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HIGH ENERGY RESOLVENT ESTIMATES FOR ACOUSTIC PROPAGATORS IN A STRATIFIED MEDIA

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§1 Introduction.

Let \( n \geq 2 \) and \( x = (y, z) \in \mathbb{R}^{n-1} \times \mathbb{R} \). In this report we study the following operator:

\[
L_0 = -a_0(z)^2 \Delta,
\]

where

\[
a_0(z) = \begin{cases}
  c_+ & (z \geq h) \\
  c_h & (0 < z < h) \\
  c_- & (z \leq 0),
\end{cases}
\]

and \( c_\pm, c_h \) and \( h \) are positive constants and

\[
\Delta = \sum_{j=1}^{n-1} \frac{\partial^2}{\partial y_j^2} + \frac{\partial^2}{\partial z^2}.
\]

We consider only the case \( c_h < \min(c_+, c_-) \) because we can find the guided waves (cf. Wilcox [9] or Weder [6]). It seems that there are no works dealing with high energy resolvent estimates for acoustic propagators in stratified media. Here we shall prove high energy resolvent estimates for the case \( c_h < c_+ = c_- \).

Kikuchi-Tamura [3] and Kadowaki [2] have proved low energy resolvent estimates for the case \( c_h < \min(c_+, c_-) \) and \( c_+ \neq c_- \) and the case \( c_h < c_+ = c_- \) respectively. Both works were used Mourre’s commutator method (cf. Mourre[4]). But the conjugate operator in Kadowaki [2] is different from Kikuchi-Tamura [3]. Kikuchi-Tamura [3] took the generator of dailation in \( \mathbb{R}^3 \) as the conjugate operator. They dealt with only media of \( \mathbb{R}^3 \) but their result can be extended for media of \( \mathbb{R}^n (n \geq 3) \) (cf. Kadowaki [2]). Kadowaki [2] has constructed the conjugate operator by using the generator of dailation in \( \mathbb{R}^n \) and \( \mathbb{R}^{n-1} (n \geq 3) \) together with the generalized Fourier transform of a related operator (cf. Weder [6]). The generator of dailation in \( \mathbb{R}^{n-1} \) has been used to estimate the guided wave (see §2). In this report, we also use Mourre’s method. Our conjugate operator is similar to Kadowaki [2](see §2).

Let \( \mathcal{H}_0 = L^2(\mathbb{R}^n; a_0^{-2}(z)dx) \) be Hilbert space with inner products

\[
\langle u, v \rangle_0 = \int_{\mathbb{R}^n} u(x) \overline{v(x)} a_0^{-2}(z) dx.
\]
In particular $L^2(\mathbb{R}^n_x)$ is the usual $L^2$ space defined on $\mathbb{R}^n_x$ with inner products

$$<u, v>_{L^2(\mathbb{R}^n_x)} = \int_{\mathbb{R}^n_x} u(x)\overline{v(x)}dx$$

and the corresponding norms $| \cdot |_{L^2(\mathbb{R}^n_x)}$.

$L_0$ admits a unique self-adjoint realizations in $\mathcal{H}_0$. Then $L_0$ is a non-negative operator (zero is not an eigenvalue) and the $D(L_0)$ is given by $H^2(\mathbb{R}^n_x)$, $H^s(\mathbb{R}^n_x)$ being Sobolev space of order $s$ over $\mathbb{R}^n_x$. We also denote by $R(z; L_0)$ the resolvent $(L_0 - z)^{-1}$ of $L$ for $\text{Im}z \neq 0$.

$A$ is considered as an operator from $L^2(\mathbb{R}^n_x)$ into itself, then its norm is denoted by the notation $\|A\|$.

We also denote by $R(z; L_0)$ the resolvent $(L_0 - z)^{-1}$ of $L$ for $\text{Im}z \neq 0$.

Our result is:

**Theorem 1.1.** Let $\alpha > 1/2$. Assume that $c_h < c_+ = c_-$. Then, we have

$$\|X_{\alpha}R(\lambda \pm i\kappa; L_0)x\| = O(\lambda^{-\frac{1}{2}}) \quad (\lambda \to \infty),$$

uniformly in $\kappa > 0$, where $X_{\alpha} = (1 + |x|^2)^{-\alpha/2}$.

