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Kyoto University
Introduction to Resolution of Singularities

Pierre Milman (University of Toronto)

We will present an elementary proof of a canonical resolution of singularities in characteristic zero (at least in the hypersurface case) including detailed examples illustrating some (elementary, but important) applications and the constructive features of the "local to global" argument. The proof is by introduction of a discrete local invariant whose maximum locus determines a smooth centre of blowing up, leading to desingularization.

Lecture 1
Blow ups, Desingularization Theorems and examples.

Lecture 2
Proof of Weak Desingularization (in all details). From local to global: local properties of an invariant that imply a global desingularization.

Lecture 3
Constructive definition of the invariant for desingularization and an example illustrating the construction.
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PART I. BRIEF HISTORY OF DESINGULARIZATION AND THE MAIN FEATURES (OF MY WORK WITH BIERSTONE)

History.

The problem of resolution of singularities appeared in the middle of the nineteenth century, although in the 1-dimensional case it existed earlier in the guise of finding good parametrization of curves. At first, the problem was only considered for $\mathbb{R}$ and $\mathbb{C}$.


$\dim X = 2$, char $k = 0$: Beppo Levi 1897, Chisini 1921, Albanese 1924, R.J. Walker – Jung 1935, Zariski 1939, 1942.

Local Uniformization Theorem, any dimension, char $k = 0$: Zariski, 1940.

$\dim X = 3$, char $k = 0$: Zariski, 1944.
DESINGULARIZATION HISTORY (very brief):

A. By blow ups:

NEWTON ... ZARISKII ... HIRONAKA
in char$k=0$ (60 - 90)
BLOW UPS INTRODUCED BY:
MAX NOETHER ... ZARISKII

B. By projections, alterations:

ALBANESE ... de JONG
char$k=0$ (from '96 - '97)
DRAWBACK: POOR CONTROL OF THE
DESINGULARIZATION MAP.

C. By Nash blow ups:

NASH ... HIRONAKA (SPIVAKOVSKII)
Hironaka's theorem on the existence of resolutions of singularities for any algebraic or analytic variety \( V \) over a field of characteristic zero is an outstanding achievement of twentieth-century mathematics, by virtue of the depth both of its proof and of its applications. ...The history of the problem of existence of resolutions goes back more than a century ... . And the history is ongoing ... novel approaches to global desingularization have just been developed by A. J. de Jong et al ..., leading to a new generation of short, but non-constructive, proofs. Hironaka's proof is lengthy, difficult, and non-constructive. Influential as the proof has been, few people can have checked it through entirely, even after some subsequent enhancements of the machinery .... Simplified, more algorithmic proofs are important not only for imparting better understanding of what is really involved in this great theorem, but also for their potential value in unearthing basic features of singularities and their classification. The challenge of finding more straightforward algorithmic approaches was taken up by Zariski, Abhyankar, and others, and successfully met only in the past decade by Bierstone, Milman, and Villamayor.
THE MAIN FEATURES (Bierstone-Milman, 88-97)

1. CANONICAL DESINGULARIZATION

2. LOCAL PROPERTIES (OF AN INVARIANT) \Rightarrow \text{GLOBAL RESOLUTION}.

3. IN THE HYPERSURFACE CASE INDUCTION DOES NOT INVOLVE PASSING TO CODIMENSION \( > 1 \).

IN CHARACTERISTIC ZERO (THOUGH THE REDUCTION TO THE HYPERSURFACE CASE WORKS EVEN IN NONZERO CHARACTERISTIC)

4. APPLIES (IN THE LANGUAGE OF LOCALLY RINGED SPACES) TO SPACES \( S' \) THAT ARE LOCALLY \( S' \to U \) SMOOTH ETALE "COORDINATE CHARTS", WHICH WE DEFINE BELOW. (WE ALSO REQUIRE THAT \( O_S \) - COHERENT, \( |S'| \) - LOCALLY
Noetherian and one more technical property — "privileged nbhd. property".

Elements of \( O(U) \) called regular functions, e.g. polynomials, analytic, quasianalytic, ...

