and of non-torus type. For the computational reason, we take the substitution $y \mapsto y\sqrt{-3}$ to make the equation to be defined over rational numbers: Then $f_0(x, y)$ and $f_3(x, y)$ change into:

$$
f_0(x, y) := -27y^6 + (-162 + 162x)y^5 + (315 - 684x + 342x^2)y^4

+ (-216x + 324x^3 - 108x^3)y^3 + (282x^2 - 600x^3 + 309x^4)y^2

+ (-54x^5 + 126x^4 - 72x^3)y + 68x^5 + 64x^4 - 133x^4

f_3(x, y) := -(x - 1)(3y^2 + x^2)
$$

Summarizing the discussion, we have

Theorem 3.9. The quotient moduli space $\mathcal{N}/G$ is one dimensional and consists of two components.

(1) The component $\mathcal{N}_{\text{torus}}/G$ has the normal forms represented by the family of sextics $C_s = \{f(x, y, s) = 0\}$ where $f(x, y, s) = f_2(x, y)^3 + s f_3(x, y)^2$ for $s \in C^*$ and $s \neq 0, 27$ where

$$
f_2(x, y) = y^2 + x^2 - 2x, \quad f_3(x, y) = (y^2 - x^2)(x - 1)
$$
The curve $C_{54}$ is a unique curve in $N/G$ which is self-dual.

(2) The component $N_{gen}/G$ of sextics of non-torus type has the normal form: $f_{gen}(x, y, s) = f_{0}(x, y) + sf_{3}(x, y)^{2}$ where $f_{3}$ is as above and the sextic $f_{0}(x, y) = 0$ is contained in $N_{gen}$. This component has no self-dual curve.

Proof of Theorem 3.9. We need only prove the assertion for the dual curves. The proof will be done by a direct computation of dual curves using the method of §2, [O2] and the above parametrizations. We use Maple V for the practical computation. Here is the recipe of the proof. Let $X^{*}, Y^{*}, Z^{*}$ be the dual coordinates of $X, Y, Z$ and let $(x^{*}, y^{*}) := (X^{*}/Z^{*}, Y^{*}/Z^{*})$ be the dual affine coordinates.

1. Compute the defining polynomials of the dual curves $C^{*}_{s}$ and $D^{*}_{s}$ respectively, using the method of Lemma 2.4, [O2]. Put $g_{torus}(x^{*}, y^{*}, s)$ and $g_{gen}(x^{*}, y^{*}, s)$ the defining polynomials.

2. Let $G_{e}(X^{*}, Y^{*}, Z^{*}, s)$ be the homogenization of $g_{e}(x^{*}, y^{*}, s)$, $e = torus$ or $gen$. Compute the discriminant polynomials $\Delta_{Y^{*}}(G)$ which is a homogeneous polynomial in $X^{*}, Z^{*}$ of degree 30 (cf. Lemma 2.8, [O1]). Recall that the multiplicity of the pencil $X^{*} - \eta Z^{*} = 0$ passing through a singular point is generically given by $\mu + m - 1$ where $\mu, m$ are the Milnor number and the multiplicity of the singularity ([O2]). Thus the contribution from a $(2,3)$-cusp (respectively from a $(3,4)$-cusp) is 3 (resp. 8). Thus if $C_{s}^{*}$ has three $(3,4)$ cusps, it is necessary that $\Delta_{Y^{*}}(G) = 0$ has three linear factors with multiplicity at least 8.

(3-1) For the non-torus curves, it is not possible to get a degeneration into 3 $(3,4)$-cuspidal sextic.

(3-2) For the torus curves, we can see that $s = 54$ is the only possible parameter. Thus it is enough to show that $C_{54}^{*} \cong C_{54}$.

(4) The dual curve $C_{54}^{*}$ of $C_{54}$ is defined by the homogeneous polynomial

$$G(X^{*}, Y^{*}, Z^{*}) := 128X^{*5}Z^{*} + 1376X^{*4}Z^{*2} - 192X^{*3}Y^{*2}Z^{*} + 4664X^{*3}Z^{*3} - 2X^{*2}Y^{*4} - 1584X^{*2}Y^{*2}Z^{*2} + 7090X^{*2}Z^{*4} + 58X^{*4}Z^{*3} - 3060X^{*2}Z^{*3} + 5050X^{*2}Z^{*5} + Y^{*6} + 349Y^{*4}Z^{*2} - 1725Y^{*2}Z^{*4} + 1375Z^{*6}$$

We can see that $C_{54}^{*}$ has also 3 $(3,4)$-cusps. Moreover we can see that $C_{54}^{*}$ is isomorphic to $C_{54}$ as $(C_{54}^{*})^{A} = C_{54}$ where

$$A = \begin{pmatrix} -4/3 & 0 & -5/3 \\ 0 & 1 & 0 \\ -5/3 & 0 & -13/3 \end{pmatrix}$$