We define the self-adjoint operator $L_0(\lambda)$ on $L^2(\mathbb{R}^n_x)$,

$$\begin{cases} 
L_0(\lambda) &= -\triangle - \lambda(a_0^-2(z) - c_+^{-2}) \\
D(L_0(\lambda)) &= H^2(\mathbb{R}^n_x).
\end{cases}$$

This operator has been introduced by Weder [7]. Theorem 1.1 is obtained as an immediate consequence of the following proposition

**Proposition 1.2.** Assume that $c_h < c_+ = c_-$. Then we have

$$\|X_{\alpha}G_\kappa(0; \lambda)x\| = O(\lambda^{-\frac{1}{2}}) \quad (\lambda \to \infty),$$

uniformly in $\kappa > 0$, where

$$G_\kappa(0; \lambda) = (L_0(\lambda) - \lambda c_+^{-2} - i\kappa a_0^-2(z))^{-1}$$

for $\kappa > 0$.

In §§2,3 we shall give the proof of above proposition.

We give a comment for the assumption of Theorem 1.1. This follows from our method. Applying Mourre's method to the original operator, $L_0$, we do not get the Mourre's estimates on the neighborhood of thresholds of $L_0$ (cf. Wilcox [9] or Weder [6]). The conjugate operator for $L_0$ is constructed by using generator of dilation in $\mathbb{R}^n$ and exterior domains of ball in $\mathbb{R}^{n-1}$ together with the generalized Fourier transform for $L_0$ (cf. Kadowaki [1]). While, since $L_0(\lambda)$ dose not have thresholds on $[0, \infty)$ (see Weder [7]) , we can obtain Mourre's estimates. But, to prove only Lemma 3.6 in §3, we need the assumption $c_h < c_+ = c_-$. In brief we deal with only $c_h < c_+ \leq c_-$. 
As an application of our theorem, we can consider scattering problem for wave equations with dissipative terms in stratified media. This is due to Mochizuki [4]. He has proved existence of scattering states for wave equations with dissipative terms in the case $c_h = c_+ = c_- = 1$. His idea is due to Kato’s smooth perturbation theory together with low and high energy resolvent estimates for Laplacian in $\mathbb{R}^n (n \neq 2)$. To consider scattering problem for stratified media, we need low energy estimates which is required in Mochizuki [4]. Kikuchi-Tamura [3] and Kadowaki [2] have proved low energy estimates in perturbed stratified media. But the 3-dimensonal case in Kadowaki [3] and Kikuchi-Tamura [2] do not satisfy Mochizuki’s condition (for detail see Mochizuki [4]). For Kikuchi-Tamura’s result, we can remake it to satisfy Mochizuki’s condition (see Kadowaki [3]). We will give low energy estimates for stratified media of $\mathbb{R}^n (n \geq 2)$ elsewhere and consider scattering problem.

§2 Conjugate operator and Mourre’s estimates.

In this section we construct the conjugate operators and show Mourre’s estimates (2.1). First we define conjugate operator, $D(\lambda)$, as follows:

$$D(\lambda) = F_0(\lambda)^* (-D_n) F_0(\lambda) + F_1(\lambda)^* (-D_{n-1}) F_1(\lambda)$$

$$+ \sum_{j=1}^{Q(\lambda)} G_j(\lambda)^* (-D_{n-1}) G_j(\lambda),$$

where $k = (\overline{k}, k_0) \in \mathbb{R}^{n-1} \times \mathbb{R}$, $F_0(\lambda), F_1(\lambda)$ and $G_j(\lambda)$ are partially isometric operators for $L_0(\lambda)$ (see Appendix) and

$$D_n = \frac{1}{2i} (k \cdot \nabla_k + \nabla \cdot k), \quad D_{n-1} = \frac{1}{2i} (\overline{k} \cdot \nabla_{\overline{k}} + \nabla \cdot \overline{k}).$$

We consider the commutator $i[L_0(\lambda), D(\lambda)]$ as a form on $H^2(\mathbb{R}_x^n) \cap D(D(\lambda))$ as follows:

$$<i[L_0(\lambda), D(\lambda)] u, u>_{L^2(\mathbb{R}_x^n)} = i (<D(\lambda)u, L_0(\lambda)u>_{L^2(\mathbb{R}_x^n)} - <L_0(\lambda)u, D(\lambda)u>_{L^2(\mathbb{R}_x^n)})$$

for $u \in H^2(\mathbb{R}^n) \cap D(D(\lambda))$. Then Lemma A of the Appendix implies that