**Definition** \( U \) is a smooth etale "coordinate chart" iff exist \( x_1, \ldots, x_n \in O(U) \), which we call "coordinates", and a ("Taylor") homomorphism \( T : O_U \to O_U[[X]] \) into the ring of formal power series expansions in \( X = (X_1, \ldots, X_n) \) such that letting

\[
\sum_{\alpha} f_\alpha x_1^{\alpha_1} \cdots x_n^{\alpha_n} \overset{\text{def}}{=} T f
\]

we have:

1. \( f_0 = f \) ;  
2. \( Tx_j = x_j + x_j \forall j \);  
3. Denote \( D_\alpha f \overset{\text{def}}{=} f_\alpha \) and for a power series expansion \( F(X) \) let \( D_\alpha F(X) \)

be defined by \[
\sum_{\alpha} D_\alpha F(X) \cdot Y^\alpha = F(X+Y)
\]
THEN \[ T \circ D_\alpha = D_\alpha \circ T \quad \forall \alpha \; ,\]

4. LET \( T_\alpha : O_\alpha \to \mathbb{F}_\alpha [[X]] \) BE DEFINED BY EVALUATION OF COEFFICIENTS AT \( \alpha \in U \), i.e.
\[ T_\alpha f = \sum \alpha \cdot f_\alpha(\alpha) \cdot X^\alpha , \text{where } \mathbb{F}_\alpha \overset{\text{def}}{=} O_\alpha / m_\alpha , \]
AND \( \hat{T}_\alpha : \hat{O}_\alpha \to \mathbb{F}_\alpha [[X]] \) BE THE INDUCED MAP OF COMPLETIONS. THEN \( \hat{T}_\alpha \) IS AN ISOMORPHISM.

An illustrating example, e.g. in \( k \)-algebraic case (or schemes of finite type):

\( U = \{ (x, y) : P(x, y) = 0 \} \subset \mathbb{A}^2_k \)

\( P = (P_1, \ldots, P_{n-k}) \) polynomials

and \( \pi \) is etale, i.e.
\[ \det \frac{\partial P}{\partial y} \neq 0 \text{ on } U \]

\( x_1, \ldots, x_n \) - "coordinates"
AND (via implicit differentiation) IT IS EASY TO DEFINE A HOMOMORPHISM \( T \).
PART 2. Blow up, examples.

**Blow up along** \( I \subset \mathcal{O}_U \)

\( I = (f_1, \ldots, f_k) \subset \mathcal{O}(U), \text{U-smooth} \)

\[ U - V(f_1, \ldots, f_k) \xrightarrow{[f]} \mathbb{P}^{k-1} \]

\[ x \mapsto [f_1(x) : \ldots : f_k(x)] \]

\( \text{Bl}_I U := \text{closure}\_\text{graph}[f] \subset U \times \mathbb{P}^{k-1} \)

1. \( \text{Bl}_I U \) does not depend on the choice of generators of \( I \)

2. If \( I = I_C \), where \( C \subset M \)

\( \text{Bl}_C M := \text{Bl}_I M \)

3. Even for smooth \( M \) (unless \( C \) smooth \( \rightarrow M \)) \( \text{Bl}_C M \) may have singularities

Example:

\[ e.g. M = \mathbb{C}k^2 \quad C = \mathbb{C}o^2 \quad \text{Bl}_0 1k^2 \]
Blowing up.

The plane at a point:

0 is removed and replaced by $\mathbb{P}^1$ parametrizing all the lines in $k^2$ passing through 0. If $k = \mathbb{R}$, we get the Möbius band.

We lift every line through the origin with the slope $z = \frac{y}{x}$ (y-axis has slope $\infty$) to the height $z$. Union of all these lines, say $M'$, is the blow up of the plane at 0.

$\pi$ induces an isomorphism $M' \setminus \pi^{-1}(0) \cong k^2 \setminus \{0\}$. 
Example 1

Let \( x, y \) be coordinates on \( k^2 \). Algebraically, \( M' \) is glued together from two coordinate charts, with coordinates \((x, \frac{y}{x})\) and \((y, \frac{x}{y})\), respectively.

