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TORIC MODIFICATIONS OF LINE SINGULARITIES ON SURFACES

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ABSTRACT. We study the topology of the Milnor fibre $F$ of a function $f$ with critical locus a smooth curve $L$ on a surface $X$, where $X$ has an isolated complete intersection singularity and contains $L$. We use toric modification to resolve the non-isolated singularity $V = X \cap f^{-1}(0)$. Then we compute the Euler-Poincaré characteristic of $F$. Some examples are worked out.

INTRODUCTION

Let $(X, 0) \subset (\mathbb{C}^{n+1}, 0)$ be a germ of an ics (isolated complete intersection singularity) and contain a smooth curve $L$, which will be called a line in this article. We are interested in the topology of the Milnor fibre $F_f$ of a function $f$ whose zero level hypersurface passes $L$ or is tangent to the regular part of $X$ along a line $L$. Hence the critical locus of $f$ contains $L$ if its zero hypersurface is tangent to $X_{\text{reg}}$ along $L$. Since toric modification is used, we assume that $f$ together with the defining equations $h_1, \ldots, h_{n-1}$ of $X$ form a non-degenerate complete intersection.

Let $L$ be the $x$-axis in a local coordinate system defined by the ideal $\mathfrak{g} = (y_1, \ldots, y_n)$. Since $L \subset X$, their defining ideals satisfy the relation $\mathfrak{h} := (h_1, \ldots, h_{n-1}) \subset \mathfrak{g}$. This implies that $X$ is not convenient or commode in French. Also the zero level hypersurface defined by $f$ contains $L$, so $V := X \cap f^{-1}(0)$ is not convenient either. If $X$ also contains another axis of the local coordinate system, then there is a point $Q$ in the dual Newton diagram $\Gamma^*(h_1, \ldots, h_{n-1}, f)$ of $V$ such that $Q$ is not strictly positive, not on the axes and the minimal value $d(Q; \mathfrak{h})$ of the linear function determined by the covector $Q$ on the Newton polyhedron $\Gamma_{+}(h_1, \ldots, h_{n-1})$ is positive, but $d(Q; f) = 0$ on $\Gamma_{+}(h_1, \ldots, h_{n-1}, f)$. This means that the assumptions, called $\sharp$ and $\sharp'$ conditions in the literature (see for example [12, P.128, P.205]), are not satisfied. These "sharp" requirements seem essential in order to get the good resolution and zeta function along the last "principal direction" of the non-degenerate complete intersection $(h_1, \ldots, h_{n-1}, f)$.

Nevertheless, in case $X$ is a surface with isolated singularity, we really can replace these "sharp" conditions by a weaker one and obtain a good resolution of $f$ on $X$. By A’Campo’s theorem, we are able to compute the zeta function of the algebraic monodromy of $F_f$ and the Euler-Poincaré characteristic of $F_f$.

In case $f$ is a generic function contained in $\mathfrak{g}$, our work also supplies some information on the hypersurface intersection of $X$ along the line contained therein.

If $f \notin \mathfrak{g}^2$ and the transversal singularity type of $f$ along $L$ is $A_1$, practically to get the Euler-Poincaré characteristic of $F_f$ we do not need to resolve the function $f$ (which might be very general) since the theory developed in [6]. For example we can consider the Milnor fibre $F_q$ of a generic quadric form $q$ in the variables $y_1, \ldots, y_n$. The Euler-Poincaré characteristic of $F_q$ can be expressed by: the Euler-Poincaré characteristic of $F_q$, the number of Morse points outside $L$, and the number of $D_{\infty}$ points on $L \setminus 0$ of the Morsification $f + q$. If, moreover, $F_q$ is connected, $F_f$ is also connected, hence a bouquet of one circles.

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As applications, we prove that $F_t$ is homotopically a bouquet of one cycles if $f \in \mathfrak{g}^2$ has transversal $A_1$ singularity along $L$ and $X$ is an $A_k - D_k - E_6 - E_7$ type surface singularity. We also prove that $F_t$ is in general not connected when $X$ is a Brieskorn-Pham surface.

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1. PRELIMINARIES

1.1. Let $\mathcal{O}_{\mathbb{C}^m}$ be the structure sheaf of $\mathbb{C}^m$. The stalk $\mathcal{O}_{\mathbb{C}^m,0}$ of $\mathcal{O}_{\mathbb{C}^m}$ at 0 is denoted by $\mathcal{O}$. Let $(X,0)$ be a reduced analytic space germ in $(\mathbb{C}^m,0)$ defined by the radical ideal $\mathfrak{h}$ of $\mathcal{O}$. Let $(L,0)$ be the germ of a subspace of $X$ defined by the radical ideal $\mathfrak{g}$ of $\mathcal{O}$. Denote $\mathcal{O}_X := \mathcal{O}/\mathfrak{h}$, $\mathcal{O}_L := \mathcal{O}/\mathfrak{g}$.

Let $\text{Der}(\mathcal{O})$ denote the $\mathcal{O}$-module of germs of analytic vector fields on $\mathbb{C}^m$ at 0. Define $D_X := \text{Der}_h(\mathcal{O}) = \{\xi \in \text{Der}(\mathcal{O}) \mid \xi(\mathfrak{h}) \subset \mathfrak{h}\}$, which is the $\mathcal{O}$-module of logarithmic vector fields along $(X,0)$ (cf. [1]). Geometrically, $D_X$ consists of all germs of vector fields that are tangent to the smooth part of $X$. Equipped with $X$ there is a so called logarithmic stratification induced by logarithmic vector fields [1]. Especially, when $X$ is purely dimensional and has isolated singularity in 0, then $\{0\}$ and the connected components of $X \setminus \{0\}$ form a holonomic logarithmic stratification of $X$. And this stratification is a Whitney stratification.

Let $S = \{S_\alpha\}$ be an analytic stratification of $X$, $f : (X,0) \rightarrow (\mathbb{C},0)$ an analytic function germ. The critical locus $L_\alpha^f$ of $f$ relative to the stratification $S$ is the union of the closure of the critical loci of $f$ restricted to each of the strata $S_\alpha$, namely, $L_\alpha^f = \bigcup L_{f|S_\alpha}$. If $\dim L_\alpha^f = 0$, we say $f$ is (or defines) an isolated singularity on $(X,0)$. Otherwise, $f$ is (or defines) a nonisolated singularity on $(X,0)$. If $L_\alpha^f$ is a smooth curve, we say $f$ is (or defines) a line singularity on $(X,0)$. In this article, we always use the logarithmic stratification to define singularities of functions.

All the functions whose critical loci contain $L$ form an ideal of $\mathcal{O}$

$$\int_X \mathfrak{g} := \{f \in \mathcal{O} \mid (f) + J_X(f) \subset \mathfrak{g}\},$$

called the primitive ideal of $\mathfrak{g}$ (cf. [15, 14, 6]). This ideal collects all the functions whose zero level surfaces are tangent to $X_{\text{reg}}$ along $L$. Obviously $\mathfrak{g}^2 + \mathfrak{h} \subset \int_X \mathfrak{g} \subset \mathfrak{g}$.

