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Kyoto University
HIGH ENERGY RESOLVENT ESTIMATES FOR
ACOUSTIC PROPAGATORS IN A STRATIFIED MEDIA

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§1 Introduction.
Let $n \geq 2$ and $x = (y, z) \in \mathbb{R}^{n-1} \times \mathbb{R}$. In this report we study the following operator:

$$L_0 = -a_0(z)^2 \Delta,$$

where

$$a_0(z) = \begin{cases} 
  c_+ & (z \geq h) \\
  c_h & (0 < z < h) \\
  c_- & (z \leq 0), 
\end{cases}$$

and $c_{\pm}, c_h$ and $h$ are positive constants and

$$\Delta = \sum_{j=1}^{n-1} \frac{\partial^2}{\partial y_j^2} + \frac{\partial^2}{\partial z^2}.$$

We consider only the case $c_h < \min(c_+, c_-)$ because we can find the guided waves (cf. Wilcox [9] or Weder [6]). It seems that there are no works dealing with high energy resolvent estimates for acoustic propagators in stratified media. Here we shall prove high energy resolvent estimates for the case $c_h < c_+ = c_-$. Kikuchi-Tamura [3] and Kadowaki [2] have proved low energy resolvent estimates for the case $c_h < \min(c_+, c_-)$ and $c_+ \neq c_-$ and the case $c_h < c_+ = c_-$ respectively. Both works were used Mourre’s commutator method (cf. Mourre[4]). But the conjugate operator in Kadowaki [2] is different from Kikuchi-Tamura [3]. Kikuchi-Tamura [3] took the generator of dailation in $\mathbb{R}^3$ as the conjugate operator. They dealt with only media of $\mathbb{R}^3$ but their result can be extended for media of $\mathbb{R}^n (n \geq 3)$ (cf. Kadowaki [2]). Kadowaki [2] has constructed the conjugate operator by using the generator of dailation in $\mathbb{R}^n$ and $\mathbb{R}^{n-1} (n \geq 3)$ togerther with the generalized Fourier transform of a related operator (cf. Weder [6]). The generator of dailation in $\mathbb{R}^{n-1}$ has been used to estimate the guided wave (see §2). In this report, we also use Mourre’s method. Our conjugate operator is similar to Kadowaki [2]( see §2 ).

Let $\mathcal{H}_0 = L^2(\mathbb{R}^n; a_0^{-2}(z)dx)$ be Hilbert space with inner products

$$<u, v> = \int_{\mathbb{R}^n} u(x)\overline{v(x)}a_0^{-2}(z)dx.$$
In particular $L^2(\mathbb{R}^n_x)$ is the usual $L^2$ space defined on $\mathbb{R}^n_x$ with inner products

$$<u, v>_{L^2(\mathbb{R}^n_x)} = \int_{\mathbb{R}^n_x} u(x)\overline{v(x)}dx$$

and the corresponding norms $|\cdot|_{L^2(\mathbb{R}^n_x)}$.

$L_0$ admits a unique self-adjoint realizations in $\mathcal{H}_0$. Then $L_0$ is a non-negative operator (zero is not an eigenvalue) and the $D(L_0)$ is given by $H^2(\mathbb{R}^n_x), H^s(\mathbb{R}^n_x)$ being Sobolev space of order $s$ over $\mathbb{R}^n_x$. We also denoted by $R(z; L_0)$ the resolvent $(L_0 - z)^{-1}$ of $L$ for $\text{Im}z \neq 0$.

Let $\alpha > 1/2$. Assume that $c_h < c_+ = c_-$. Then, we have

$$|X_\alpha R(\lambda \pm i\kappa; L_0)X_\alpha| = O(\lambda^{-\frac{1}{2}}) \quad (\lambda \to \infty),$$

uniformly in $\kappa > 0$, where $X_\alpha = (1 + |x|^2)^{-\alpha/2}$.

We define the self-adjoint operator $L_0(\lambda)$ on $L^2(\mathbb{R}^n_x)$,

$$\begin{cases} L_0(\lambda) = -\Delta - \lambda(a_0^{-2}(z) - c_+^{-2}) \\
D(L_0(\lambda)) = H^2(\mathbb{R}^n_x). \end{cases}$$

This operator has been introduced by Weder [7]. Theorem 1.1 is obtained as an immediate consequence of the following proposition

**Proposition 1.2.** Assume that $c_h < c_+ = c_-$. Then we have

$$|X_\alpha G_\kappa(0; \lambda)X_\alpha| = O(\lambda^{-\frac{1}{2}}) \quad (\lambda \to \infty),$$

uniformly in $\kappa > 0$, where

$$G_\kappa(0; \lambda) = (L_0(\lambda) - \lambda c_+^{-2} - i\kappa a_0^{-2}(z))^{-1}$$

for $\kappa > 0$.