**Example.** Resolution of singularities of the curve \( C \) defined by \( y^2 - x^2 - x^3 = 0 \).

\[
x' = x, \quad y' = \frac{y}{x}.
\]

\[
y^2 - x^2 - x^3 = (x'y')^2 - x'^2 - x'^3 = x'^2(y'^2 - 1 - x').
\]
Example 2. Resolution of singularities of the curve $C$ defined by $y^2 - x^3 = 0$.

$x' = x, \ y' = \frac{y}{x}$.

$y^2 - x^3 = (x'y')^2 - x'^3 = x'^2(y'^2 - x')$.

i.e. $\tilde{C} : x' = y'^2$.

or in $(x, y, y')$ - space

$x = y'^2 \quad y = y'^3$

Not normal crossing (yet ...)
Example 3

\[ X : x^2 - y^2 - z^2 = 0 \]

\[ (x, \frac{y}{x}, \frac{z}{x}) = (u, v, w) \quad \text{THEN} \quad g : \begin{cases} x = u \\ y = u \cdot v \\ z = u \cdot w \end{cases} \]

\[ \sigma^{-1}(X) : u^2(1 - v^2 - w^2) = 0 \]

\[ \begin{align*}
E' : & \quad u = 0 \quad \text{EXPECTED}
\end{align*} \]

\[ \begin{align*}
X' : & \quad v^2 + w^2 = 1 \\
& \quad \text{STRICT TRANSFORM} \]
Example 4

**Whitney Umbrella**

\[ X: \quad y^2 - z \cdot x^2 = 0 \]

**Blow up of 0 does not desingularize:**

\[ \text{in} \left( \frac{x}{z}, \frac{y}{z}, z \right) =: (x_1, y_1, z_1) \text{ coordinates} \]

\[ y^2 - z \cdot x^2 = z_1^2 \cdot y_1^2 - z_1^2 \cdot x_1^2 = z_1^2 \cdot (y_1^2 - z_1 \cdot x_1^2) \]

**But blow up of z-axis does:**

\[ \text{in} \quad (x, \frac{y}{x}, z) =: (x', y', z') \text{ coordinates} \]

\[ y^2 - z \cdot x^2 = x'^2 \cdot y'^2 - z' \cdot x'^2 = x'^2 \cdot (y'^2 - z') \]
Example 5

$X : z^3 - x^2yz - x^4 = 0$

$\sigma: \begin{align*}
x &= u \\
y &= v \\
z &= uw
\end{align*}$

(BLOWING-UP IN LOCAL COORDINATES)

$\sigma^{-1}(X): u^3(w^3 - vw - u) = 0$

$E': u = 0$

$X': u = w^3 - vw$
PART 3  EQUIMULTIPLE BLOWUPS.

BLOW UP ALONG SMOOTH $C$:

$M \leftarrow \mathcal{E} M' = B_{L_c} M$

$U \leftarrow U$

$U \leftarrow \mathcal{E}^{-1}(U) = U' = B_{L_{unc}} U$

coord. chart

$U \cap C = \{x_1 = \ldots = x_m = 0\}$

$U' = \{(x_1, z) \text{ s.th. } [x_1: \ldots : x_m] = z^2 \} \subset U \times \mathbb{P}^{m-1}$

$\begin{array}{c}
\frac{x_i}{z_j} = \frac{x_j}{z_i} \\
\searrow \mathcal{G} \\
\downarrow \pi \\
U
\end{array}$

$U' = U \cup_{j \leq j \leq m}$

each $U_j = \{\frac{z_j}{z_i} \neq 0\}$

1. each $U_j$ coord. chart with coord.

