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Analytic Smoothing Effect and Single Point Conormal Regularity for the Semilinear Dispersive Type Equations

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1. Introduction

We study the smoothing effect for a general form of the following dispersive semilinear equation:

$$
\begin{cases} i\partial_t u + Q(D_x)u = f(u, \partial_x u), & t, x \in \mathbb{R}, \\ u(0,x) = \phi(x), \end{cases} \tag{1.1}
$$

where $Q(D_x)$ is a differential operator of homogeneous degree m. A typical example of the above type equation is the Korteweg-de Vries equation, nonlinear Schrödinger equation, the Benjamin-Ono equation and derivative nonlinear Schrödinger equation. Those equations mainly arise from the water wave theory and typically, the solution $u(t,x) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ descries the surface displacement of the water wave.

We assume that the linear differential operator is homogeneous of order m. That is $Q(D_x)$ is defined by the Fourier transform

$$
Q(D_x)u = \mathcal{F}^{-1}q(\xi)\mathcal{F}u,
$$

where $q(\xi)$ satisfies $q(\lambda\xi) = \lambda^m q(\xi)$ for $\lambda > 0$.

When we consider the well-posedness of those type of equation, L^2 based (Sobolev) space is considered and the regularity of solution is then, derived as much as the same order of regularity of the initial data ϕ. Namely if the initial data $\phi \in H^s(\mathbb{R})$ for some $s \in \mathbb{R}$, then the solution expected up to $H^s(\mathbb{R})$. This is because the singularities of the solution come from the infinity and regularity is never to be gained by time evolution.

However, it is studied by many cases that the local or some restricted version of smoothing effect holds for those type of equations. Among others, the smoothing effect from the low initial regularity solution to the analyticity is our main concern. Especially, to the weak solution constructed in the Fourier restriction space $X_b^s = \{f \in \mathcal{S}'(\mathbb{R}^2); \langle i\partial_t + Q\rangle^b \langle D_x\rangle^s f \in L^2(\mathbb{R}; L^2(\mathbb{R}))\}$, it is possible to prove the regularity of solution reaches up to analytic in both space and time variable by an operation of the conformal vector fields. More specifically, we introduce the linear complimentary (variable coefficient) operator $P = mt\partial_t + x\partial_x$ that plays an role of the compensating part where the main linear operator $L = i\partial_t + Q(D_x)$ can not gain the regularity. In what follows, we restrict our problem to the simplest dispersive case $q(\xi) = -\xi^m$ i.e., $Q(D_x) = -(i\partial_x)^m$. Note that usual derivative is given by $\partial_x = iD_x$.

Our goal is to obtain the smoothing effect for a single point singularity. To make it simply, we further restrict the situation like the following. We assume that $f(u, \partial_x u)$ is a polynomial of u and $\bar{u}$ of order p but not depends on $\partial_x u$ nor $\partial_x \bar{u}$. That is the equation we discuss is the following simpler one:

$$\begin{cases} i\partial_t u + Q(D_x)u = f(u, \bar{u}), & t, x \in \mathbb{R}, \\ u(0, x) = \phi(x). \end{cases} \tag{1.2}$$

The following is our main theorem.

Theorem 1.1. *Let $s \geq 0$. Suppose that for some $A_0 > 0$, the initial data $\phi \in H^s(\mathbb{R})$ and satisfies*

$$\sum_{k=0}^{\infty} \frac{A_0^k}{k!} \|(x\partial_x)^k \phi\|_{H^s} < \infty,$$

then there exist $T > 0$ and a unique solution u of the nonlinear dispersive equation (1.1) such that for some $b \in (1/2, 1)$, $u \in C((-T, T), H^s) \cap X_b^s$. Besides for any $(t, x) \in \{(-T, 0) \cup (0, T)\} \times \mathbb{R}$, $u(t, \cdot)$ is a real analytic function in both space and time variable.

Remark 1. For the case of KdV equation, that is $Q(D_x) = i\partial_x^3$ and $f(u, u_x) = \partial_x(u^2)$, or nonlinear Schrödinger equation (NLS), $Q(D_x) = \partial_x^2$ and $f(u, u_x) = u^2$ or $\bar{u}^2$, we have obtained a much stronger result of the smoothing effect ([12] [13]). In that case, the initial data can be taken such as the Dirac measure or principal value of $1/x$. Also, for the nonlinear Schrödinger equation with exotic power nonlinearities like u^2, $\bar{u}^2$, u^3, $\bar{u}^3$...., we may conclude that the same stronger result like Theorem 1.1 holds.