In §§2,3 we shall give the proof of above proposition.

We give a comment for the assumption of Theorem 1.1. This follows from our method. Applying Mourre's method to the original operator, $L_0$, we do not get the Mourre's estimates on the neighborhood of thresholds of $L_0$ (cf. Wilcox [9] or Weder [6]). The conjugate operator for $L_0$ is contructed by using generator of dailation in $\mathbb{R}^n$ and exterior domains of ball in $\mathbb{R}^{n-1}$ together with the generalized Fourier transform for $L_0$ (cf. Kadowaki [1]). While, since $L_0(\lambda)$ dose not have thresholds on $[0, \infty)$ (see Weder [7]) , we can obtain Mourre's estimates. But, to prove only Lemma 3.6 in §3, we need the assumption $c_h < c_+ = c_-$. In brief we deal with only $c_h < c_+ \leq c_-$. 
As an application of our theorem, we can consider scattering problem for wave equations with dissipative terms in stratified media. This is due to Mochizuki [4]. He has proved existence of scattering states for wave equations with dissipative terms in the case \( c_h = c_+ = c_- = 1 \). His idea is due to Kato's smooth perturbation theory together with low and high energy resolvent estimates for Laplacian in \( \mathbb{R}^n (n \neq 2) \).

To consider scattering problem for stratified media, we need low energy estimates which is required in Mochizuki [4]. Kikuchi-Tamura [3] and Kadowaki [2] have proved low energy estimates in perturbed stratified media. But the 3-dimensional case in Kadowaki [3] and Kikuchi-Tamura [2] do not satisfy Mochizuki's condition (for detail see Mochizuki [4]). For Kikuchi-Tamura's result, we can remake it to satisfy Mochizuki's condition (see Kadowaki [3]). We will give low energy estimates for stratified media of \( \mathbb{R}^n (n \geq 2) \) elsewhere and consider scattering problem.

\section{2 Conjugate operator and Mourre's estimates.}

In this section we construct the conjugate operators and show Mourre's estimates (2.1). First we define conjugate operator, \( D(\lambda) \), as follows:

\[
D(\lambda) = F_0(\lambda)^* (D_n F_0(\lambda)) + F_1(\lambda)^* (D_{n-1} F_1(\lambda)) + \sum_{j=1}^{Q(\lambda)} G_j(\lambda)^* (D_{n-1} G_j(\lambda)),
\]

where \( k = (\overline{k}, k_0) \in \mathbb{R}^{n-1} \times \mathbb{R} \), \( F_0(\lambda), F_1(\lambda) \) and \( G_j(\lambda) \) are partially isometric operators for \( L_0(\lambda) \) (see Appendix) and

\[
D_n = \frac{1}{2i} (k \cdot \nabla_k + \nabla_k \cdot k), \quad D_{n-1} = \frac{1}{2i} (\overline{k} \cdot \nabla_{\overline{k}} + \nabla \cdot \overline{k} \overline{k}).
\]

We consider the commutator \( i[L_0(\lambda), D(\lambda)] \) as a form on \( H^2(\mathbb{R}_x^n) \cap D(D(\lambda)) \) as follows:

\[
< i[L_0(\lambda), D(\lambda)]u, u >_{L^2(\mathbb{R}_x^n)} = i(< D(\lambda)u, L_0(\lambda)u >_{L^2(\mathbb{R}_x^n)} - < L_0(\lambda)u, D(\lambda)u >_{L^2(\mathbb{R}_x^n)})
\]

for \( u \in H^2(\mathbb{R}_x^n) \cap D(D(\lambda)) \). Then Lemma A of the Appendix implies that

\[
< i[L_0(\lambda), D(\lambda)]u, u >_{L^2(\mathbb{R}_x^2)} = i\{ < |k|^2 F_0(\lambda)u, D_n F_0(\lambda)u >_{L^2(\mathbb{R}_x^n)} - < D_n F_0(\lambda)u, |k|^2 F_0(\lambda)u >_{L^2(\mathbb{R}_x^n)} \\
+ < |\overline{k}|^2 F_1(\lambda)u, D_{n-1} F_1(\lambda)u >_{L^2(\Omega_0)} - < D_{n-1} F_1(\lambda)u, |\overline{k}|^2 F_1(\lambda)u >_{L^2(\Omega_0)} \\
+ \sum_{j=1}^{Q(\lambda)} (< |\overline{k}|^2 G_j(\lambda)u, D_{n-1} G_j(\lambda)u >_{L^2(\mathbb{R}_x^{n-1})} - < D_{n-1} G_j(\lambda)u, |\overline{k}|^2 G_j(\lambda)u >_{L^2(\mathbb{R}_x^{n-1})}) \}
\]