$y_i = \frac{x_i}{z_j} = \frac{z_i}{z_j}$ if $i \neq j$, $1 \leq i \leq m$

$y_s = x_s$ if $s = j$ or $m < s < n$

2. $U_{\text{\acute{a}}} = U' - \{x_j = 0\}$

3. in coord. on $U_j$, $\{x_i = 0\} = \{y_i = 0\}$ $i \neq j$
**Blow up along smooth C:**

\[ U \cap C = \{ x_1 = \ldots = x_m = 0 \} \]

\[ \mathcal{U}' = \mathcal{U} \cup \bigcup_{1 \leq j \leq m} U_j \]

\[ \{ U_j \text{ coord. charts} \}
\]

\[ y = (y_1, \ldots, y_n) \]

\[ \varphi_j : x_j = y_j \]

\[ \mathcal{O}/U_j : x_i = y_j \cdot y_i \quad i \neq j, \quad 1 \leq i \leq m \]

\[ x_s = y_s \quad m < s \leq n \]

**IF** \( X = V(f) \) **THEN** \( X' = V(f') \)

WHERE \( (f') := (y_{\text{exc}})^{-d} \cdot (f \circ \varphi) \)

**on** \( U_j \) \( y_{\text{exc}} = y_j \), \( \{ y_{\text{exc}} = 0 \} := \mathcal{O}(C) \)

\[ d = \text{maximal integer such that } f' \text{ is regular} \]

**In general, codim X > 1:**

**IF** \( X \subset \mathcal{M} \) **THEN** \( X' := \mathcal{O}(\mathcal{M}(X - C)) \)

AND \( I_{X'} = (f' \text{ s.th. } f \in I_X) \subset \mathcal{O}_{\mathcal{M}}' \)
THE EFFECT OF AN EQUIMULTIPLE BLOW UP:

HYPERSURFACE $f(x) = 0$

$d = \text{ord}_a f$

\[
f(x) = c_0(x) + \cdots + c_{d-1}(x)x_n^{d-1} + c_d(x)x_n^d
\]

$0$, by ASSUME $\equiv 1$

COMPLETING $d$'TH POWER

EQUIMULTIPLE LOCUS

$S_d := \{ x : \text{ord}_x f = d \}$

$= \{ x : x_n = 0, \text{ ord}_{\bar{x}} c_q \geq d - q \}$

$\min_{q} \text{ord}_a c_q / (d - q)$
EFFECT OF A BLOWING-UP

\[ \sigma, \text{ CENTRE } C \subset S_d \]

\[ C = Z_{i} := \{ x_n = 0, \ x_i = 0, \ i \in I \} \]

WHERE \( I \subset \{1, \ldots, n-1\} \)

EFFECT \( \sigma \text{ in chart } \mathcal{U}_{i}, \ i \in I \) :

\[ \sigma|_{\mathcal{U}_{i}} : x_j = y_i y_j \quad j \in I \cup \{n\} \setminus \{i\} \]

\[ x_j = y_j \quad \text{OTHER } j \]

\( i \in I : \quad f(\sigma(y)) = y_i^d f'(y) \),

\[ f'(y) = 
\]

\[ c_0'(\tilde{y}) + \cdots + c_{d-2}'(\tilde{y}) y_n^{d-2} + y_n^{d} \]

\[ c_q'(\tilde{y}) = \gamma_i^{-(d-q)} c_q(\tilde{\sigma}(\tilde{y})) \]
**EFFECT** ... **IN CHART** \( U'_h - \bigcup_{i \neq n} U'_i \)

**in** \( U'_n \) **coordinates** \( y = (y_1, \ldots, y_n) \)

\[
\begin{align*}
\mathcal{O}_{U'_n} & : \quad x_h = y_n \\
x_j = y_j y_i, & \quad j \in I \\
x = y & \quad \neq I \setminus \{n\}
\end{align*}
\]

\( W := U'_n - \bigcup_{i \in I} U'_i = \{ y_i = 0 \ \forall i \in I \} \)

\( U \cap C = \{ x_h = 0, \ x_j = 0 \ \forall j \in I \} \)

\((*)\) \quad \text{ord}_c c_k \geq d - k > 0 \quad k = 1, 2, \ldots, d - 2

\[
\begin{align*}
f' = y_h^{-d}(f \circ \sigma) = (c_d \circ \sigma)(y) + \sum_{0 \leq k < d} c_k'(y) \\
\rightarrow \neq 0 \text{ on } W
\end{align*}
\]