1.2. For $f \in \int_X \mathfrak{g}$ we define an ideal $J_X(f) := \{\xi(\mathfrak{f}) \mid \xi \in D_X\}$, called Jacobian ideal of $f$. Call $\mathfrak{g}/(J_X(f) + \mathfrak{h})$ the Jacobian module of $f$ on $X$, and its dimension over $\mathbb{C}$ is called the Jacobian number of $f$ on $X$ and is denoted by $j(f) := \dim_{\mathbb{C}} \mathfrak{g}/(J_X(f) + \mathfrak{h})$. If $X_{\text{sing}} \subset \{0\}$ and $\dim L = 1$, it is known [6] that $j(f) < \infty$ if and only if the transversal singularity type of $f$ along $L \setminus \{0\}$ is $A_1$.

The $\mathcal{O}_L$-module $M := \overline{\mathfrak{g}}/\mathfrak{g}^2 \cong \mathfrak{g}/(\mathfrak{g}^2 + \mathfrak{h})$ is called the conormal module of $\mathfrak{g}$ (as an ideal of $\mathcal{O}_X$). Denote $T := \int_X \mathfrak{g}/(\mathfrak{g}^2 + \mathfrak{h})$, $N := \mathfrak{g}/\int_X \mathfrak{g}$. We have the exact sequence of $\mathcal{O}_L$-modules

$$0 \rightarrow T(M) \rightarrow M \rightarrow N \rightarrow 0.$$

If $L$ does not contain any irreducible components of $X_{\text{sing}}$, $T(M)$ is the torsion submodule of $M$. In case $\dim L = 1$, $T(M)$ has finite length, called torsion number of $(L,X)$, denoted by $\lambda(L,X)$. See [9] for generalizations of primitive ideals and torsion numbers.

1.3. Let $L$ be a line (i.e. smooth curve). We choose $L$ to be the $x$-axis of the local coordinate system in $(\mathbb{C}^{n+1},0)$. Then $L$ is defined by ideal $\mathfrak{g} = (y_1, \ldots, y_n)$. For a function $f \in \mathfrak{g}^2$, we have $f = \sum_{k,l=1}^n h_{kl}y_ky_l$ with $h_{kl} = h_{lk}$. Let $U = \{u := (u_{kl}) \in \mathbb{C}^{n^2} \mid u_{kl} = u_{lk}\}$, and $V = \mathbb{C}^{mn}$.
with coordinates \( v = (v_{jk})_{1 \leq j \leq m, 1 \leq k \leq n} \). Let \( s = (u, v) \) be the coordinates of \( S = U \times V \). Define \( f_s(z) := f + q(s, z) \) with

\[
q(s, z) := \sum_{k,l=1}^{n} u_{kl} + \sum_{j=1}^{m} z_j v_{jk} \delta_{kl}
\]

where \( \delta_{kl} \) is Kronecker’s delta, \( z_0 := x, z_j := y_j (1 \leq j \leq n) \) are the local coordinates of \((\mathbb{C}^{n+1}, 0)\).

The following proposition is a generalization of a result due to Siersma-Pellikaan \cite{17, 16}, and the proof is similar to \cite{16}(cf. \cite{6}).

**Proposition 1.** Let \( (X, 0) \subset (\mathbb{C}^{n+1}, 0) \) be an icis of pure dimension \( n - p + 1 \), and \( (L, 0) \subset (\mathbb{C}^{n+1}, 0) \) be a line. If \( j(f) < \infty \). Then there exists a Zariski open dense subset \( S' \subset U \times V \) such that

1. For any \( s \in S' \), \( f_s \) has only isolated Morse points on \( X \setminus L \), and only \( A_{\infty} \) and \( D_{\infty} \) type singularities on \( L \setminus \{0\} \);
2. The module \( N \) is free, and if the images of \( y_{p+1}, \ldots, y_n \) form the basis of \( N \),

\[
\delta := \dim F_{\infty} = \dim_{\mathbb{C}} \frac{\mathcal{O}_L}{(\det(h_{kl}p_{1 \leq k, l \leq n})};
\]

We say that \( f_s \) is a good deformation of \( f \).

1.4. Let \( B_{\varepsilon} \) denote an open ball of radius \( \varepsilon \) centered at \( 0 \), \( \Delta_\eta \) denote an open disk in \( \mathbb{C} \) with center \( 0 \) and radius \( \eta \). Let \( \varepsilon \) and \( \eta \) be admissible for the Milnor fibration of \( f \). Namely, there exists the following local trivial topological fibration, the *Milnor fibration*

\[
f : \overline{B}_{\varepsilon}(0) \cap X \cap f^{-1}(\Delta_{\eta}^*) \longrightarrow \Delta_{\eta}^*,
\]

where \( \overline{\Delta}_{\eta}^* = \Delta_{\eta} \setminus \{0\} \). The fibre \( F \) of this fibration is called the *Milnor fibre* of \( f \). The Milnor fibre \( F^c \) of \( f_s \) is called the *central type* of the Milnor fibre \( F \) of \( f \). The following proposition is a generalization of a result of Siersma \cite{17, 18}. The proof of it can be found in \cite{6}.

**Proposition 2.** Let \( L \) and \( X \) be the same as in Proposition 1, and \( f_s \) be a good deformation of \( f \). Let \( \varepsilon \) and \( \eta \) be admissible for the Milnor fibration of \( f \). Then for \( s \in S' \) with \( |s|, \eta \) and \( \varepsilon \) sufficiently small, the map

\[
f_s : \overline{B}_{\varepsilon} \cap f_{s}^{-1}(\Delta_{\eta}) \longrightarrow \overline{\Delta}_{\eta}^*
\]

has the following properties:

1. For all \( t \in \overline{\Delta}_{\eta}^* \), \( f_{s}^{-1}(t) \cap \partial \overline{B}_{\varepsilon} \) (as stratified spaces);
2. For every \( t \in \partial \overline{\Delta}_{\eta}^* \), and hence for every \( t \in \overline{\Delta}_{\eta} \setminus \{ \text{critical values of } f_s \} \), there is a homeomorphism: \( F := f^{-1}(t) \cong \overline{F} := f_{s}^{-1}(t) \);
3. There is a homeomorphism: \( f_{s}^{-1}(\Delta_{\eta}) \cong \Delta_{\eta} \);
4. Let \( F^0 \) be the intersection of \( \overline{\Delta}_{\eta}^* \) with a sufficiently small tubular neighborhood \( T \) of \( L \) such that \( f_s \) is not Morse type points. Then \( F^0 \) can be obtained from \( F \) by attaching \( n \)-cells along a transversal vanishing cycle of \( F \), the number of the \( n \)-cells is \( 2\dim_{\mathbb{C}} \);
5. If \( \dim X = n - p + 1 > 3 \) and \( F^c \) is simply connected \( \overline{F} \cong F^0 \cup S^{n-p} \cup \cdots \vee S^{n-p} \), the number of \( S^{n-p} \) is the number of Morse point on \( X \setminus L: h_{A_1} \);
6. If \( \dim X = 2 \) and \( F^c \) is connected \( \overline{F} \cong F^0 \cup \cdots \vee S^1 \), the number of \( S^1 \) is \( h_{A_1} + 2\dim_{\mathbb{C}} - 1 \).
1.5. Let \( g : (\mathbb{C}^{n+1}, 0) \longrightarrow (\mathbb{C}, 0) \) be an analytic function germ. Let \( g := \sum_{\nu} a_{\nu} z^{\nu} \) be the Taylor expansion of a representative of \( g \). The Newton polyhedron \( \Gamma_{+}(g) \) (with respect to the local coordinate \( z \)) is by definition the convex hull of \( \bigcup_{\nu | a_{\nu} \neq 0} \{ \nu + \mathbb{R}^{n+1} \} \). The Newton boundary \( \Gamma(g) \) (with respect to the local coordinate \( z \)) is by definition the collection of all the compact facets of \( \Gamma_{+}(g) \).

Let \( P \in \text{Hom}_{\mathbb{Z}}(\mathbb{Z}^{n+1}, \mathbb{Z}) \) with non-negative integral coordinates \( p_{0}, \ldots, p_{n} \), and be denoted by \( P = T(p_{0}, \ldots, p_{n}) \geq 0 \), called positive covector. As an \( \mathbb{R} \)-linear function on \( \mathbb{R}^{n+1} \), the restriction of \( P \) to \( \Gamma_{+}(g) \) has a minimal value, denoted by \( d(P; g) \). Denote also \( \Delta(P; g) = \{ z \in \Gamma_{+}(g) \mid P(z) = d(P; g) \} \). The face function of \( g \) with respect to \( P \) is by definition \( g_{P}(z) = g_{\Delta(P; g)} := \sum_{\nu \in \Delta(P; g)} a_{\nu} z^{\nu} \).

Let \( X \) be a complete intersection defined by \( \mathcal{O} \)-regular sequence \( h_{1}, \ldots, h_{p} \). The Newton polyhedron \( \Gamma_{+}(h_{1}, \ldots, h_{p}) \) of \( X \) is by definition the mixed sum of \( \Gamma_{+}(h_{j}) \), and the Newton boundary \( \Gamma(h_{1}, \ldots, h_{p}) \) of \( X \) is the mixed sum of \( \Gamma(h_{j}) \).

Two positive covector \( P, Q \) are equivalent if and only if \( \Delta(P; h_{j}) = \Delta(Q; h_{j}) \) for \( j = 1, \ldots, p \). The dual Newton diagram \( \Gamma^{*}(h_{1}, \ldots, h_{p}) \) of \( X \) is a collection of all the equivalent classes of positive covectors under the aforementioned equivalence.

\( X \) is called a non-degenerate complete intersection (with respect to the local coordinate \( z \)) if \( X \cap \mathbb{C}^{n+1} \) is a reduced non-singular complete intersection in the complete torus \( \mathbb{C}^{n+1} \).

For more systematical introduction to toric modifications of non-degenerate complete intersections, we refer the reader to [12], where the notions and notations used in this article without explanations can be found.

2. Lines on Singular Spaces

2.1. Let \( (X, 0) \) be a reduced analytic space germ in \( (\mathbb{C}^{n+1}, 0) \). A smooth curve germ \( (L, 0) \) in \( (\mathbb{C}^{n+1}, 0) \) is called a line. If \( L \setminus \{0\} \subset X_{\text{reg}} \), we say that \( X \) contains (or has) a line passing through \( O \).

On a singular space \( X \) in \( \mathbb{C}^{n+1} \) one can not always find a line passing through (not contained in) the singular locus of \( X \). Gonzalez-Sprinberg and Lejune-Jalabert [4, 5] proved a criterion for the existence of smooth curve on any (two dimensional) surface.

The existence and number of lines on surfaces with isolated simple singularities and on Brieskorn-Pham surfaces have been studied in [8, 7].

2.2. Let \( \mathcal{R} \) be the group of all the local automorphisms of \( (\mathbb{C}^{n+1}, 0) \). \( \mathcal{R}_{L} := \{ \phi \in \mathcal{R} | \phi(L) = L \} \) is a subgroup of \( \mathcal{R} \). Define \( L\mathcal{K} := \mathcal{R}_{L} \times \mathcal{C} \), the semi-product of \( \mathcal{R}_{L} \) with the contact group \( \mathcal{C} \) [11]. This group has an action on the space \( \text{mg}^{\mathcal{O}^{P}} \) consisting of mapping germs \( h : (\mathbb{C}^{n+1}, 0) \longrightarrow (\mathcal{O}, 0) \) with components \( h_{j} \in \text{mg} \). For \( h = (h_{1}, \ldots, h_{p}) \in \text{mg}^{\mathcal{O}^{P}} \), we define an analytic space \( X = \mathcal{V}(h) \), where \( h \) is the ideal generated by \( h_{1}, \ldots, h_{p} \). The image of the morphism:

\[ \mathcal{O}^{P+1} \longrightarrow \mathcal{O}^{P} \]

is denoted by \( \text{th}(h) \), where \( dh \) is the differential of \( h \). Define

\[ \tilde{\lambda} := \tilde{\lambda}(L, X) = \dim_{\mathcal{C}} \frac{\mathcal{O}^{P}}{\text{th}(h) + g_{\mathcal{O}^{P}}} \].

Remark that \( \tilde{\lambda} \) is \( L\mathcal{K} \)-invariant.

Let \( x, y_{1}, \ldots, y_{n} \) be the local coordinates of \( (\mathbb{C}^{n+1}, 0) \). Let \( L \) be the \( x \)-axis defined by \( g = (y_{1}, \ldots, y_{n}) \). Then \( L \) can be defined by \( g = (y_{1}, \ldots, y_{n}) \).
2.3. Theorem. Let $L$ be a line on an icis $X$ defined by $\mathfrak{g}$ and $\mathfrak{h}$ as above. Then $h$ is $LK$-equivalent to a mapping germ with components

$$\tilde{h}_j = b_j y_j \mod g^2, \quad j = 1, \ldots, p$$

(2.3.1)

where $b_j \notin \mathfrak{g}$. Moreover

$$\lambda(L, X) = \tilde{\lambda}(L, X) = \dim_{\mathbb{C}} \frac{\mathcal{O}_L}{(\overline{b})} = \sum_{j=1}^{p} \lambda_j.$$

(2.3.2)

where $\overline{b}$ is the image of $b := b_1 \cdots b_p$ in $\mathcal{O}_L$, and $\lambda_j$ is the order of $\overline{b}_j$ in $\mathcal{O}_L$.