Thus we have by integral by parts

\[
< i[L_0(\lambda), D(\lambda)]u, u >_{L^2(\mathbb{R}_x^2)} = < 2(F_0(\lambda)^* |k|^2 F_0(\lambda) + F_1(\lambda)^* |\overline{k}|^2 F_1(\lambda) + \sum_{j=1}^{Q(\lambda)} G_j(\lambda)^* |\overline{k}|^2 G_j(\lambda)) u, u >_{L^2(\mathbb{R}_x^2)}
\]
for $u \in H^2(\mathbb{R}^n_+ \cap D(D(\lambda)))$. Thus the form $i[L_0(\lambda), D(\lambda)]$ can be extended to a bounded operator from $H^1(\mathbb{R}^n_+)$ to $H^{-1}(\mathbb{R}^n_+)$ which is denoted by $i[L_0(\lambda), D(\lambda)]^0$.

Let $\lambda > 1$, take $f_\lambda(r) \in C_0^\infty(\mathbb{R})$, $0 \leq f_\lambda \leq 1$ such that $f_\lambda$ has support in $((c_+^{-2} - c_-^{-2}/2)\lambda, 2c_+^{-2}\lambda)$ and $f_\lambda = 1$ on $[(c_+^{-2} - c_-^{-2}/4)\lambda, 3c_+^{-2}\lambda/2]$. Noting that

$$f_\lambda(L_0(\lambda))i[L_0(\lambda), D(\lambda)]^{0}f_\lambda(L_0(\lambda)) = 2(F_0(\lambda)^*|k|^2f_\lambda(|k|^2 + q_-\lambda)^2F_0(\lambda) + F_1(\lambda)^*|\overline{k}|^2f_\lambda(|\overline{k}|^2 - k_0^2 + q_-\lambda)^2F_1(\lambda) + \sum_{j=1}^Q G_j(\lambda)^*|\overline{k}|^2f_\lambda(|\overline{k}|^2 - \omega_j^2(\lambda))^2G_j(\lambda).$$

Then there exists a positive constant $C$ which is independent of $\lambda$ such that

$$(2.1) \quad f_\lambda(L_0(\lambda))i[L_0(\lambda), D(\lambda)]^{0}f_\lambda(L_0(\lambda)) \geq C\lambda f_\lambda(L_0(\lambda))^2$$
in the form sense.

§3 Proof of Proposition 1.2.

Proposition 1.2 follows from lemmas in this section. But we omit the proof of lemmas and give only a comment of the proof.

We can prove the following lemmas in the same way as in the proof of Lemma 2.5 of Weder [7].

Lemma 3.1. Let $f \in C_0^\infty(\mathbb{R})$. Then

(i) $f(L_0(\lambda))$ sends $D(D(\lambda))$ into $D(D(\lambda))$.

(ii) $[f(L_0(\lambda)), D(\lambda)]$ defined as operator on $D(D(\lambda))$ is extended to a bounded operator on $L^2(\mathbb{R}^n_+)$ which is denoted by $[f(L_0(\lambda)), D(\lambda)]^0$.

It follows from (2.1) that $M_0(\lambda)$ is non-negative and hence we define an operator, $G_\kappa(\epsilon; \lambda)$, on $L^2(\mathbb{R}^n_+)$ by

$$(3.1) \quad G_\kappa(\epsilon; \lambda) = (L_0(\lambda) - \lambda c_+^{-2} - ic_0^{-2}(z) - i\epsilon M_0(\lambda))^{-1}$$
for $\kappa > 0$ and $\epsilon > 0$. Using (2.1), we can prove the following lemma (for detail, see that of Lemma 5.3 of Kikuchi-Tamura [3]).

Lemma 3.2. For $\epsilon > 0$, as $\lambda \to \infty$, one has

$$\|G_\kappa(\epsilon; \lambda)\| = \epsilon^{-1}O(\lambda^{-1}), \quad (\lambda \to \infty)$$
uniformly in $\kappa > 0$.