WHERE

\[
\begin{align*}
c_k'(y) & := y_h^{-(d-k)}(c_k \circ \sigma)(y) \\
& \neq 0 \text{ on } W \text{ due to } (*)
\end{align*}
\]

Hence,

\[
\begin{align*}
f' & \neq 0 \text{ on } U'_n - \bigcup_{i \in I} U'_i
\end{align*}
\]
\[ C_d(x) \neq 1 \]

**SUMMARY:**

\[ f = c_0(x) + \ldots + c_{d-1}(x)x_{n-1}^d + c_d(x)x_n^d \]

\[ \frac{d^{d-1}f}{dx_n^{d-1}} \approx x_n \Rightarrow c_{d-1} \equiv 0, \ c_d(x) \neq 0 \]

\[ C = \mathbb{Z}_p \]

\[ \left\{ U_i \subseteq x_j = y_{i,j} \quad j \in \{j \mid i \in I \} \right\} \]

\[ x_j = y_j \quad \text{other } j \]

\[ f'(y) = y_i^{-d} \cdot (f \circ x)(y) \quad \text{in } U_i \]

\[ f' = c_0(y) + \ldots + c_{d-2}(y)y_n^{d-2} + c_d(y)y_n^d \]

\[ C_q := y_i^{-(d-q)} \cdot (C_q \circ x) \]

**THEN:**

\[ \frac{1}{d!} \frac{d^d f}{dy_n^d} = y_n \cdot c_d(y) + y_n^2 \cdot c_d(y) \ldots \approx y_n \]

\[ \text{ord}_y f' \leq d \quad \text{AND} \quad \text{ord}_y f' = d \quad \iff \]

\[ y_n = 0 \quad \text{AND} \quad \text{ord}_y C_q \geq d-q, q = 0, 1, \ldots \]

**NOTE:** \[ \{ y_n = 0 \} = \{ x_n = 0 \} \]
PARTY "WEAK" DESINGULARIZATION THEOREM. (PROOF IN ALL DETAILS.) WE WILL NEED THE FOLLOWING:

**DEFINITION OF \( \psi \triangleq \psi_{x \cdot id} \) FOR \( \psi : N' \to N \) THAT OCCUR IN THE INDUCTIVE STEP OF PROOF BELOW:**

For a coord. chart \( U \) and \( N \triangleq \{ x : g(x) = 0 \} \) with \( \frac{\partial g}{\partial x^i} \neq 0 \) on \( U \) and a smooth
\[
C \triangleq \{ x \in N : x_i = 0, i \in I \} = : \mathbb{Z}_I, 1 \leq i \leq n - I, \]
and \( \psi \triangleq \sigma : N' \triangleq \mathcal{B}_{\mathcal{L}} \to N \) a blow up map

**LET \( \psi \triangleq \psi_{x \cdot id} : U' \to U \) DENOTE THE BLOW UP MAP \( U' \triangleq \mathcal{B}_{\mathcal{L}} U \to U \) with
\[
\widetilde{C} \triangleq \{ x \in U : x_i = 0, i \in I \}.
\]

**FOR A Coord. Chart \( U \) and \( N \triangleq \{ x : g(x) = 0 \} \) with \( \frac{\partial g}{\partial x^i} \neq 0 \) on \( U \) and for an open
finite covering \( N' = \bigcup_{i \in I} N_i' \) with each
\[
N_i' \triangleq N' - \{ x_i = 0 \} \quad \text{and} \quad \psi \triangleq \bigcup_{i \in I} N_i' \to \bigcup_{i \in I} N_i',
\]
a "covering" map, let \( \psi \triangleq \psi_{x \cdot id} : M' \to M \),

where \( M \triangleq U' \), denote the "covering"
map \( M' \triangleq \bigcup_{i \in I} U_i' \to \bigcup_{i \in I} U_i' = U' \triangleq \mathcal{B}_{\mathcal{L}} U \),
where each \( U_i' = U' - \{ x_i = 0 \} \). NOTE:

\( \sigma : U'_i \to U \) in coordinates is \( \sigma : [x_j = y_j \text{ for } j \in I - \{ i \}] \)
\( [x_i = y_i \text{ otherwise}] \)
\( U'_i \) and \( N'_i = \{ x \in U'_i : y_i \cdot g(0)(y) = 0 \} \subset U'_i \).
"Weak" Design Thm.: For $(\mathfrak{m})$ on smooth $M$ exists $\psi : M' \to M$ a composite of blow ups (along smooth centers) and of "coverings" (i.e. maps $\sqcup U_j \to U_j \cup U_j$, open $U_j$) such that $(\psi^*(x))(\psi(x)) = (x_1^{n_1}, \ldots, x_n^{n_2})$ locally (the so-called NCR).