Proof. Since $L \subset X$, $\mathfrak{h} \subset \mathfrak{g}$. Then for a given generator set $\{h_1, \ldots, h_p\}$ of $\mathfrak{h}$, we have

$$h_k \equiv \sum \overline{b}_{kj} y_j \mod g^2, \quad k = 1, \ldots, p.$$ 

where $\overline{b}_{kj} \in \mathcal{O}_L$, and for fixed $k$, $\overline{b}_{kj}$'s are not all zero since $X_{\text{sing}} = \{0\} \not\subset L$. Since $\mathcal{O}_L$ is a principal ideal domain, by changing the indices, we can assume $\overline{b}_{11} | b_{kj}$. Let

$$y_1' = y_1 + \sum_{j=2}^{n} \frac{\overline{b}_{1j}}{\overline{b}_{11}} y_j.$$ 

Then

$$h_1 \equiv \overline{b}_{11} y_1' \mod g^2.$$ 

Let

$$h_k' = h_k - \frac{\overline{b}_{k1}}{\overline{b}_{11}} h_1, \quad k = 2, \ldots, p.$$ 

Repeat the above argument will prove the first part of the theorem.

Consider the exact sequence

$$\mathcal{O}^{n+1} \xrightarrow{dh^*} \mathcal{O}^p \longrightarrow \text{coker}(dh^*) \longrightarrow 0.$$ 

By tensoring with $\mathcal{O}_L$, we have exact sequence

$$\mathcal{O}_L^{n+1} \xrightarrow{dh^*} \mathcal{O}_L^p \longrightarrow \text{coker}(dh^*) \longrightarrow 0.$$ 

However by the expression of $\tilde{h}_k$'s above, this is just

$$\mathcal{O}_L^{p} \xrightarrow{dh^*} \mathcal{O}_L^p \longrightarrow \frac{\mathcal{O}^p}{th(h) + g\mathcal{O}^p} \longrightarrow 0.$$ 

Since $\overline{b} \neq 0$, by [3, A.2.6], we have the formula for $\lambda$. 

3. TORIC MODIFICATIONS OF LINE SINGULARITIES ON SURFACES

3.1. In this section we study the toric modifications of functions with lines singularities on surfaces. Let $z_0 := x, z_1 := y_1, \ldots, z_n := y_n$ be the local coordinates of $(\mathbb{C}^{n+1}, 0)$. Let $L = \{y_1 = \cdots = y_n = 0\}$ be contained in $X = \{z \in \mathbb{C}^{n+1} \mid h_1(z) = \cdots = h_{n-1}(z) = 0\}$, the germ at 0 of a two dimensional irreducible non-degenerate icis. Assume that $h_1$ takes the form in (2.3.1).

Consider a function germ $f : (X, 0) \longrightarrow (\mathbb{C}, 0)$ such that $V = X \cap f^{-1}(0)$ is a one dimensional non-degenerate complete intersection. Let $\hat{\pi} : \tilde{X} \longrightarrow \mathbb{C}^{n+1}$ be the admissible toric modification for $V$ associated with a small admissible regular simplicial cone subdivision $\Sigma^*$. Denote by $\hat{X}$ the strict transform of $X$ by $\hat{\pi}$. We denote by $E_j$ the unit vector along the $j$-th axis of $\mathbb{R}^{n+1}$.

For $P \in \Sigma^*$, denote by $\hat{E}(P)$ the exceptional divisor of $\hat{\pi}$, and $D(P) := \hat{E}(P) \cap \tilde{X}$. For a vector $Q = (q_0, q_1, \ldots, q_n)$, define $I(Q) := \{j \mid q_j = 0\}$. Let $|A|$ denote the cardinality of a finite set $A$.

The following theorem generalizes [12, III(6.2)].
3.2. Theorem. Let $X$ be a 2-dimensional non-degenerate projective variety defined by $\mathfrak{h} = (h_1, \ldots, h_{n-1})$ with the form of (2.3.1).

(1) There exists at least one primitive integral covector $Q = (0, p_1, \ldots, p_n)$ in $\Gamma^*(\mathfrak{h})$, the dual Newton diagram of $\mathfrak{h}$, such that $\dim(\Delta(Q; \mathfrak{h}) \cap \Gamma(\mathfrak{h})) = n - 1$.

(2) Assume that $X$ is $(n-1)$-convenient. If on each cone $\text{Cone}(E_0, \ldots, E_{i-1}, E_i, E_{i+1}, \ldots, E_n)$, there exist at most one point $Q$ which belongs to $\Gamma^*(\mathfrak{h})$, such that $\dim(\Delta(Q; \mathfrak{h}) \cap \Gamma(\mathfrak{h})) \geq n - 1$,

then the small toric modification $\pi : \tilde{X} \to X$ for $\mathfrak{h}$ is a good resolution of $X$.

Proof. The first statement follows straightforward from Theorem 2.3.

The proof of (2) is similar to that of [12, III(6.2)].

Suppose $X$ is not convenient, for each vertex $Q \in \text{vertex}(\Sigma^*)$ with $|I(Q)| = 1$, $\hat{\pi} : \hat{E}(Q) \to \mathbb{C}^{I(Q)} := \{z \in \mathbb{C}^{n+1} \mid z_j = 0 \text{ if } j \notin I(Q)\}$ is a surjective morphism with fibre $\mathbb{P}^1$. Since $X$ is $(n - 1)$-convenient, $\pi$ is biholomorphic over $X \cap \left( \mathbb{C}^{n+1} \setminus \bigcup_{|I| = 1} \mathbb{C}^I \right)$. Take such a point $Q$ on, for instance, $\text{Cone}(E_1, \ldots, E_n)$ with $\dim(\Delta(Q; \mathfrak{h}) \cap \Gamma(\mathfrak{h})) \geq n - 1$ by assumption. Hence the exceptional divisor $D(Q)$ is the only non-empty divisor which is surjectively mapped onto $L = \{y_1 = \cdots = y_n = 0\}$. As $\dim(D(Q)) = 1$, the fibre of $\pi$ on $L$ consists of finite points. Indeed it contains exactly one point by [12, III(6.2.1)]. By using Riemann's removable singularity theorem, $\pi : \tilde{X} \setminus \pi^{-1}(0) \to X \setminus \{0\}$ is a biholomorphism.

The following theorem is a slight generalization of [12, III(3.4.11)].