Remark 2. It is well-known that the global in time solution has been obtained (see [4], [10]) to the spacial dispersive nonlinear equations by the inverse scattering method. Also the analyticity for the inverse scattering solution of KdV equation with a weighted initial data was obtained by Tarama [25]. However, since our method is based on the fact that the solution is in H^s, we don't know if our result is true globally in time. We also notice that the small data global analticity was shown by Nakamitsu [23] for the nonlinear Schrödinger equations in higher dimensions for some weighted initial data.

By a almost similar argument of Theorem 1.1, one can also show the following weaker theorem.

Theorem 1.2. *Let $s \geq 0$. Suppose that for some $A_0 > 0$, the initial data $\phi \in H^s(\mathbb{R})$ and satisfies*

$$\sum_{k=0}^{\infty} \frac{A_0^k}{(k!)^m} \|(x\partial_x)^k \phi\|_{H^s} < \infty,$$

then there exist $T > 0$ and a unique solution u of the dispersive equation (1.1) such that $u \in C((-T, T), H^s) \cap X_b^s$ for some $b \in (1/2, 1)$. Moreover for any $t \in (-T, 0) \cup (0, T)$,

$u(t,\cdot)$ is analytic function in space variable and for $x \in \mathbb{R}$, $u(\cdot,x)$ is of Gevrey m as a time variable function.

Remark 3. In both Theorems, the assumption on the initial data implies the analyticity and Gevrey m regularity except the origin respectively. In this sense, those results are stating that the singularity at the origin immediately disappear after $t > 0$ or $t < 0$ up to analyticity.

Remark 4. Some related results are obtained for the linear and nonlinear Schrödinger equations. For linear variable coefficient case, see Kajitani-Wakabayashi [11] and for nonlinear case, Chihara [3]. They are giving a global weighted uniform estimates of the solution with arbitrary order derivative in space variable. In our case, it is still unknown if the weighted uniform bounds are possible or not.

2. Method

Our method is based on the following observation. To make description simple, we consider the following simplest case:

$$\begin{cases} i\partial_t u + D_x^m u = \mu u^2, & t, x \in \mathbb{R}, \\ u(0,x) = \phi(x). \end{cases} \tag{2.1}$$

Firstly, we introduce the generator of the dilation $P = mt\partial_t + x\partial_x$ for the linear part of the dispersive equation. Noting the commutation relation with the linear dispersive operator $L = \partial_t + Q(D_x)$:

$$[L, P] = mL,$$

it follows

$$LP^k = (P+m)^k L, \tag{2.2}$$

for any $k = 1, 2, \cdots$. Applying $P = mt\partial_x + x\partial_x$ to the equation (2.1), we have

$$i\partial_t(P^k u) + Q(D_x)(P^k u) = (P+m)^k Lu = \mu(P+m)^k(u^2) \tag{2.3}$$

We set $u_k = P^k u$ and $F_k(u, \bar{u}) = \mu(P+m)^k u^2$. Then noting that

$$\begin{aligned} (P+m)^l u &= (P+m)^{l-1} Pu + m(P+m)^{l-1} u = \cdots \\ &= \sum_{j=0}^{l} \frac{l!}{j!(l-j)!} m^{l-j} P^j u \end{aligned} \tag{2.4}$$

we see

$$
\begin{aligned}
F_k(u,\bar{u}) =&\mu(P+m)^k(u^2) = \mu\sum_{l=0}^{k}\binom{k}{l}(P+m)^l uP^{k-l}u \\
=&\mu\sum_{l=0}^{k}\sum_{m=0}^{l}\binom{k}{l}\binom{l}{j}m^{l-j}P^j uP^{k-l}u \\
=&\mu\sum_{k=k_0+k_1+k_2}\frac{k!}{k_0!k_1!k_2!}m^{k_1}u_{k_2}u_{k_3}
\end{aligned}
$$

The nonlinear terms $F_k(u,u)$ maintain a similar structure of original nonlinear term. This is because the Leibniz law can be applicable for an operation of P. If we consider the slightly general (monolyal) nonlinearity, $f(u,\bar{u}) = \mu u^{p_1}\bar{u}^{p_2}$ $(p = p_1 + p_2)$, it is easy to see that

$$
F_k(u,\bar{u}) = \mu\sum_{k=k_0+k_1+\cdots+k_p}\frac{k!}{\prod_{i=0}^{p}k_i!}m^{k_0}\prod_{i=1}^{p_1}u_{k_i}\cdot\prod_{i=1}^{p_2}\bar{u}_{k_i}
$$