3.3. Involution $\tau$ on $C_{54}$. For the later purpose, we change the coordinates of $G$ so that the three cusps of $C_{s}$ are at $O_{x} := (0, 0, 1), O_{y} := (0, 1, 0), O_{z} := (1, 0, 1)$. New normal form in affine space is given by $f(x, y, s) = f_{2}(x, y)^{3} + sf_{3}(x, y)^{2}$ where

$$f_{2}(x, y) := xy - x + y, \quad f_{3}(x, y) := -xy$$
and $C_{54}$ is defined by $f(x, y) = (xy - x + y)^3 - 54x^3y^3 = 0$. In this coordinate, $C_{54}^*$ is defined by

$$-28y^3 - 17x^4y^2 - 17x^2y^4 - 28x^3y^3 - 2y^5 + 1788x^3y + 1788x^2y - 17y^4 - 17x^4$$

$$+ 262xy + 1788x^2 + 262xy - 2y5 + 1 - 17y^2 - 17x^2 + 2x^5 + 2x - 2y + x^6 + y^6 = 0$$

It is easy to see that $(C_{54}^*)^{A_1} = C_{54}$ where

$$A_1 := \left(\frac{-1}{3}, \frac{7}{3}, \frac{1}{3}\right)$$

Let $F(X, Y, Z)$ be the homogenization of $f(x, y)$. Then the Gauss map induces an automorphism $\text{dual}_C : C_{54} \to C_{54}^*$ which is defined by $(X, Y, Z) \mapsto (F_X, F_Y, F_Z)$, where $F_X, F_Y, F_Z$ are partial derivatives. We define an isomorphism $\tau : C_{54} \to C_{54}$ by the composition of $\text{dual}_{C_{54}}$ and the linear map $\varphi_{A_1} : C_{54}^* \to C_{54}$ which is defined by the multiplication by $A_1$ from the right. $\tau$ is given by the restriction of the rational mapping: $\Psi : C^2 \to C^2$.

$$\Psi(x, y) = (x_{d}, y_{d})$$

Observe that $\tau$ is defined over $Q$. $C_{54}$ has three flexes of order 2 at $F_1 := (1, -1/4, 1)$, $F_2 := (1/4, -1, 1)$, $F_3 := (4, -4, 1)$ and $\tau$ exchanges flexes and cusps:

$$\begin{align*}
\tau(O_X) &= F_1, \tau(O_Y) = F_2, \tau(O_Z) = F_3, \\
\tau(F_1) &= O_X, \tau(F_2) = O_Y, \tau(F_3) = O_Z
\end{align*}$$

Furthermore we assert that

**Proposition 3.12.** The morphism $\tau$ is an involution $C_{54}$.

For the proof, we prepare a lemma. Let $C$ be a given irreducible curve in $\mathbb{P}^2$ defined by a homogeneous polynomial $F(X, Y, Z)$ and let $B \in \text{GL}(3, \mathbb{C})$. Then $C_B$ is defined by $G(X, Y, Z) := F((X, Y, Z)B^{-1})$. Let dual$_C : C \to C^*$ be the Gauss map which is defined by $(X, Y, Z) \mapsto (F_X(X, Y, Z), F_Y(X, Y, Z), F_Z(X, Y, Z))$.

**Lemma 3.13.** Two curves $(C_B)^*$ and $(C^*)^B$ coincide and the following diagram commutes.

$$\begin{array}{ccc}
C & \xrightarrow{\text{dual}_B} & C^* \\
\varphi_B & \downarrow & \varphi_B^{-1} \\
C_B & \xrightarrow{\text{dual}_{C^*}B} & (C^*)^B
\end{array}$$

**Proof.** This is essentially the same as Lemma 2, [02]. The assertion follows from the following equalities. Let $(a, b, c) \in C$.

$$\text{dual}_{OB}(\varphi_B(a, b, c)) = (G_X(\varphi_B(a, b, c)), G_Y(\varphi_B(a, b, c)), G_Z(\varphi_B(a, b, c)))$$

$$= (F_X(a, b, c), F_Y(a, b, c), F_Z(a, b, c))^B = \varphi_{B^{-1}}(\text{dual}_C(a, b, c)) \square$$
Proof of Proposition 3.12. By the definition of $\tau$, we have $(C := C_{54})$:

$$\tau \circ \tau = (\varphi_{A_{1}^{-1}} \circ \text{dual}_{C})^2 = (\text{dual}_{C_{A_{1}}} \circ \varphi_{A_{1}}) \circ (\varphi_{A_{1}^{-1}} \circ \text{dual}_{C}) = \text{id}$$

as $A_{1}$ is a symmetric matrix.

Of course, the same assertion is true for $C_{54}$ in the old normal form. $C_{54}$ has another obvious involution $\iota : C_{54} \rightarrow C_{54}$ which is defined by $(x, y) \mapsto (x, -y)$ in the old normal form. For the application to arithmetic property of cubic curves, see [O3].

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