Palais-Smale Condition for Some Semilinear Parabolic Equations

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1 Introduction

In this paper we are concerned with the following mixed problem to semilinear parabolic equation:

\[ u_t(t, x) - \Delta u(t, x) = |u(t, x)|^{p-1}u(t, x), \quad (t, x) \in (0, T) \times \Omega, \quad (1) \]

\[ u(0, x) = u_0(x), \quad x \in \Omega, \quad (2) \]

\[ u|_{\partial \Omega} = 0, \quad t \in (0, T). \quad (3) \]

Here, \( 1 < p \leq \frac{N+2}{N-2}, \ \Omega \subset \mathbb{R}^N (N \geq 3) \) is a bounded domain with smooth boundary \( \partial \Omega \). In the case when \( 1 < p < \frac{N+2}{N-2} \), of course, we can treat the low dimensional case \( N = 1, 2 \), but for simplicity we restrict our attention to the above mentioned case. For large initial data \( u_0 \) in some sense, it is well-known that the solution \( u(t, x) \) to the problem (1)-(3) blows up in a finite time (see Ikehata-Suzuki[7], Ishii[9], Levine[10], Otani[11], Tsutsumi[16], and Payne-Sattinger[12]), meanwhile for small initial data, exponentially decaying solutions are obtained (see [7] and the references therein). In this paper, we have much interest in solutions to (1)-(3) which neither blowup nor decay. In that occasion, we proceed our argument based on the following local well-posedness theorem due to [7] (see also, Hoehino-Yamada[5]). In the following, \( \| \cdot \|_q (1 \leq q \leq \infty) \) means the usual (real) \( L^q(\Omega) \)-norm.

**Proposition 1.1** For each \( u_0 \in H_0^1(\Omega) \), there exists a number \( T_m > 0 \) such that the problem (1.1)-(1.3) has a unique solution \( u \in C([0, T_m]; H_0^1(\Omega)) \) which becomes classical on \((0, T_m)\). Furthermore, if \( T_m < +\infty \), then

\[ \lim_{t \uparrow T_m} \| u(t, \cdot) \|_{\infty} = +\infty, \]

and in particular, in the case when \( 1 < p < \frac{N+2}{N-2} \) one also has

\[ \lim_{t \uparrow T_m} \| \nabla u(t, \cdot) \|_2 = +\infty. \]

Set

\[ X = H_0^1(\Omega), \]

\[ J(u) = \frac{1}{2} \| \nabla u \|_2^2 - \frac{1}{p+1} \| u \|_{p+1}^{p+1}, \]
\[ I(u) = \|\nabla u\|_{2}^{2} - \|u\|_{p+1}^{p+1}, \]
\[ N = \{ v \in X \setminus \{0\} | I(v) = 0 \} , \]
\[ d_{p} = \inf_{v \in N} J(v) = \inf \{ \sup_{\lambda \geq 0} J(\lambda v) | v \in X \setminus \{0\} \}. \]

It is easy to show that the potential depth \( d_{p} \) (see Sattinger[13]) satisfies \( d_{p} > 0 \) because of the Sobolev continuous embedding \( X \hookrightarrow L^{p+1}(\Omega) \) \((1 < p \leq \frac{N+2}{N-2})\). The stable and unstable sets are defined as usual:

\[ W = \{ u \in X | J(u) < d_{p}, I(u) > 0 \} \cup \{0\}, \]
\[ V = \{ u \in X | J(u) < d_{p}, I(u) < 0 \}. \]

Furthermore, for later use we define the following notations.

\[ E = \{ u \in X | -\Delta u = |u|^{p-1}u \text{ in } \Omega, u|_{\partial\Omega} = 0 \}, \]
\[ E^{*} = \{ u \in D^{1,2}(R^{N}) | -\Delta u = |u|^{p-1}u \text{ in } R^{N} \}, \]
\[ E_{+}^{*} = \{ u \in E^{*} | u \geq 0 \text{ in } R^{N} \}, \]
\[ J_{*}(u) = \frac{1}{2} \int_{R^{N}} |\nabla u(x)|^{2} dx - \frac{1}{p+1} \int_{R^{N}} |u(x)|^{p+1} dx. \]

Here \( D^{1,2}(R^{N}) \) denotes the closure of \( C_{0}^{\infty}(R^{N}) \) with respect to the norm \( \|\nabla u\|_{L^{2}(R^{N})} \). In particular, in the case when \( p = \frac{N+2}{N-2} \), because of the Sobolev embedding \( S\|u\|_{L^{p+1}(R^{N})} \leq \|\nabla u\|_{L^{2}(R^{N})} \) for \( u \in D^{1,2}(R^{N}) \), one also has

\[ d^{*} = \inf_{\lambda \geq 0} \{ \sup_{\lambda \geq 0} J_{*}(\lambda v) | v \in D^{1,2}(R^{N}) \setminus \{0\} \} = \frac{1}{N} S^{N} > 0. \]

Note that \( d^{*} = d_{p} \) with \( p = \frac{N+2}{N-2} \).

**Remark 1.1** In the case when \( p = \frac{N+2}{N-2} \), it is well-known (Struwe[14]) that the family \( \{u_{\epsilon}^{*}(x)\} \) such as

\[ u_{\epsilon}^{*}(x) = \frac{[N(N-2)\epsilon^{2}]^{\frac{N-2}{4}}}{[\epsilon^{2} + |x|^{2}]^{\frac{N-2}{2}}}, \epsilon > 0 \]

satisfies

\[ -\Delta u = |u|^{p-1}u \text{ in } R^{N}, \]

so that \( E_{+}^{*} \setminus \{0\} \neq \emptyset \).