Thus each of u_k satisfies the following system of equations;

$$
\begin{cases} i\partial_t u_k + Q(D_x)u_k = F_k(u,\bar{u}), & t,x\in\mathbb{R}, \\ v_k(0,x) = (x\partial_x)^k\phi(x). \end{cases} \tag{2.5}
$$

Therefore we firstly establish the local well-posedness of the solution to the following infinitely coupled system of dispersive equation in a suitable weak space:

$$
\begin{cases} i\partial_t u_k + Q(D_x)u_k = F_k(u,\bar{u}), & t,x\in\mathbb{R}, \\ u_k(0,x) = \phi_k(x). \end{cases} \tag{2.6}
$$

Then taking $\phi_k = (x\partial_x)^k\phi(x)$, the uniqueness and local well-posedness allow us to say $u_k = P^k u$ for all $k = 0, 1, \cdots$.

3. Linear and Nonlinear Estimates

We firstly consider the corresponding linear equation

$$
\begin{cases} i\partial_t u + Q(D_x)u = 0, & t,x\in\mathbb{R}, \\ u(0,x) = \phi(x). \end{cases} \tag{3.1}
$$

Proposition 3.1. *Let $e^{-itD_x^m}$ be the unitary operator generated by the linear dispersive equation of the space order m. Then we have*

$$
\|e^{-itD_x^m}\phi\|_p \le Ct^{-\gamma}\|\phi\|_q, \tag{3.2}
$$

where $2\le p\le\infty$, $1\le q\le 2$ and

$$
-\frac{m-1}{p}+\frac{m}{2}\le\frac{1}{q}\le-\frac{1}{p(m-1)}+\frac{m}{2(m-1)}.
$$

This fact follows from the asymptotic behavior of the fundamental solution $E_{e^{-itD_x^m}}(x,y,t)$. Namely we see via the stationary phase method that

$$|E_{e^{-itD_x^m}}(x,y,t)| \le Ct^{-1/k}\langle x/t^{1/m}\rangle^{-\frac{m-2}{2(m-1)}}.$$

Therefore the fundamental solution belongs to L^κ_w where $\kappa = \frac{2(k-1)}{k-2}$. Hence Hausdorff-Young inequality gives

$$\|E_{e^{-itD_x^m}} * \phi\|_p \le \|E_{e^{-itD_x^m}}(x,y,t)\|_{L^\kappa_w}\|\phi\|_q,$$

where $1/p = 1/q + 1/\kappa - 1$.

Proposition 3.2. *For the free evolution $U_k(t)$, we have*

$$\|e^{-itD_x^m}\phi\|_{L^\theta(I;L^p)} \le C_0\|\phi\|_2,$$

where

$$\frac{2}{\theta} = \frac{n}{m}\left(1-\frac{2}{p}\right)$$

and

$$\|\int_0^t e^{-i(t-s)D_x^m}F(s)ds\|_{L^\theta(I;L^p)} \le C_1\|F\|_{L^{\rho'}(I;L^{q'})}$$

where

$$\frac{1}{\theta}+\frac{1}{\rho} = \frac{n}{m}\left(1-\frac{1}{p}-\frac{1}{q}\right)$$

According to the Strichartz type linear estimate in Proposition 3.2, we have the bilinear estimate for the nonlinear term:

The following estimates of linear and nonlinear part due to Bourgain [2] and refined by Kenig-Ponce-Vega [18] are our essential tools.

Lemma 3.3. *Let $s \in \mathbb{R}$, $a, a' \in (0,1/2)$, $b \in (1/2,1)$and $\delta < 1$. Then for any $k = 0,1,2,\cdots$, we have*

$$\|\psi_\delta\phi_k\|_{X^s_{-a}} \le C\delta^{(a-a')/4(1-a')}\|\phi_k\|_{X^s_{-a'}}, \tag{3.3}$$

$$\|\psi_\delta e^{-itD_x^m}\phi_k\|_{X^s_b} \le C\delta^{1/2-b}\|\phi_k\|_{H^s} \tag{3.4}$$

$$\|\psi_\delta\int_0^t e^{-i(t-t')D_x^m}F(t')dt'\|_{X^s_b} \le C\delta^{1/2-b}\|F\|_{X^s_b} \tag{3.5}$$

Proof of Lemma 3.3. See [17]. ❑

The core part of the nonlinear estimate is to establish the bilinear estimate in the space of X^s_b, which is established by Bourgain [2] and Kenig-Ponce-Vega [19] [20]. The following is the somewhat arranged version of them.