3.3. Theorem. Let $X, V, \hat{\pi}$ be as above. Suppose that $X$ is $(n-1)$-convenient, and on each cone $\text{Cone}(E_0, \ldots, E_{i-1}, E_i, E_{i+1}, \ldots, E_n)$, there exist at most one point $Q$ which belongs to $\Gamma^*(\mathfrak{h})$, such that $\dim(\Delta(Q; \mathfrak{h}) \cap \Gamma(\mathfrak{h})) \geq n - 1$. Assume that

\[(#') Q \in \text{vertex}(\Sigma^*), 1 < |I(Q)| < n, \hat{E}(Q) \cap \tilde{X} \neq \emptyset \implies d(Q; f) > 0.\]

Then the restriction $\pi : \tilde{X} \to X$ of $\hat{\pi}$ is a good resolution of $f$.

Proof. Since the dual Newton diagram $\Gamma^*(h_1, \ldots, h_{n-1}, f)$ is finer than $\Gamma^*(h_1, \ldots, h_{n-1})$, the smoothness of $\tilde{X}$ is obvious by [12, III(3.4)] as $\Sigma^*$ is admissible for $\Gamma^*(h_1, \ldots, h_{n-1})$. And the map

\[\pi : \tilde{X} \cap \left( X \setminus \bigcup_{|I(\sigma)| < n} \hat{E}(\sigma) \right) \to X \cap \left( \mathbb{C}^{n+1} \setminus \bigcup_{|I| < n} \mathbb{C}^I \right)\]

is biholomorphic.

However, for $Q \in \text{vertex}(\Sigma^*) \setminus \{E_0, \ldots, E_n\}$, if $1 < |I(Q)| < n$, then $\hat{E}(Q) \cap \tilde{X}$ is included in the zero locus of $f \circ \pi$. If $|I(Q)| = 1$, then $Q$ is on a cone $\text{Cone}(E_0, \ldots, E_{i-1}, E_i, E_{i+1}, \ldots, E_n)$. Hence even though $\hat{E}(Q) \cap \tilde{X}$ might not be included in the zero locus of $f \circ \pi$, $\pi$ is still bijective on $\hat{E}(Q) \cap \tilde{X}$ by Theorem 3.2. Hence $\pi : \tilde{X} \to X$ is a good resolution of $f$.

3.4. Remark. Note that in case $X$ is a surface in $\mathbb{C}^3$, the $(#')$ condition is empty.

3.5. The zeta function. Let $F_f$ be the Milnor fibre of $f$. We are interested in the zeta function $\zeta_f(t)$ of the Milnor fibration of $f$.

Let $f, X, V$ be the same as before. Assume $(#')$. Let $\Sigma^*$ be the small regular simplicial subdivision of $\Gamma^*(\mathfrak{h}, f)$, where $\mathfrak{h} = (h_1, \ldots, h_{n-1})$. Let $\hat{\pi} : X \to \mathbb{C}^{n+1}$ be the associated toric modification map. By Theorem 3.3, the restriction $\pi : \tilde{X} \to X$ of $\hat{\pi}$ to the strict transform $\tilde{X}$ of $X$ is a good resolution of $f$.

For $P \in \text{vertex}(\Sigma^*)$, denote by

\[D(P) := \hat{E}(P) \cap \tilde{X}, \quad E(P) := \hat{E}(P) \cap \tilde{V},\]
$$D(P)^* := D(P) \setminus \left( \bigcup_{P' \neq P} D(P') \right), \quad E(P)^* := E(P) \setminus \left( \bigcup_{P' \neq P} E(P') \right),$$

$$V^+(f) := \{ P \in \text{vertex}(\Sigma^*) \mid d(P; f) > 0 \}.$$

The total transform is

$$\tilde{V}^{\text{tot}} = \tilde{V} + \sum_{P \in V^+(f)} d(P; f) D(P).$$

Note that the multiplicity of $\pi^* f$ along $D(P)$ is $d(P; f)$. Let

$$\mathcal{V}^{+}(f) := \{ P \in \text{vert}(\Sigma^*) \mid d(P; f) > 0 \}.$$

The total transform is

$$\tilde{V}^{\text{tot}} = \tilde{V} + \sum_{P \in V^+(f)} d(P; f) D(P).$$

By A'Campo formula we have the zeta function and Lefschetz number

$$\zeta_f(t) = \prod_{P \in \mathcal{V}^{+}(f)} (1 - t^{d(P; f)})^{-1} \cdot x(D(P)),$$

$$\Lambda_f^k = \sum_{d(P; f) \mid k} d(P; f) x(D(P)) \quad (k \geq 1).$$

3.6. Since $\tilde{X}$ is a surface, $D(P)$ is a smooth curve. Hence $D(P) \cap D(Q)$ and $E(P)$ are at most zero dimensional for all $P, Q \in V^+(f)$. Define $e(P) := |E(P)|$, the cardinality of the set $E(P)$. Let $P \in \mathcal{V}^{+}(f)$ be strictly positive. By [12, IV(6.2)], we know that

1) $e(P) = 0$ if and only if $\{ \Delta(P; h_1), \ldots, \Delta(P; h_{n-1}), \Delta(P; f) \}$ satisfies the $(A_0)$ condition;
2) $\tilde{e}(P, Q) > 0$ if and only if both $\{ \Delta(P; h) \}$ and $\{ \Delta(Q; \mathfrak{h}) \}$ satisfies the $(A_0)$ condition, $\text{Cone}(P, Q) \subset \Sigma^*$ and $\dim \Delta(P; \mathfrak{h}) \cap \Delta(Q; \mathfrak{h}) \geq n - 2$.

Hence we have (see [12, IV§7])

$$e(P) = \chi(E(P)) = \chi(E^+(P)) = n! V_n(\Delta(P; h_1), \ldots, \Delta(P; h_{n-1}), \Delta(P; f)),$$

where $V_n(\cdots)$ is the Minkowski's mixed volume.

Let $\sigma := \text{Cone}(P, Q, P_2, \ldots, P_n) \in \Sigma^*$ be a regular simplex. By [12, III(3.4.10)], in the coordinate chart $\mathbb{C}_{\sigma}^{n+1}$

$$\tilde{E}(P) \cap \tilde{E}(Q) \cap \tilde{X} = \{ (0, 0, y') \mid \tilde{h}_{1, P, \sigma}(y') = \cdots = \tilde{h}_{n-1, P, \sigma}(y') = 0 \},$$

$$= \{ y' \in \mathbb{C}^{n+1} \mid \tilde{h}_{1, P, \sigma}(y') = \cdots = \tilde{h}_{n-1, P, \sigma}(y') = 0 \},$$

where $\tilde{P} = P + Q$, and $\tilde{h}_{\alpha, P, \sigma}(y') := h_{\alpha, P, \sigma}(\hat{\pi}_{\sigma}(y'))/\prod_{j=0}^{n-1} y_{\sigma,j}^{d(P_j; h_\alpha)}$. Hence

$$\tilde{e}(P, Q) = \chi(\tilde{E}(P) \cap \tilde{E}(Q) \cap \tilde{X}) = (n-1)! V_{n-1}(\Delta(\tilde{P}; \mathfrak{h})).$$

If $P \in V^+(f)$ is not strictly positive. By [12, IV(6.5)],

3) $e(P) = 0$ since $E(P)$ is empty;
4) $\tilde{e}(P, Q) > 0$ if and only if both $\{ \Delta(P; h_1, P), \ldots, \Delta(P; h_{n-1}, P) \}$ and $\{ \Delta(Q; \mathfrak{h}) \}$ satisfies the $(A_0)$ condition, $\text{Cone}(P, Q) \subset \Sigma^*$ and $\dim \Delta(P; \mathfrak{h}) \cap \Delta(Q; \mathfrak{h}) \geq n - 2$. 