We write

$$F_\kappa(\epsilon; \lambda) = \lambda^{\frac{3}{2}}Z_\alpha(\epsilon, \lambda)G_\kappa(\lambda^{-\frac{1}{2}}\epsilon; \lambda)Z_\alpha(\epsilon, \lambda),$$
where $Z_\alpha(\epsilon, \lambda) = (\lambda^{\frac{3}{2}} + |D(\lambda)|)^{-\alpha}(\lambda^{\frac{3}{2}} + \epsilon|D(\lambda)|)^{\alpha-1}$.

This is due to Yafaev [8]. But we do not use the scaling argument for $\lambda$ (cf. (3.1)).

Let $g_\lambda(p) = 1 - f_\lambda(p)$. We write in brief $f_\lambda$ and $g_\lambda$ for $f_\lambda(L_0(\lambda))$ and $g_\lambda(L_0(\lambda))$ respectively.
Noting that
\[ G_\kappa(\epsilon; \lambda) D(D(\lambda)) \subset D(D(\lambda)) \cap H^2(\mathbb{R}^n) \]
(cf. Kadowaki [2]), we decompose \((d/d\epsilon) F_\kappa(\epsilon; \lambda)\) as a form on \(L^2(\mathbb{R}_x^n)\)

\[ (d/d\epsilon) F_\kappa(\epsilon; \lambda) = \sum_{j=1}^8 Y^j_\kappa(\epsilon; \lambda), \]

where
\[
\begin{align*}
Y^1_\kappa &= i Z_\alpha(\epsilon, \lambda) G_\kappa(\lambda^{-\frac{1}{2}}; \lambda) g_\lambda [L_0(\lambda), D(\lambda)]^0 f_\lambda G_\kappa(\lambda^{-\frac{1}{2}}; \lambda) Z_\alpha(\epsilon, \lambda), \\
Y^2_\kappa &= i Z_\alpha(\epsilon, \lambda) G_\kappa(\lambda^{-\frac{1}{2}}; \lambda) g_\lambda [L_0(\lambda), D(\lambda)]^0 g_\lambda G_\kappa(\lambda^{-\frac{1}{2}}; \lambda) Z_\alpha(\epsilon, \lambda), \\
Y^3_\kappa &= i Z_\alpha(\epsilon, \lambda) G_\kappa(\lambda^{-\frac{1}{2}}; \lambda) f_\lambda [L_0(\lambda), D(\lambda)]^0 g_\lambda G_\kappa(\lambda^{-\frac{1}{2}}; \lambda) Z_\alpha(\epsilon, \lambda), \\
Y^4_\kappa &= -i Z_\alpha(\epsilon, \lambda) \{D(\lambda) G_\kappa(\lambda^{-\frac{1}{2}}; \lambda) + G_\kappa(\lambda^{-\frac{1}{2}}; \lambda) D(\lambda)\} Z_\alpha(\epsilon, \lambda), \\
Y^5_\kappa &= \kappa Z_\alpha(\epsilon, \lambda) G_\kappa(\lambda^{-\frac{1}{2}}; \lambda) [a_0^{-2}(z), D(\lambda)]^0 f_\lambda Z_\alpha(\epsilon, \lambda), \\
Y^6_\kappa &= \epsilon Z_\alpha(\epsilon, \lambda) G_\kappa(\lambda^{-\frac{1}{2}}; \lambda) [M_0(\lambda), D(\lambda)]^0 Z_\alpha(\epsilon, \lambda), \\
Y^7_\kappa &= \lambda^{-\frac{1}{2}} \{d/d\epsilon Z_\alpha(\epsilon, \lambda)\} G_\kappa(\lambda^{-\frac{1}{2}}; \lambda) Z_\alpha(\epsilon, \lambda), \\
Y^8_\kappa &= \lambda^{-\frac{1}{2}} Z_\alpha(\epsilon, \lambda) G_\kappa(\lambda^{-\frac{1}{2}}; \lambda) \frac{d}{d\epsilon} Z_\alpha(\epsilon, \lambda).
\end{align*}
\]

We need the following lemmas (Lemma 3.3~Lemma 3.5) to estimate each term of the right side of (3.2).

Note that there is \(c_0, c_0 > 0\) such that \((L_0(\lambda) + c_0\lambda)^{-1}\) exists.