$$<i[L_0(\lambda), D(\lambda)] u, u>_{L^2(\mathbb{R}_x^n)}$$

$$= i \{ <\kappa^2 F_0(\lambda) u, D_n F_0(\lambda) u>_{L^2(\mathbb{R}_k^n)} - <D_n F_0(\lambda) u, \kappa^2 F_0(\lambda) u>_{L^2(\mathbb{R}_k^n)}$$

$$+ <\kappa^2 F_1(\lambda) u, D_{n-1} F_1(\lambda) u>_{L^2(\Omega_0)} - <D_{n-1} F_1(\lambda) u, \kappa^2 F_1(\lambda) u>_{L^2(\Omega_0)}$$

$$+ \sum_{j=1}^{Q(\lambda)} (<\kappa^2 G_j(\lambda) u, D_{n-1} G_j(\lambda) u>_{L^2(\mathbb{R}_{k_0}^{n-1})}$$

$$- <D_{n-1} G_j(\lambda) u, \kappa^2 G_j(\lambda) u>_{L^2(\mathbb{R}_{k_0}^{n-1})}) \}.$$

Thus we have by integral by parts

$$<i[L_0(\lambda), D(\lambda)] u, u>_{L^2(\mathbb{R}_x^n)}$$

$$= <2 F_0(\lambda)^* \kappa^2 F_0(\lambda) + F_1(\lambda)^* \kappa^2 F_1(\lambda) + \sum_{j=1}^{Q(\lambda)} G_j(\lambda)^* \kappa^2 G_j(\lambda)) u, u>_{L^2(\mathbb{R}_x^n)}$$
for \( u \in H^2(\mathbb{R}^n) \cap D(D(\lambda)) \). Thus the form \( i[L_0(\lambda), D(\lambda)] \) can be extended to a bounded operator from \( H^1(\mathbb{R}^n) \) to \( H^{-1}(\mathbb{R}^n) \). Let \( \lambda > 1 \), take \( f_\lambda(r) \in C_0^\infty(\mathbb{R}), 0 \leq f_\lambda \leq 1 \) such that \( f_\lambda \) has support in \( ((c_+^{-2} - 2c_-^{-2})/\lambda, 2c_+^{-2}\lambda) \) and \( f_\lambda = 1 \) on \( [((c_+^{-2} - 2c_-^{-2})/4)\lambda, 3c_+^{-2}\lambda/2] \). Noting that

\[
\begin{align*}
\int_{\mathbb{R}^n} f_\lambda(L_0(\lambda)) i[L_0(\lambda), D(\lambda)]^{0} f_\lambda(L_0(\lambda))
&= 2(\int_{\mathbb{R}^n} f_\lambda(\lambda)^* |k|^2 f_\lambda(|k|^2 + q_-(\lambda))^2 F_0(\lambda) + F_1(\lambda)^* |\overline{k}|^2 f_\lambda(|\overline{k}|^2 - k_0^2 + q_-(\lambda))^2 F_1(\lambda)
+ \sum_{j=1}^{Q(\lambda)} G_j(\lambda)^* |\overline{k}|^2 f_\lambda(\overline{k}^2 - \omega^2(j)\lambda)^2 G_j(\lambda).
\end{align*}
\]

Then there exists a positive constant \( C \) which is independent of \( \lambda \) such that

\[
(2.1) \quad f_\lambda(L_0(\lambda)) i[L_0(\lambda), D(\lambda)]^{0} f_\lambda(L_0(\lambda)) \geq C\lambda f_\lambda(L_0(\lambda))^2
\]
in the form sense.

§3 Proof of Proposition 1.2.

Proposition 1.2 follows from lemmas in this section. But we omit the proof of lemmas and give only a comment of the proof.

We can prove the following lemmas in the same way as in the proof of Lemma 2.5 of Weder [7].

**Lemma 3.1.** Let \( f \in C_0^\infty(\mathbb{R}) \). Then

(i) \( f(L_0(\lambda)) \) sends \( D(D(\lambda)) \) into \( D(D(\lambda)) \).

(ii) \( [f(L_0(\lambda)), D(\lambda)] \) defined as operator on \( D(D(\lambda)) \) is extended to a bounded operator on \( L^2(\mathbb{R}^n) \) which is denoted by \( [f(L_0(\lambda)), D(\lambda)]^0 \).

It follows from (2.1) that \( M_0(\lambda) \) is non-negative and hence we define an operator, \( G_\kappa(\epsilon; \lambda) \), on \( L^2(\mathbb{R}^n) \) by

\[
G_\kappa(\epsilon; \lambda) = (L_0(\lambda) - \lambda c_+^{-2} - i\kappa a_0^{-2}(z) - i\epsilon M_0(\lambda))^{-1}
\]

for \( \kappa > 0 \) and \( \epsilon > 0 \). Using (2.1), we can prove the following lemma (for detail, see that of Lemma 5.3 of Kikuchi-Tamura [3]).