Proposition 3.4. *For $u_i \in X_b^s$ $(s \geq 0)$ then we have*

$$\|\prod_{i=1}^{l} u_i\|_{X_{b-1}^s} \leq C\prod_{i=1}^{l}\|u_i\|_{X_b^s}.$$

From Proposition 3.4, it is immediately obtained by the bilinear estimate for the nonlinearity of the system.

Corollary 3.5. *Let $s \geq 0$, $b, b' \in (1, 1/2)$ with $b < b'$ and $\delta < 1$. Then, we have*

$$\|F_k(u,\bar{u})\|_{X_{b'-1}^s} \leq C\delta^{1/2-b} \sum_{k=k_0+k_1+\cdots+k_p} m^{k_0}\frac{k!}{\prod_{i=0}^{p} k_i!}\prod_{i=1}^{p}\|u_{k_i}\|_{X_b^s} \tag{3.6}$$

4. Construction of the solution

According to Bourgain [2], we introduce the Fourier restriction space as

$$X_b^s = \{f \in \mathcal{S}'(\mathbb{R}^2); \|f\|_{X_b^s} < \infty\},$$

where

$$\|f\|_{X_b^s}^2 = c\iint \langle\tau - \xi^m\rangle^{2b}\langle\xi\rangle^{2s}|\hat{f}(\tau,\xi)|^2 d\tau d\xi = \|e^{itD_x^m} f\|_{H_t^b(\mathbb{R};H_x^s)}^2.$$

The space where we solve the system is infinite sum of this space. Let $f = (f_0, f_1, \cdots, f_k, \cdots)$ denotes the infinity series of distributions and define

$$\mathcal{A}_{A_0}(X_b^s) = \{f = (f_0, f_1, \cdots, f_k, \cdots), f_i \in X_b^s \quad (i = 0, 1, 2, \cdots) \text{ such that} \|f\|_{\mathcal{A}_{A_0}} < \infty\},$$

where

$$\|f\|_{\mathcal{A}_{A_0}} \equiv \sum_{k=0}^{\infty}\frac{A_0^k}{k!}\|f_k\|_{X_b^s}.$$

The system will be shown to be well-posed in the above space if $s \geq 0$.

The well-posedness is derived by utilizing the contraction principle argument to the corresponding system of integral equations:

$$\psi(t)u_k(t) = \psi(t)e^{-itD_x^m}\phi_k - \psi(t)\int_0^t e^{-i(t-t')D_x^m}\psi_T(t')F_k(u,\bar{u})(t')dt'. \tag{4.1}$$

Proposition 4.1. *Let $s \geq 0$, $b \in (1/2, 1)$. Suppose that for some $A_0 > 0$, the initial data $\phi \in H^s(\mathbb{R})$ and satisfies*

$$\sum_{k=0}^{\infty}\frac{A_0^k}{k!}\|\phi_k\|_{H^s} < \infty,$$

then there exist $T > 0$ and the integral equation (4.1) associated with the nonlinear dispersive equation (1.1) is wellposed in the class [1]$\mathcal{C}((-T,T); H^s) \cap \mathcal{A}_{A_0}(X_b^s)$.

---

[1]$\mathcal{C}(I; X)$ denotes a space of a sequence of function $f = \{f_i\}_{i=0}^{\infty}$ with $f_i \in C(I; X)$ for each i.

The outline of the proof is the following: Let a map $\Phi : \{u_k\}_{k=0}^\infty \to \{u_k(t)\}_{k=0}^\infty$ such that $\Phi = (\Phi_0, \Phi_1, \cdots)$ and

$$\Phi_k(u) \equiv \psi e^{-itD_x^m}\phi_k - \psi \int_0^t e^{-it(t-t')D_x^m} F_k(u, \bar{u})(t')dt'.$$

Then it is shown that $\Phi_k : \mathcal{A}_{A_0}(H^s) \to \mathcal{A}_{A_1}(X_b^s)$ is a contraction.