Proof in complete details below.

Definition: $\text{Sing}_V(f)$ a iff $(f(x)) = (g(x))$ for $\text{ord}_a f = d$ "near a", $\text{ord}_a g = 1$.

Setup and other basic notions:

"Year" $x \in M_j \xrightarrow{\delta_j} \cdots \xrightarrow{\delta_{i+1}} M_i \xrightarrow{\delta_{i+1}} M_i \xrightarrow{\delta_{i+1}} M_i \to M_0 = M$ "era" of higher $d = \cdots$

$E_j$: exc. hypersurfaces on $M_j$ — all smooth.

$X^d_j = \{x : f_0(x) = 0\}$, $\Sigma = \sum_j \text{Sing}_V X^d_j$

$d = \max\{\text{ord}_{x, f_0} : x \in \Sigma\}$

$S_d = \{x : \text{ord}_{x, f_0} = d\}$ prop. $a \in \sum \cap S_d \Rightarrow S_d \subseteq \Sigma$

$S(x) = \# \{H \in E_j : H \ni x \text{ and } \text{"comes" from } \text{"era" of higher } d\}$

Proof of Prop: choose coord s.th. $\frac{\partial f_0}{\partial x_\alpha}(a) \neq 0$ then

$N = \{x : \frac{\partial f_0}{\partial x_\alpha}(x) = 0\} \supseteq S_d = \{x \in N : \text{ord}_{x, f_0} c_k \geq d - k, \alpha k < a\}$

where $c_k = \frac{\partial f_0}{\partial x_\alpha} |_N$. For $x \in S_d$ $x \in \Sigma$ iff some $c_k \neq 0$ done.
Proof of "W.D. THM." Induction on \( n = \dim M \) and \((d,s)\), \(s \overset{\text{def}}{=} \max\{s(x) : x \in S_d \cap \Sigma_3^d\} \), namely:

- In the "year" \( l+1 \) (when \( d \) just decreased) choose \( a \in S_d \cap \Sigma_3^d \) and coord. chart: \( \frac{\partial^d f}{\partial x^d} \neq 0 \), \( \frac{\partial b}{\partial x} \neq 0 \), \( \forall \lambda \in \mathbb{R} \).

Let \( N \overset{\text{def}}{=} \{ x : \frac{\partial^d f}{\partial x^d} \neq 0 \}, \quad c_k \overset{\text{def}}{=} \frac{\partial f}{\partial x^k} \bigg|_N, \quad b_h \overset{\text{def}}{=} \frac{\partial b}{\partial x} \bigg|_N, \quad \forall \lambda \in E'(a) \).

Let \( A = \prod k \forall c_k \overset{\text{def}}{=} \sum_{d-k} \text{all } c_k \overset{\text{def}}{=} \sum_{d-k} \text{all } b_h \cdot \text{all their differences } \neq 0 \).

\( A \in \mathcal{O}(N) \), \( \dim N = n-1 < n \) \( \Rightarrow \) \( \exists \psi, N' \rightarrow N \) \( \vdash A \psi \in \mathcal{N}. \) Then

\[ \psi \overset{\text{def}}{=} \psi_{x \mapsto \text{id}} : M' \rightarrow M \quad \therefore \quad \text{"} a \rightarrow a_0 \overset{\text{def}}{=} a \text{"} \]

Lemma 1:

Each \( c_k^{d-k} \approx \sum \gamma \), \( b_h^{d-k} \approx \sum \gamma \).

\[ \exp \overset{\text{def}}{=} \{ \ldots \sum k, \ldots \} \leq n^{n-1} \text{ is totally ordered!} \]