**Lemma 3.2.** For \( \epsilon > 0 \), as \( \lambda \to \infty \), one has

\[
\|G_\kappa(\epsilon; \lambda)\| = \epsilon^{-1} O(\lambda^{-1}), \quad (\lambda \to \infty)
\]

uniformly in \( \kappa > 0 \).

We write

\[
F_\kappa(\epsilon; \lambda) = \lambda^{\frac{1}{2}} Z_\alpha(\epsilon, \lambda) G_\kappa(\lambda^{-\frac{1}{2}} \epsilon; \lambda) Z_\alpha(\epsilon, \lambda),
\]

where \( Z_\alpha(\epsilon, \lambda) = (\lambda^{\frac{1}{2}} + |D(\lambda)|)^{-\alpha}(\lambda^{\frac{1}{2}} + \epsilon|D(\lambda)|)^{\alpha^{-1}} \).

This is due to Yafaev [8]. But we do not use the scaling argument for \( \lambda \) (cf. (3.1)).

Let \( g_\lambda(p) = 1 - f_\lambda(p) \). We write in brief \( f_\lambda \) and \( g_\lambda \) for \( f_\lambda(L_0(\lambda)) \) and \( g_\lambda(L_0(\lambda)) \) respectively.
Noting that

\[ G_\kappa(\epsilon; \lambda)D(D(\lambda)) \subset D(D(\lambda)) \cap H^2(\mathbb{R}^n) \]

(cf. Kadowaki [2]), we decompose \((d/d\epsilon)F_\kappa(\epsilon; \lambda)\) as a form on \(L^2(\mathbb{R}^n_x)\)

\[
(3.2) \quad (d/d\epsilon)F_\kappa(\epsilon; \lambda) = \sum_{j=1}^{8} Y_\kappa^j(\epsilon; \lambda),
\]

where

\[
Y_\kappa^1 = iZ_\alpha(\epsilon, \lambda)G_\kappa(\lambda^{-\frac{1}{2}}\epsilon; \lambda)g_\lambda[L_0(\lambda), D(\lambda)]g_\lambda[G_\kappa(\lambda^{-\frac{1}{2}}\epsilon; \lambda)Z_\alpha(\epsilon, \lambda),
\]

\[
Y_\kappa^2 = iZ_\alpha(\epsilon, \lambda)G_\kappa(\lambda^{-\frac{1}{2}}\epsilon; \lambda)g_\lambda[L_0(\lambda), D(\lambda)]g_\lambda[G_\kappa(\lambda^{-\frac{1}{2}}\epsilon; \lambda)Z_\alpha(\epsilon, \lambda)
\]

\[
Y_\kappa^3 = iZ_\alpha(\epsilon, \lambda)G_\kappa(\lambda^{-\frac{1}{2}}\epsilon; \lambda)f_\lambda[L_0(\lambda), D(\lambda)]g_\lambda[G_\kappa(\lambda^{-\frac{1}{2}}\epsilon; \lambda)Z_\alpha(\epsilon, \lambda)
\]

\[
Y_\kappa^4 = -iZ_\alpha(\epsilon, \lambda)(D(\lambda)G_\kappa(\lambda^{-\frac{1}{2}}\epsilon; \lambda) + G_\kappa(\lambda^{-\frac{1}{2}}\epsilon; \lambda)D(\lambda))Z_\alpha(\epsilon, \lambda)
\]

\[
Y_\kappa^5 = \kappa Z_\alpha(\epsilon, \lambda)G_\kappa(\lambda^{-\frac{1}{2}}\epsilon; \lambda)[a_0(\lambda)^{-2}\lambda]Z_\alpha(\epsilon, \lambda)
\]

\[
Y_\kappa^6 = \epsilon Z_\alpha(\epsilon, \lambda)G_\kappa(\lambda^{-\frac{1}{2}}\epsilon; \lambda)[M_0(\lambda), D(\lambda)]Z_\alpha(\epsilon, \lambda)
\]

\[
Y_\kappa^7 = \lambda^{-\frac{1}{2}}\frac{d}{d\epsilon}Z_\alpha(\epsilon, \lambda)\}
\]

\[
Y_\kappa^8 = \lambda^{-\frac{1}{2}}Z_\alpha(\epsilon, \lambda)G_\kappa(\lambda^{-\frac{1}{2}}\epsilon; \lambda)\frac{d}{d\epsilon}Z_\alpha(\epsilon, \lambda).
\]

We need the following lemmas (Lemma 3.3~Lemma 3.5) to estimate each term of the right side of (3.2).