In fact, by using Lemma 3.3 and Corollary 3.5, we easily see that

$$\begin{aligned}
\|\Phi\|_{\mathcal{A}_{A_1}(X_b^s)} =& \sum_{k=0}^\infty \frac{A_1^k}{k!}\|u_k\|_{X_b^s} \\
\le& C_0 \sum_{k=0}^\infty \frac{A_0^k}{k!}\|\phi_k\|_{H^s} + C_1 T^\kappa \sum_{k=0}^\infty \frac{A_0^k}{k!} \sum_{k=k_0+k_1+\cdots+k_p} m^{k_0}\frac{k!}{\prod_{i=0}^p k_i!}\prod_{i=1}^p \|u_{k_i}\|_{X_b^s} \\
\le& C_0\|u\|_{\mathcal{A}_{A_0}(H^s)} + C_1 T^\kappa \sum_{k_0=0}^\infty m^{k_0}\frac{A_0^{k_0}}{k_0!}\prod_{i=1}^p \{\sum_{k_i=0}^\infty \frac{A_0^{k_i}}{k_i!}\|u_{k_i}\|_{X_b^s}\}.
\end{aligned}$$

Hence, it follows

$$\|\Phi(u)\|_{\mathcal{A}_{A_1}(X_b^s)} \le C_0\|\phi\|_{\mathcal{A}_{A_0}(H^s)} + C_1 e^{2A_0}T^\kappa \|u\|_{\mathcal{A}_{A_1}(X_b^s)}^p$$

and also we have the estimate for the difference

$$\|\Phi(u^{(1)}) - \Phi(u^{(2)})\|_{\mathcal{A}_{A_1}(X_b^s)} \le C_1 e^{2A_0}T^\kappa (\|u^{(1)}\|_{\mathcal{A}_{A_1}(X_b^s)}^{p-1} + \|u^{(2)}\|_{\mathcal{A}_{A_1}(X_b^s)}^{p-1})\|u^{(1)} - u^{(2)}\|_{\mathcal{A}_{A_1}(X_b^s)}.$$

Choosing T small enough, the map Φ is contraction from

$$X_T = \{f = (f_0, f_1, \cdots); f_i \in X_b^s, \sum_0^\infty \frac{A_0^k}{k!}\|f_k\|_{X_b^s} \le 2C_0M_0\}$$

to itself, where $M_0 = \|u\|_{\mathcal{A}_{A_0}(H^s)}$. This shows the well-posedness.

5. Bootstrap argument

We have constructed a weak solution to the dispersive equation (1.1) satisfying the following extra conormal regularity:

$$\|P^k u\|_{X_b^s} \le CA_0^k k! \quad k = 0, 1, \cdots,$$

under the condition to the initial data ϕ:

$$\|(x\partial_x)^k\phi\|_{H^s} \le CA_1^k k! \quad k = 0, 1, \cdots.$$

Now by the localization argument, the operator P can be regarded as a vector field $P_0 = 3t_0\partial_t + x_0\partial_x$ where $(t_0, x_0) \in \{(-T, 0) \cup (0, T)\} \times \mathbb{R}$ is any fixed point. Since the Fourier restriction norm originally contains the regularity with the characteristic derivative $L^b = \langle i\partial_t + D_x^m\rangle^b$, we combine the both derivative L^b and P_0^k (and by the localization argument) to derive the regularity. If we set a smooth cut-off $a(t,x)$ whose support is around the point (t_0,x_0) with $\operatorname{supp} a \subset B_{2\varepsilon}$. Then we firstly derive

$$\|aP^k u\|_{L^2_{t,x}(\mathbb{R}^2)} \le CA_2^k k! \quad k=0,1,2,\cdots. \tag{5.1}$$

This estimate is obtained by the following lemma which plays a key role in this bootstrap argument.

Lemma 5.1. *Let $P = mt\partial_t + x\partial_x$ be the generator of the dilation, $Q(D_x)$ is the differential operator of order m and $D_{t,x}$ be defined by $\mathcal{F}^{-1}_{t,x}\langle|\tau|+|\xi|\rangle\mathcal{F}_{t,x}$. For a fixed point (t_0,x_0), we suppose that $a(t,x)\in C_0^\infty(B_\varepsilon(t_0,x_0))$ and $f\in H^\nu(\mathbb{R}^2_{t,x})$ with $tQ(D_x)f$, $P^3 f\in H^{\nu-k}(\mathbb{R}^2_{t,x})$. Then for $\nu\in\mathbb{R}$, there exist a constant $C>0$ such that*

$$\|af\|_{H^\nu(\mathbb{R}^2_{t,x})} \le C\left\{\|af\|_{H^{\nu-m}(\mathbb{R}^2_{t,x})} + \|tQ(D_x)(af)\|_{H^{\nu-m}(\mathbb{R}^2_{t,x})} + \|P^m(af)\|_{H^{\nu-m}(\mathbb{R}^2_{t,x})}\right\}$$

where the constant C depends on (t_0,x_0) and ε.