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4. Lines singularities on certain surfaces

4.1. Lemma. Let $(X, 0)$ be a 2-dimensional ics containing the line $L$. Let $f \in \mathfrak{g}$ be a function with $j(f) < \infty$ such that $h_1, \ldots, h_{n-1}, f$ define a complete intersection. Then for generic $s \in S$, the Milnor fibre $F_s$ of $f_s$ is homotopy equivalent to the Milnor fibre of $q(s, z)$, if the Milnor fibre of $q$ is connected. Proof. We give the outline of the proof. Note that in the case $N = M/T(M)$ is free $\mathcal{O}_L$-module, and $q$ is defined by

$$q(s, z) := \sum_{k,l=1}^{n} \left( u_{kl} + \sum_{j=0}^{m} z_j v_{jk} \delta_{kl} \right) y_k y_l,$$

where $z_0 = x$, and $z_j = y_j$ for $j > 0$. Hence for generic parameter value $f_s$ is a good deformation of $f$. Fix such an $s$, define $f_t := t \cdot f + q(s, z)$. Then one proves that for $t \in \mathbb{C} \setminus \{\text{finite points} \neq 0, 1\}$, $f_t$ has no critical points outside $L$ and has only $A_\infty$ type singularity on $L \setminus \{0\}$ in a small neighborhood of $0$. By using a generalized version of additivity of vanishing homology (see e.g. [18, 6]), one proves that $F^q$ and the Milnor fibre of $q$ have the same homology, which implies that they also have the same homotopy type since we assume the connectedness of the Milnor fibre of $q$.

4.2. Denote by

$$q_1(u, z) := \sum_{k,l=1}^{n} u_{kl} y_k y_l.$$

Note that all the terms in $q - q_1$ are "above" the Newton boundary $\Gamma(q_1)$ of $q_1$. The following lemma is a corollary of Damon [2, Corollary 1].

4.3. Lemma. If, for a fixed $u, h_1, \ldots, h_{n-1}, q_1$ define a non-degenerate complete intersection, then the Milnor fibres of $q$ and $q_1$ are homeomorphic.

4.4. In the remainder of this section we study certain functions whose zero level surfaces have higher order contact with a surface along a line contained therein. Let $L$ be a line in $\mathbb{C}^3$ defined by $y = (y, z)$, and contained in a surface $X$ defined by $h = (h) \subset \mathfrak{g}$. Assume that $X_{\text{sing}} = \{0\}$. Define $f(s) = \sum_{i=0}^{c} a_i y^{-i} z^i \in \mathfrak{g}^c$, where $(a_0, \ldots, a_c) \in \mathbb{C}^{c+1}$ are generic.

Let $\Sigma^* \subset \Gamma^*(h)$, the dual Newton diagram of $h$, such that the restriction of $\Sigma^*$ to each two dimensional cone $\text{Cone}(P, Q)$ is obtained by the canonical way as described in [12, III2]. Associated with this $\Sigma^*$ there is a toric modification $\hat{\pi} : \hat{X} \rightarrow \mathcal{C}^{3}$. The restriction $\pi$ of $\hat{\pi}$ to the strict transform $\hat{X}$ of $X$ under $\pi$ is called the canonical toric modification of $X$. Denote by $\Gamma^*(h)_{2}^{+} \subset \Sigma^*$ be the union of two dimensional cones $\sigma_2 = \text{Cone}(P, Q)$ of $\Gamma^*(h)$ such that for any $P_1 \in \sigma_2 \cap \Sigma^* \setminus \{P, Q\}$, $P_1 >> 0$ and $\dim(\Delta(P_1; h)) \geq 1$. Let $\mathcal{G}_X^*$ be the graph of $\Sigma^* | \Gamma^*(h)_{2}^{+}$. The dual resolution graph $\mathcal{G}_X$ of $X$ can be obtained from $\mathcal{G}_X^*$ in the way described by [12, III(6.3)].

Now we study $\Gamma_+(h)$ more carefully. In $\Gamma_+(h)$ we have a non-compact face $Q : qy + z = q$ by [12, III(6.1)] with vertices $A(a, 1, 0)$ and $C(c, 0, q)$ (see the proof of loc. cit.). Let $P : ax + \beta y + \gamma z = \delta$ be the face in $\Gamma_+(h)$ which intersects with $Q$ along $AC$. Assume that $\gcd(\alpha, \beta, \gamma) = 1$. Hence in the dual Newton diagram $\Gamma^*(h)$ we have the point $Q = \Gamma_1(0, 1, q)$ on the edge $E_2 E_3$. And the $\text{Cone}(P, Q)$ also belongs to $\Gamma^*(h)$. One sees that $P = \Gamma_1(\alpha, \delta - \alpha a, \frac{\delta - \alpha a}{q})$ and det $P Q = \alpha$.

4.5. Lemma. The divisor $E(Q)$ is a reduced smooth curve on $X$ intersecting the exceptional divisor $E(Q_1)$ transversally, and is biholomorphic to $L$ under $\pi$. And $d(\mathcal{Q}; f^{(c)}) = c$.

Proof. Let $Q_1 = \frac{1}{\alpha}(P + k_1 Q) = \Gamma_1(1, q_1, q_2)$ be the first point ("near" $Q$) in the canonical subdivision of $P Q$. One sees that

$$q_1 = \frac{\delta - \alpha a + k_1 q}{\alpha}, \quad q_2 = \frac{\delta - \alpha c + k_1 q}{q\alpha},$$
where $k_1$ is the smallest integer such that $0 < k_1 < \alpha$, and both $q_1$ and $q_2$ are integers. Then the simplex $\sigma$ determined by $QQ_1 E_2$ is regular for $h$. The restriction of $\hat{\pi}$ to this chart is

$$\hat{\pi}_\sigma : x = v, y = u^q v^{a_1} w, z = u v^{a_2},$$

then

$$h \circ \hat{\pi}_\sigma = uv^{a_1+q_1}(1 + w + \cdots).$$

One sees that $u = 0, v = t$ defines $\tilde{L}$, which is mapped on to $L$ biholomorphically. \hfill \Box

4.6. As the Newton polyhedron $\Gamma_+(f^{(c)})$ consists of one non-compact face: $U : y + z = \zeta$, we assume from now on that $f^{(c)} = a_0y^s + a_\zeta z^s$. The Newton polyhedron $\Gamma_+(h, f^{(c)})$ consists of two kind of faces: 1) certain faces coming from the parallel transformations of the faces of $\Gamma_+(h) \cup \Gamma_+(f^{(c)}); 2)$ the faces spanned by the parallel transformations in $y$-direction and $z$-direction of the edges of $\Gamma(h)$. A calculation shows that each face from class 2) has equation of the form $F : ax + \beta y + \beta' z = \gamma'$. Hence the dual Newton diagram $\Gamma^+(h, f^{(c)})$ is a subdivision of $\Gamma^+(h)$ by adding the point $U = \tau(0, 1, 1)$ to $E_2 E_3$ and certain points of the form $P' = \tau(\alpha', \beta', \beta')$ to some two dimensional cone of $\Gamma^+(h)$.