Note that there is \(c_0, c_0 > 0\) such that \((L_0(\lambda) + c_0\lambda)^{-1}\) exists.

**Lemma 3.3.** As \(\lambda \to \infty\), one has:

(i) \(\|g_\lambda G_\kappa(\lambda^{-\frac{1}{2}}\epsilon; \lambda)\| = O(\lambda^{-1})\),

(ii) \(\|(L_0(\lambda) + c_0\lambda)\lambda^{1/2}f_\lambda Z_\alpha(\epsilon, \lambda)\| = \epsilon^{-1/2}\|F_\kappa\|^{1/2}O(1)\),

(iii) \(\|(L_0(\lambda) + c_0\lambda)\lambda^{1/2}g_\lambda Z_\alpha(\epsilon, \lambda)\| = \lambda^{-1}\|F_\kappa(\epsilon; \lambda)\| = O(\lambda^{-1}),\)

(iv) \(\|F_\kappa(\epsilon; \lambda)\| = \epsilon^{-1}(\lambda^{-1}),\)

uniformly in \(\kappa > 0\).

For a proof of Lemma 3.5 (i), see that of Lemma 5.4 of Kikuchi- Tamura [3]. Also, for a proof of (ii) and (iii), see that of Lemma 5.5 of Kikuchi-Tamura [3]. (ii) and (iii) imply (iv).

**Lemma 3.4.** Assume that \(c_h < c_+ = c_-\). Then \([a_0^{-2}(z), D(\lambda)]\) defined as a form on \(D(D(\lambda))\) is extended to a bounded operator from \(H^1(\mathbb{R}^n_x)\) to \(H^{-1}(\mathbb{R}^n_x)\) which is denoted by \([a_0^{-2}(z), D(\lambda)]^0\). Moreover we have

\[
i[a_0^{-2}(z), D(\lambda)]^0
\]

\[
= (c_h^{-2} - 1)(n-1)\chi_0 < z < h(z) - (\partial_{k_0} F_0(\lambda))\chi_0 < z < h(z))k_0 F_0(\lambda)
\]

\[
- (k_0 F_0(\lambda))\partial_{k_0} F_0(\lambda)\chi_0 < z < h(z) + F_0(\lambda) F_0(\lambda)
\]

and

\[
\|(L_0(\lambda) + c_0\lambda)^{-1/2}i[a_0^{-2}(z), D(\lambda)]^0(L_0(\lambda) + c_0\lambda)^{-1/2}\| = O(\lambda^{-\frac{1}{2}}) \quad (\lambda \to \infty).
\]
proof. Noting that the representation of $F_0(\lambda)$, we show this lemma by straightforward calculation (cf. Kadowaki [2]).

Using Lemma 3.1 and the representation of $i[L_0(\lambda), D(\lambda)]^0$ we show the following lemma.

**Lemma 3.5.** As $\lambda \to \infty$, one has:

$$\| [M_0(\lambda), D(\lambda)]^0 \| = O(\lambda).$$

Using Lemma 3.2 ~ 3.5, we can evaluate the norm of $Y^j_\kappa, 1 \leq j \leq 8$ (see Kikuchi-Tamura [3]). Thus we obtain the following differential inequality:

$$(3.3) \quad \| (d/d\epsilon)F_\kappa(\epsilon; \lambda) \| \leq C(\lambda^{-1}\epsilon^{\alpha-1} + \lambda^{-\frac{1}{2}}\epsilon^{\alpha-\frac{3}{2}}\|F_\kappa\|^{1/2} + \|F_\kappa\|)$$

It follows from Lemma 3.3(iv) and (3.3) that

$$(3.4) \quad \| (\lambda^{\frac{1}{2}} + |D(\lambda)|)^{-\alpha}G_\kappa(0; \lambda)(\lambda^{\frac{1}{2}} + |D(\lambda)|)^{-\alpha} \| = O(\lambda^{-\frac{1}{2} - \alpha}), \quad (\lambda \to \infty),$$

uniformly in $\kappa > 0$.