Proof of Lemma 5.1. Note that $\langle|\tau|+|\xi|\rangle^m \le C_1(t_0^{-1},\langle x_0\rangle^{-1})(1+|t_0 q(\xi)|+|kt_0\tau + x_0\xi|^m)$, which implies

(5.2)

$$\begin{aligned}\|\langle D_{t,x}\rangle^\nu(af)\|_{L^2(\mathbb{R}^2)} \le & C_1\{\|\langle D_{t,x}\rangle^{\nu-m}(af)\|_{L^2(\mathbb{R}^2)}\\ &+\|\langle D_{t,x}\rangle^{\nu-m}t_0Q(D_x)(af)\|_{L^2(\mathbb{R}^2)} + \|\langle D_{t,x}\rangle^{\nu-m}P_0^m(af)\|_{L^2(\mathbb{R}^2)}\}\end{aligned}$$

for $f\in H^k$ and $P_0 = mt_o\partial_t + x_0\partial_x$. Since $\operatorname{supp} a\subset B_\varepsilon(t_0,x_0)$, the second term of the R.H.S. of (5.2) can be estimated by

$$\varepsilon\|Q(D_x)(af)\|_{H^{\nu-m}} + \|tQ(D_x)(af)\|_{H^{\nu-m}}$$

and the third term by $\varepsilon\|R_m f\|_{H^{\nu-m}} + \|P^m f\|_{H^{\nu-m}}$, where R_m is a partial differential operators of order m. Hence by taking ε sufficiently small, we obtain the desired estimate (5.2). □ □

Based upon the above Lemma 5.1, we proceed to show the regularity. The first step is the following proposition.

Proposition 5.2. *Let u be the solution constructed in Proposition 4.1. For $a(t,x)\in C_0^\infty(\mathbb{R}^2)$ with $a=1$ near (t_0,x_0), u satisfies*

$$\|aP^k u\|_{H^{m/2}} \le C_3 A_3^k k!$$

for all $k=0,1,2,\cdots,$.

Sketch of Proof of Proposition 5.2 . Taking $\nu=1$ and $f=u_k$ in Lemma 5.1, it follows that

(5.3)

$$\|\langle D_{t,x}\rangle au_k\|_{L^2(\mathbb{R}^2_{t,x})} \le C\left\{\|au_k\|_{L^2(\mathbb{R}^2_{t,x})} + \|tQ(D_x)(au_k)\|_{H^{1-m}(\mathbb{R}^2_{t,x})} + \|P^m(au_k)\|_{H^{1-m}(\mathbb{R}^2_{t,x})}\right\}.$$

The first and third term in the R.H.S. of (5.3) is easily estimated by the terms of u_k and u_{k+l}. The second is the essential part which is estimated by

$$\|aQ(D_x)u_k\|_{H^{1-m}(\mathbb{R}^2)} + \|[Q(D),a]u_k\|_{H^{1-m}(\mathbb{R}^2)}.$$

Since the commutator $[Q(D), a]$ is a differential operator of order $m-1$, we see

$$\|[Q(D),a]u_k\|_{H^{1-m}(\mathbb{R}^2)} \leq C_3\|au_k\|_{L^2(\mathbb{R}^2)} \leq C_3A_2^k k!. \tag{5.4}$$

While the first term can be dominated via the relation

$$itQ(D_x)u_k = \frac{1}{m}Pu_k - \frac{1}{m}x\partial_x u_k + itF_k(u,\bar{u}), \tag{5.5}$$

where $u_k = P^k u$, such that

$$\begin{aligned}&\|taQ(D_x)u_k\|_{H^{1-m}(\mathbb{R}^2)}\\ &\leq C_4(m^{-1})\big\{\|au_{k+1}\|_{H^{1-m}(\mathbb{R}^2)} + \|ax\partial_x u_k\|_{H^{1-m}(\mathbb{R}^2)} + \|atF_k(u,\bar{u})\|_{H^{1-m}(\mathbb{R}^2)}\big\}\end{aligned} \tag{5.6}$$