Note that if all the points of form $P'$ which are qualified to be added to $\Gamma^+(h)$ are equal to some points in $\Gamma^+(h)$, then the canonical toric modification of $X$ is also a good resolution of $f^{(c)}$. And $V$ and $X$ have the same resolution graph (including the self intersection numbers of the exceptional divisors). Although in general this is not the case, the dual resolution graph $\mathcal{G}_V$ and the total dual resolution graph $\mathcal{G}_V^{\text{tot}}$ of $f^{(c)}$ can be obtained from $\mathcal{G}_X$ by adding some vertices. To do this one only needs to identify the faces of the form $P'$. In the remainder of this section we will do this for certain classes of surfaces.

4.7. Theorem. If $X$ is a surface with isolated simple singularity and contains a line, the toric modification of $X$ is already a good resolution of $f^{(c)}$ and the Milnor fibre of $f^{(c)}$ is a bouquet of 1-cycles for any integer $t > 0$. In particular, the Milnor fibre of any function $f$ with $j(f) < \infty$ is a bouquet of 1-cycles. The zeta function $\zeta_{f^{(c)}}(t)$ and Milnor number $\mu(f^{(c)})$ are listed in table 1.

<table>
<thead>
<tr>
<th>Type of $X$</th>
<th>Equations</th>
<th>$\lambda$</th>
<th>$\zeta_{f^{(c)}}(t)$</th>
<th>$\mu(f^{(c)})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_{k, l}$</td>
<td>$x^k y + x^l z + y z = 0$</td>
<td>1</td>
<td>$(1-t^{k+l+1})(1-t^2)^2$</td>
<td>$t^2 + (t-2)s + 1$</td>
</tr>
<tr>
<td>$D_{k, 2}$</td>
<td>$x^2 y + y^{k-1} + z^2 = 0$</td>
<td>2</td>
<td>$(1-t^{2k+1})(1-t^{k+2})^2$</td>
<td>$2k^2 + 1$</td>
</tr>
<tr>
<td>$D_{5, 2}$</td>
<td>$x^2 y + x z^2 + y^2 = 0$</td>
<td>2</td>
<td>$(1-t^{5k+1})(1-t^{2k+1})^2$</td>
<td>$2k^2 + 1$</td>
</tr>
<tr>
<td>$D_{2k, l}$</td>
<td>$x^l y + x y^2 + z^2 = 0$</td>
<td>1</td>
<td>$(1-t^{2k+1})(1-t^{l+1})^2$</td>
<td>$k(s+1) - 2s + 1$</td>
</tr>
<tr>
<td>$D_{2k, 1, l}$</td>
<td>$x^l y + x z^2 + y^2 = 0$</td>
<td>1</td>
<td>$(1-t^{2k+1})(1-t^{l+1})^2$</td>
<td>$k(s+1) - 2s + 1$</td>
</tr>
<tr>
<td>$E_{6, 2}$</td>
<td>$x^3 y + y^3 + z^2 = 0$</td>
<td>2</td>
<td>$(1-t^{6k})(1-t^{3k})^2$</td>
<td>$2k^2 + 1$</td>
</tr>
<tr>
<td>$E_{7, 3}$</td>
<td>$x^3 y + y^3 + z^2 = 0$</td>
<td>3</td>
<td>$(1-t^{7k})(1-t^{3k+1})^2$</td>
<td>$3k^2 + s + 1$</td>
</tr>
</tbody>
</table>

Proof. By studying $\Gamma(h, f^{(c)})$ case by case, one sees that the resolution of $X$ is already a good resolution of $f^{(c)} : X \to \mathbb{C}$. One only need to resolve $X$. By toric modification (cf. [12]), we obtain a “canonical” resolution of $X$. The dual resolution graph $\mathcal{G}$ can be obtained by the way described in §4.4.

Note that the strict transform of $f^{(c)}$ only intersects with the reduced components of $Z_X$. The weight of each component $E(T)$ of $Z_X$ can be computed on the line $x + y = \zeta$, the only compact 1-facet of $\Gamma^+(f^{(c)})$.

We include the total resolution graph of $f^{(c)}$. In the graphs, a bullet • denotes an (compact) exceptional divisor of the resolution of $X$. A small circle ○ denotes a branch of the strict transform
of $V$. A circled circle $\odot$ denotes the lifting of $L$, the divisor corresponding to the point $Q$ in §4.4. Each number in the parentheses denotes the multiplicity of $f^{(\sigma)} \circ \pi$ along the divisor to which the number attached.

\[ G_{A_{k},\ell}^{\text{tot}} : \]
\[ G_{D_{k},2}^{\text{tot}} : \]
\[ G_{D_{2l,\iota}}^{\text{tot}} : \]
\[ G_{D_{52}^{*}}^{\text{tot}}, \]

\[ \mathcal{G}_{A_{k},\ell}, \mathcal{G}_{D_{k},2}, \mathcal{G}_{D_{2l,\iota}}, \mathcal{G}_{D_{52}^{*}}^{\text{tot}}, \ldots \]
From the total resolution graphs we see immediately the zeta functions and the Euler-Poincaré characteristics.

4.8. Remark. Among simple surface singularities only $A_k - D_k - E_6 - E_7$ type surfaces have lines and their definition equations are given in the table 1 (cf. [8]). If $\zeta = 1$, the above theorem gives information about the hyperplane intersections of $X$ by a generic plane passing through the line. If $\zeta = 2$, the zeta functions and Milnor numbers are those of the central type of a function with line singularity and $j(f) < \infty$. One sees clearly how the torsion number $(\lambda = l)$ enters the resolution data. The theorem also provides information about the topology of generic functions coming from $g^\sigma/g^{\sigma+1}$. 