Noting Lemma 3.1 we rewrite $D(\lambda)f_\lambda X_1$ as

$$\frac{1}{i}(f_\lambda \nabla_y \cdot yx_1 + \frac{n-1}{2}f_\lambda x_1) - \frac{1}{i}(f_\lambda F_0(\lambda)^*k_0 \partial k_0 F_0(\lambda)x_1 + \frac{1}{2}f_\lambda F_0(\lambda)^*F_0(\lambda)x_1) + [D(\lambda), f_\lambda]^0x_1.$$  

We can show that

$$(3.6) \quad \| [D(\lambda), f_\lambda]^0 \| = O(1), \quad (\lambda \to \infty),$$

(for proof, see that of Lemma 5.6 of Kikuchi-Tamura [3]).

By straightforward calculation we can show next lemma.

**Lemma 3.6.** As $\lambda \to \infty$, one has:

$$\| f_\lambda F_0(\lambda)^*k_0 \partial k_0 F_0(\lambda)x_1 \| = O(\lambda^{\frac{1}{2}})$$

It follows from (3.5), (3.6) and Lemma 3.8 that

$$\| D(\lambda)f_\lambda X_1 \| = O(\lambda^{\frac{1}{2}}) \quad (\lambda \to \infty).$$

Thus we obtain by interpolation

$$(3.7) \quad \| (\lambda^{\frac{1}{2}} + |D(\lambda)|)^{\alpha}f_\lambda X_\alpha \| = O(\lambda^{\frac{\alpha}{2}}) \quad (\lambda \to \infty).$$

Note that

$$\| g_\lambda G_\kappa(0; \lambda) \| = O(\lambda^{-1}) \quad (\lambda \to \infty).$$

(3.4) and (3.7) imply that

$$\| X_\alpha G_\kappa(0; \lambda)X_\alpha \| = O(\lambda^{-\frac{3}{2}}) \quad (\lambda \to \infty).$$

Now the proof of Proposition 1.2 is complete.
Appendix.

In this Appendix we state the generalizied Fourier transform of $L_{0}(\lambda)$ established by Weder (cf. Weder [6]).

For $\lambda \gg 1$ large enough, we consider the following operator:

$$\begin{cases}
    h(\lambda) = -\frac{d^2}{dz^2} - \lambda(a_0^{-2}(z) - c_+^{-2}), \\
    D(h(\lambda)) = H^2(\mathbb{R}_z).
\end{cases}$$

This is the self-adjoint operator in $L^2(\mathbb{R}_z)$.

$h(\lambda)$ has finite number $Q(\lambda) \in \mathbb{N}$, of eigenvalues, $-\omega_j^2(\lambda), 0 < \omega_j^2(\lambda) < q_h(\lambda) = \lambda(c_h^{-2} - c_+^{-2}), 1 \leq j \leq Q(\lambda)$, of multiplicity one. There exist $F_0(\lambda), F_1(\lambda)$ and $G_j(\lambda)(j = 1, 2, 3 \cdots Q(\lambda))$ which are partially isometric operators from $L^2(\mathbb{R}_k^n)$ onto $L^2(\mathbb{R}_k^n)$, respectively, where $\Omega_0 = \{k \in \mathbb{R}^n; 0 < k_0 < \sqrt{q_-}(\lambda) = \sqrt{\lambda(c_+^{-2} - c_-^{-2})}\}$. Defining the operator $F(\lambda)$ as

$$F(\lambda)u = (F_0(\lambda)u, F_1(\lambda)u, G_1(\lambda)u, G_2(\lambda)u, G_3(\lambda)u \cdots G_Q(\lambda)u)$$

for $u \in L^2(\mathbb{R}_n)$, we have

**Lemma A.** $F(\lambda)$ is unitary operator from $L^2(\mathbb{R}_k^n)$ onto

$$\mathcal{H} = L^2(\mathbb{R}_k^n) \bigoplus_{j=1}^{Q(\lambda)} L^2(\Omega_0) \bigoplus L^2(\mathbb{R}_k^{n-1})$$

and for every $u \in D(L_{0}(\lambda)) = H^2(\mathbb{R}_n)$

$$F(\lambda)L_{0}(\lambda)u = ((|k|^2 + q_-(\lambda))F_0(\lambda)u, (||k||^2 - k_0^2 + q_-(\lambda))F_1(\lambda)u, (||k||^2 - \omega_1^2(\lambda))G_1(\lambda)u, (||k||^2 - \omega_2^2(\lambda))G_2(\lambda)u, \cdots, (||k||^2 - \omega_Q^2(\lambda))G_Q(\lambda)(\lambda)u).$$

For the proof, see Weder [6].

**References**