Proof. Direct and easy.

\[ \Rightarrow \exists \min \exp =: \sum \gamma \Rightarrow S_d \cap \bigcap_{h \in E'(a)} H = \] \[ \{ x \in N : \text{ord}_x c_k > d-k \forall k, \text{ord}_x b_h > 1 \forall h \in E'(a) \} = \] \[ \{ x \in N : \text{ord}_x \sum \gamma \geq d \} = \bigcup_{I} Z_I \quad \text{EACH } Z_I \overset{\text{def}}{=} \] \[ \{ x \in N : x_i = 0 \forall i \in I \} \quad \text{AND } \text{minimal } \{ \sum \gamma \} \]

Such that \( \sum_{i \in I} \gamma_i \geq d \} \iff 0 \leq \sum_{i \in I} \gamma_i < d \forall j \in I, \).

Lemma 2. Consider blow up with \( C = Z_I \) (any): \( U' \overset{\sigma}{\rightarrow} U \)

\( U' = U \cup_{j \in I} U' \), \( U' \overset{\sigma}{=} U' \rightarrow N' \) Then \( \forall x \in U' \), \( C \cap S_d = \text{ord}_x c_k \).

If \( \text{ord}_x c_k = d \Rightarrow a \in \Sigma_3^d \cap U' \), some \( j \in I \).

Proof. Follows from "the effect of equimultiple..."
**Blow Up** Calculation.

Also, (the "Effect of Equimultiple Blow Up" Calculation), \( c'_k = y'_{d_1} \cdot (c_{k \circ \theta}) \), \( b'_h = y_{d_1}^{-1} \cdot (b_{h \circ \theta}) \).

Hence, \( \bullet \) again is valid since \( y_{d_1}^{-d_1!} (\lambda \circ \theta) = y^{-\lambda} \) with \( \sum_{i \neq j} c'_i = c'_j \) for \( i \neq j \) and \( \sum_{i \in \mathbb{I}} c'_i = \sum_{i \in \mathbb{I}} c''_i - d_1! \) (\( \forall \lambda \in \mathbb{N}^{n-1} \)).

For \( \Sigma = \min \left( \exp \text{ at } \alpha \right) \) as above, \( \sum_{i \in \mathbb{I}} c'_i < \sum_{i \in \mathbb{I}} c''_i \) (\( \forall \lambda \in \mathbb{N}^{n-1} \)) and

\( \Omega' = \min \left( \exp \text{ at } \alpha' \right) \). But \( 0 \leq \sum_{i \in \mathbb{I}} c'_i \leq \sum_{i \in \mathbb{I}} c''_i \) (and these are from in) and if \( s(\alpha') = s \) then \( \sum_{i \in \mathbb{I}} c''_i > d \).

We continue these blow ups (unless the respective \( \sum_{i \neq j} c''_i = \emptyset \)) until \( (d, s) \) decreases (need no more then \( \sum_{i \in \mathbb{I}} c''_i \) blow ups, where \( \Sigma \) is as in \( \bullet \) at \( \alpha' \)). If \( d \) decreases we start with induction on \( n = \dim M \) step all over.

\( \bigcirc \) otherwise (i.e. either \( \sum_{i \neq j} c''_i = \emptyset \) or \( s \) is smaller, \( \sum_{i \neq j} c''_i \neq \emptyset \)) we continue these blow ups (which we may since although for some \( H \in E^1(\alpha') \), \( H \neq \alpha' \), i.e.

\( \Sigma_{H'} = 0 \), \( \bullet \) remains valid with \( H' \in E^1(\alpha') \), i.e. such that \( \Sigma_{H'} \neq 0 \), and "new" \( \Sigma = \min \left( \exp \text{ at } \alpha' \right) \neq 0 \) until either \( d \)-decreases or (if \( \sum_{i \neq j} c''_i = \emptyset \)) \( s(\alpha) \leq 1 \) \( \forall \alpha \) and \( s(\alpha) = 0 \) \( \forall \alpha \in X_j \). DONE. End of Proof of Thm.