4.9. Let $X$ be a Brieskorn-Pham surface $G(p, q, r) : h = x^p + y^q + z^r = 0$. Assume that $1 < p < q < r$ and gcd$(p, q) = 1$. By [7], if $r > pq$ and $p \nmid r, q \nmid r$, there exists $\left[ \frac{r}{pq} \right]$ different families of lines on $G(p, q, r)$. Let $L_{k+1}$ be the family of lines with $\lambda = \lambda_{k+1} := (k + 1)(p - 1)q$ $(k = 0, 1, \ldots, \left[ \frac{r}{pq} \right] - 1)$. We first choose a line in $L_{k+1}$ on $G(p, q, r)$ to be the last axis in a local coordinate system $x', y', z'$ of $\mathbb{C}^3$. Then the line is defined by $g = (x', y')$. Define function $f_{k+1}^{(\sigma)} := ax'^\sigma + ay'^\sigma$, where $\sigma > 0$ is an integer as before, and $a, b$ are generic constants. Then we consider the transformed function of $f_{k+1}^{(\sigma)}$ under the inverse coordinate transformation. We still denote this function by $f_{k+1}^{(\sigma)}$. 

\[
G_{D_{2l+1}}^{\text{tot}}, \quad (l \geq 3)
\]

\[
G_{E_6}^{\text{tot}}, 2
\]

\[
G_{E_7}^{\text{tot}}, 3
\]
4.10. **Theorem.** The Milnor fibre of \( f_{k+1}^{(1)} \) is a bouquet of 1-cycles. The Milnor fibre of \( f_{k+1}^{(c)} \) is not connected and consists of \( \varsigma \) disjoint pieces. The zeta function is

\[
\zeta_{f_{k+1}^{(c)}}(t) = \frac{(1 - t^{(k+1)p})p(1 - t^{(k+1)p^2\varsigma})}{(1 - t^{p\varsigma})(1 - t^{(k+1)p^2})(1 - t^{(k+1)p^2\varsigma})},
\]

and the Euler-Poincaré characteristic of the Milnor fibre is \( \chi(f_{k+1}^{(c)}) = -\varsigma p(\lambda_{k+1} + k) \).

**Proof.** Note that \( \Gamma^*(h) \) consists of three arms: \( PE_1 \), \( PE_2 \) and \( PE_3 \). Let \( R_i, S_j \) and \( T_k \) denote the points added to these arms in order to get the canonical subdivision of the respective 2-simplices. One sees that (cf. [7]) the exceptional divisor corresponding to \( T_{k+1} = \tau((k+1)q, (k+1)p, 1), \quad (k = 0, \ldots, \left[ \frac{c}{pq} \right] - 1) \) are reduced. And they are the only reduced ones in \( Z_X \). The lines in \( L_{T_{k+1}} \) can be parameterized as

\[
x = kqu_1 \frac{1+(kp+\alpha)q}{p}, \quad y = kp, u_1 \frac{1+(kp+\alpha)q}{p}, \quad z = cu_1 t,
\]

where \( u_1 \) is a unit satisfying \( 1 + u_1 + c^{-(kp+\alpha)q}u_1^{r-(kp+\alpha)q} = 0 \), and \( 0 \leq \alpha < p \) such that \( \frac{1+(kp+\alpha)q}{p} \) is an integer. The torsion number of the lines in \( L_{T_{k+1}} \) are: \( \lambda_{k+1} := (k+1)(p-1)q \).

Then

\[
f_{k+1}^{(c)} = a(x - \bar{u}_1 z^{(k+1)\varsigma}) + b(y - \bar{u}_2 z^{(k+1)\varsigma}),
\]

where \( \bar{u}_1 \) and \( \bar{u}_2 \) are unit functions of \( z \).

From the Newton boundary \( \Gamma^*(h, f_{k+1}^{(c)}) \), one sees that the canonical toric modification of \( X \) is a good resolution of \( f_{k+1}^{(c)} \). The following is the total resolution graph.

![Resolution Graph](image)

From the total resolution graph one sees immediately the zeta function. The Milnor fibre \( F_{k+1}^{(1)} \) of \( f_{k+1}^{(1)} \) is connected since there are reduced components in \( G_{G(p,q,r)}^{\varsigma} \). In case \( \varsigma > 1 \), all the multiplicities of the divisors in \( G_{G(p,q,r)}^{\varsigma} \) have common divisor \( \varsigma \). Hence the Milnor fibre \( F_{k+1}^{(c)} \) of \( f_{k+1}^{(c)} \) is a disjoint union of \( F_{k+1}^{(1)} \).

4.11. **Remark.** The reason for the Milnor fibre \( F_{k+1}^{(c)} \) (\( \varsigma > 1 \)) being not connected is that the function \( f_{k+1}^{(c)} \) does not have \( D_\infty \) in its deformation. In the following example, the function considered has a \( D_\infty \) point in its good deformation, and its Milnor fibre is a bouquet of one cycles. This is similar to the case in which \( X \) is smooth [17, 18].

4.12. **Example.** Let \( X \) be defined by \( h = x^2 + y^3 + z^7 \). There is a line \( L \) on \( X \) parameterized by (see [7])

\[
x = -c^{21}(1 + t)^{11}t^3, \quad y = -c^{14}(1 + t)^7t^2, \quad z = -c^6(1 + t)^3t.
\]

Let \( \alpha := \alpha(\varsigma), \beta := \beta(\varsigma) \) be analytic functions such that \( \alpha(0)\beta(0) \neq 0 \) and \( x - \alpha z^3 = 0, y - \beta z^2 = 0 \) define \( L \). Consider the function \( f = (x - \alpha z^3)^2 + (y - \beta z^2)^2 \). The Newton polyhedron \( \Gamma_+(h, f) \)
is as Figure 1. The equations of the faces other than the coordinate planes in $\Gamma_+(h, f)$ are as follows.

- **FHZ**: $21x + 14y + 6z = 72 \rightsquigarrow P \in \Gamma^*(h, f)$
- **CDFH**: $3x + 2y + z = 11 \rightsquigarrow P_1 \in \Gamma^*(h, f)$
- **ABCD**: $3x + 2y + 2z = 12 \rightsquigarrow P_2 \in \Gamma^*(h, f)$
- **ADF**: $5x + 4y + 2z = 20 \rightsquigarrow R \in \Gamma^*(h, f)$
- **BC∞**: $x + 2z = 2 \rightsquigarrow Q \in \Gamma^*(h, f)$

Part of the minimal regular subdivision $\Sigma^*$ of the dual Newton diagram $\Gamma^*(h, f)$ of $V := X \cap f^{-1}(0)$ is as Figure 2, where $R_1 = T(11, 7, 3)$, $S_1 = T(7, 5, 2)$, $S_2 = T(13, 9, 4)$, $Q_1 = T(2, 1, 2)$, $Q_2 = T(4, 3, 2)$. From the total resolution graph Figure 3 we see a reduced branch. This implies the Milnor fibre $F$ of $f$ is connected and is a bouquet of $\mu = 16$ copies of $S^1$.

**Figure 1.** The Newton polyhedron $\Gamma_+(h, f)$

**Figure 2.** The dual Newton diagram $\Gamma^*(h, f)$ and part of $\Sigma^*$
Figure 3. The Total resolution graph of $V$

REFERENCES


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