The first and second terms in the RHS in (5.6) are estimated by $C_3A_2^k k!$. The term involving the nonlinear interaction is dominated as

$$\begin{aligned}
\|atF_k(u,\bar{u})\|_{H^{1-m}(\mathbb{R}^2)} \leq &\sum_{k=k_0+k_1+\cdots+k_p} \frac{k!}{\prod_{i=0}^p k_i!} m^{k_0} \|ta\prod_{i=1}^p u_{k_i}\|_{H^{1-m}(\mathbb{R}^2)}\\
\leq & C_5 \sum_{k=k_0+k_1+\cdots+k_p} \frac{k!}{\prod_{i=0}^p k_i!} m^{k_0} \prod_{i=1}^p \|\tilde{a}u_{k_i}\|_{X^0_{b-1}}\\
\leq & C_5 \sum_{k=k_0+k_1+\cdots+k_p} m^{k_0}\frac{k!}{\prod_{i=0}^p k_i!} \prod_{i=1}^p C_2A_2^{k_i}k_i!\\
\leq & C_5C_2^p \sum_{k=k_0+k_1+\cdots+k_p} \frac{k!}{k_0!} m^{k_0} A_2^{p(k-k_0)}\\
= & C_5C_2^p \sum_{k_0=0}^k \frac{k! m^{k_0} A_2^{k-k_0}}{k_0!} \sum_{k_1=0}^{k_0} \cdots \sum_{k_{p-1}=0}^{k_{p-2}} k_{p-1}\\
= & C_5C_2^p \sum_{k_0=0}^k \frac{k! m^{k_0} A_2^{k-k_0}}{k_0!} \frac{(k-k_0+p-1)!}{(k-k_0)!(p-1)!}\\
\leq & C_5C_2^p \frac{(k+p-1)!}{(p-1)!} \sum_{k_0=0}^k \frac{k!}{k_0!(k-k_0)!} m^{k_0} A_2^{k-k_0}\\
= & C_2^pC_5 \frac{(A_2+m)^k(k+p-1)!}{(p-1)!} \leq C_6A_3^k k!,
\end{aligned} \tag{5.7}$$

where we have taken the constants C_5 and C_6 are depending on $|t-t_0|$ and A_3 appropriately large. Hence by gathering the estimates (5.2)-(5.7) and changing the cut off function a

into $\tilde{a}$ if necessary, we have

$$\|au_k\|_{H^1(\mathbb{R}^2)} \le C_7 A_3^k k!, \quad k = 0,1,2,\cdots. \tag{5.8}$$

Similar but somewhat tiresome estimates yield that

$$\|\langle D_{t,x}\rangle \tilde{a} u_k\|_{H^1(\mathbb{R}^2)} \le C_7 A_4^k k!, \quad k = 0,1,2,\cdots.$$

By repeating the above argument in finite times, we conclude by changing the cut off function into $\tilde{a}$, to have

$$\|\tilde{a} u_k\|_{H^{m/2}} \le C_9 A_5^k k!, \quad k = 0,1,2,\cdots.$$

□

Based on the estimate (5.1), we forward the second step to have

$$\sup_{t_0-\varepsilon<t<t_0+\varepsilon} \|aP^k u(t)\|_{H_x^{(m-1)/2}(B_\varepsilon(x_0))} \le CA_6^k k! \quad k = 0,1,2,\cdots.$$

Note that $(m-1)/2 \ge 1$ and $H^{(m-1)/2)}(\mathbb{R}^1)$ is algebra. Then one can prove by an induction argument, that

$$\sup_t \|a\partial_x^l P^k u(t)\|_{H_x^{(m-1)/2}(B_\varepsilon(x_0))} \le CA_7^{k+l}(k+l)! \quad k,l = 0,1,2,\cdots.$$

Finally the operator P can be translated into the time derivative via $t\partial_t = k^{-1}(P - x\partial_x)$;

$$\sup_t \|a(t\partial_t)^{l_1}\partial_x^{l_2} v\|_{H^1_{t,x}(\mathbb{R}^2)} \le CA_8^{l_1+l_2}(l_1+l_2)! \quad l_1,l_2 = 0,1,2,\cdots.$$

This gives the regularity for the solution.

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