**Lemma 3.3.** As \(\lambda \to \infty\), one has:

(i) \(\|g_\lambda G_\kappa(\lambda^{-\frac{1}{2}}; \lambda)\| = O(\lambda^{-1})\),

(ii) \(\|L_0(\lambda) + c_0\lambda\}^{1/2} f_\lambda G_\kappa(\lambda^{-\frac{1}{2}}; \lambda) Z_\alpha(\epsilon, \lambda)\| = \epsilon^{-1/2}\|F_\kappa\|^{1/2} O(1),\)

(iii) \(\|L_0(\lambda) + c_0\lambda\}^{1/2} g_\lambda G_\kappa(\lambda^{-\frac{1}{2}}; \lambda) Z_\alpha(\epsilon, \lambda)\| = O(\lambda^{-1}),\)

(iv) \(\|F_\kappa(\epsilon; \lambda)\| = \epsilon^{-1} O(\lambda^{-1}),\)

uniformly in \(\kappa > 0\).

For a proof of Lemma 3.5 (i), see that of Lemma 5.4 of Kikuchi-Tamura [3]. Also, for a proof of (ii) and (iii), see that of Lemma 5.5 of Kikuchi-Tamura [3]. (ii) and (iii) imply (iv).

**Lemma 3.4.** Assume that \(c_+ < c_- = c_-\). Then \([a_0^{-2}(z), D(\lambda)]\) defined as a form on \(D(D(\lambda))\) is extended to a bounded operator from \(H^1(\mathbb{R}_x^n)\) to \(H^{-1}(\mathbb{R}_x^n)\) which is denoted by \([a_0^{-2}(z), D(\lambda)]^0\). Moreover we have

\[
i[a_0^{-2}(z), D(\lambda)]^0
= (c_+^{-2} - 1)((n - 1)\chi_{0<z<h}(z) - (\partial_{k_0} F_0(\lambda)\chi_{0<z<h}(z)) \ast k_0 F_0(\lambda)
- (k_0 F_0(\lambda)) \ast \partial_{k_0} F_0(\lambda) \chi_{0<z<h}(z) + F_0(\lambda) \ast F_0(\lambda))
\]

and

\[
\|L_0(\lambda) + c_0\lambda\}^{-1/2} i[a_0^{-2}(z), D(\lambda)]^0 (L_0(\lambda) + c_0\lambda)^{-1/2} = O(\lambda^{-\frac{1}{2}}) \quad (\lambda \to \infty).
\]
proof. Noting that the representation of \( F_0(\lambda) \), we show this lemma by straightforward calculation (cf. Kadowaki [2]).

Using Lemma 3.1 and the representation of \( i[L_0(\lambda), D(\lambda)]^0 \) we show the following lemma.

**Lemma 3.5.** As \( \lambda \rightarrow \infty \), one has:

\[
\| [M_0(\lambda), D(\lambda)]^0 \| = O(\lambda).
\]

Using Lemma 3.2 \( \sim 3.5 \), we can evaluate the norm of \( Y^j_k, 1 \leq j \leq 8 \) (see Kikuchi-Tamura [3]). Thus we obtain the following differential inequality:

\[
(3.3) \quad \| (d/d\epsilon) F_\kappa(\epsilon; \lambda) \| \leq C(\lambda^{-1} \epsilon^{\alpha-1} + \lambda^{-\frac{1}{2}} \epsilon^{\alpha-\frac{3}{2}} \| F_\kappa \|^{1/2} + \| F_\kappa \|)
\]

It follows from Lemma 3.3(iv) and (3.3) that

\[
(3.4) \quad \| (\lambda^{\frac{1}{2}} + |D(\lambda)|)^{-\alpha} G_\kappa(0; \lambda)(\lambda^{\frac{1}{2}} + |D(\lambda)|)^{-\alpha} \| = O(\lambda^{-\frac{1}{2} - \alpha}), \quad (\lambda \rightarrow \infty),
\]

uniformly in \( \kappa > 0 \).

Noting Lemma 3.1 we rewrite \( D(\lambda) f_\lambda X_1 \) as

\[
(3.5) \quad \frac{1}{i} (f_\lambda \nabla_y \cdot y X_1 + \frac{n-1}{2} f_\lambda X_1) - \frac{1}{i} (f_\lambda F_0(\lambda)^* k_0 \partial k_0 F_0(\lambda) X_1 + \frac{1}{2} f_\lambda F_0(\lambda)^* F_0(\lambda) X_1) + [D(\lambda), f_\lambda]^0 X_1.
\]

We can show that

\[
(3.6) \quad \| [D(\lambda), f_\lambda]^0 \| = O(1), \quad (\lambda \rightarrow \infty),
\]

(for proof, see that of Lemma 5.6 of Kikuchi-Tamura [3]).

By straightforward calculation we can show next lemma.

**Lemma 3.6.** As \( \lambda \rightarrow \infty \), one has:

\[
\| f_\lambda F_0(\lambda)^* k_0 \partial k_0 F_0(\lambda) X_1 \| = O(\lambda^{\frac{1}{2}})
\]

It follows from (3.5), (3.6) and Lemma 3.8 that

\[
\| D(\lambda) f_\lambda X_1 \| = O(\lambda^{\frac{1}{2}}) \quad (\lambda \rightarrow \infty).
\]

Thus we obtain by interpolation

\[
(3.3) \quad \| (\lambda^{\frac{1}{2}} + |D(\lambda)|)^\alpha f_\lambda X_\alpha \| = O(\lambda^{\frac{9}{2}}) \quad (\lambda \rightarrow \infty).
\]

Note that

\[
\| g_\lambda G_\kappa(0; \lambda) \| = O(\lambda^{-1}) \quad (\lambda \rightarrow \infty).
\]

(3.4) and (3.7) imply that

\[
\| X_\alpha G_\kappa(0; \lambda) X_\alpha \| = O(\lambda^{-\frac{1}{2}}) \quad (\lambda \rightarrow \infty).
\]

Now the proof of Proposition 1.2 is complete.
Appendix.

In this Appendix we state the generalized Fourier transform of \( L_0(\lambda) \) established by Weder (cf. Weder [6]).

For \( \lambda \gg 1 \) large enough, we consider the following operator:

\[
\begin{align*}
    h(\lambda) &= -\frac{d^2}{dz^2} - \lambda(a_0^{-2}(z) - c_+^{-2}), \\
    D(h(\lambda)) &= H^2(\mathbb{R}_z).
\end{align*}
\]

This is the self-adjoint operator in \( L^2(\mathbb{R}_z) \).

\( h(\lambda) \) has finite number \( Q(\lambda) \in \mathbb{N} \), of eigenvalues, \(-\omega_j^2(\lambda), 0 < \omega_j^2(\lambda) < q_h(\lambda) = \lambda(c_h^{-2} - c_+^{-2}), 1 \leq j \leq Q(\lambda)\) which are partially isometric operators from \( L^2(\mathbb{R}_z) \) onto \( L^2(\mathbb{R}_k^n), L^2(\Omega_0) \) and \( L^2(\mathbb{R}_k^{n-1}) \) respectively, where \( \Omega_0 = \{ k \in \mathbb{R}^n; 0 < k_0 < \sqrt{q_-(\lambda) = \sqrt{\lambda(c_+^{-2} - c_-^{-2})}} \}. \)

Defining the operator \( F(\lambda) \) as

\[
F(\lambda)u = (F_0(\lambda)u, F_1(\lambda)u, G_1(\lambda)u, G_2(\lambda)u, \cdots G_{Q(\lambda)}(\lambda)u)
\]

for \( u \in L^2(\mathbb{R}_k^n) \), we have

**Lemma A.** \( F(\lambda) \) is unitary operator from \( L^2(\mathbb{R}_k^n) \) onto

\[ \hat{\mathcal{H}} = L^2(\mathbb{R}_k^n) \bigoplus_{j=1}^{Q(\lambda)} L^2(\Omega_0) \bigoplus_{j=1}^{Q(\lambda)} L^2(\mathbb{R}_k^{n-1}) \]

and for every \( u \in D(L_0(\lambda)) = H^2(\mathbb{R}_k^n) \)

\[
F(\lambda)L_0(\lambda)u = ((|k|^2 + q_-(\lambda))F_0(\lambda)u, (|k|^2 - k_0^2 + q_-(\lambda))F_1(\lambda)u, \cdots)
\]

\[
((|\bar{k}|^2 - \omega_1^2(\lambda))G_1(\lambda)u, (|\bar{k}|^2 - \omega_2^2(\lambda))G_2(\lambda)u, \cdots)
\]

\[
((|\bar{k}|^2 - \omega_{Q(\lambda)}^2(\lambda))G_{Q(\lambda)}(\lambda)u).
\]

For the proof, see Weder [6].

**References**


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