By the way, quite recently, in [7] the following result has been shown with regard to the singularity of a global solution to the problem (1)-(3) under the assumptions below: let \( u(t, x) \) be a solution to (1.1)-(1.3) as in Proposition 1.1. Furthermore, one assumes that

(A.1) \( u_{0} \geq 0 \).

(A.2) \( p = \frac{N+2}{N-2} \).

(A.3) \( \Omega = \{ x \in R^{N} | |x| < 1 \} \).
\[(A.4) \ u(t, x) = u(t, |x|), \ u_r(t, r) < 0 \text{ on } 0 < r \leq 1 \text{ with } r = |x|.\]

Finally, assume \( T_m = +\infty \). For \( 1 < p \leq \frac{N+2}{N-2} \) set

\[C_0 = \frac{2(p+1)}{p-1} \lim_{t \to +\infty} J(u(t, \cdot)).\] \[(4)\]

Note that \( C_0 \geq 0 \) if \( T_m = +\infty \) (see [10]). Then, their results read as follows.

**Theorem 1.1** ([7]) Assume \((A.1)-(A.4)\). Let \( u(t, x) \) be a solution to (1)-(3) on \([0, T_m)\) as in Proposition 1.1. Suppose \( T_m = +\infty \) and \( C_0 > 0 \). Then, there exists a sequence \( \{t_n\} \) with \( t_n + \infty \) as \( n + \infty \) such that

1. \( |\nabla u(t_n, x)|^2 \to C_0 \delta_0 \text{ (weakly-* in } C_0(\Omega)^*),\]
2. \( u(t_n, x)^{p+1} \to C_0 \delta_0 \text{ (weakly-* in } C_0(\Omega)^*),\]

as \( n + \infty \). Here, \( \delta_0 \) means the usual Dirac measure having a unit mass at the origin.

Since \( C_0 > 0 \) if and only if \( u(t, \cdot) \notin (W \cup V) \) for all \( t \geq 0 \), their theorem states that a global orbit \( u(t, \cdot) \) which neither decay nor blowup (if this kind of solution can be constructed!) have a strong singularity at the origin. In connection with this result, we have just noticed that for such a sequence \( \{t_n\} \) constructed in Theorem 1.1 above, \( \{u(t_n, \cdot)\} \) becomes a Palais-Smale sequence so that the global compactness result due to Struwe[15] can be applied to this functional sequence. Our first result reads as follows:

**Theorem 1.2** Let \( \{u(t_n, \cdot)\} \) be a sequence as in Theorem 1.1. Under the same assumptions as in Theorem 1.1, there exist an integer \( k \in N \), a sequence of radii \( \{R_n^i\} \) with \( \lim_{n \to +\infty} R_n^i = +\infty \), a sequence \( \{x_n^i\} \in \Omega \), and \( u^i \in E_+ \setminus \{0\} \) \((1 \leq i \leq k)\) such that (taking a subsequence)

\[
\lim_{n \to +\infty} \|\nabla (u(t_n, \cdot)) - \sum_{i=1}^{k} u_n^i\|_{L^2(\Omega)} = 0,
\]

\[
\lim_{t \to +\infty} J(u(t, \cdot)) = \lim_{n \to +\infty} J(u(t_n, \cdot)) = kd^*,
\]

\[
\lim_{n \to +\infty} \|\nabla (u(t_n, \cdot))\|_{L^2(\Omega)}^2 = \sum_{i=1}^{k} \|\nabla u^i\|_{L^2(\Omega)}^2 = kS^N,
\]

where

\[
u_n^i(x) = (R_n^i)^{\frac{N-2}{2}} u^i(R_n^i(x - x_n^i)) \text{ (1 \leq i \leq k), } n = 1, 2, \cdots.
\]

**Remark 1.2** By considering scaling and translation, one finds that the compactness of \( \{u(t_n, \cdot)\} \) destroyed in Theorem 1.1 is restored once more. On the other hand, for the proof of this Theorem, we have to notice the following fact (see [14]) that each \( u^i \) is of the form \( u^i(x) = u^i_\varepsilon(x) \) (see Remark 1.1) with some \( \varepsilon \) and satisfies \( J_*(u^i) = d^* \) (least energy level).

**Remark 1.3** Under the assumptions \( \Omega = \text{star-shaped} \) and \( u_0(x) \geq 0 \), one can get the quite same results as in the radial case above. In the case when \( u_0 \) changes sign, however, even if \( \Omega \) is star-shaped, one needs a few modifications of the results above (see [14]).
The next result is concerned with the case when \(1 < p < \frac{N+2}{N-2}\). It seems not to be known that any global solutions to (1)-(3) naturally contain a subsequence which is relatively compact in \(X\) in the subcritical case. Our second result reads as follows:

**Theorem 1.3** Let \(1 < p < \frac{N+2}{N-2}\) and \(u(t, x)\) be a solution on \([0, T_m)\) as in Proposition 1.1. If \(T_m = +\infty\), then there exists a sequence \(\{t_n\}\) with \(t_n \to +\infty\) as \(n \to +\infty\) such that \(\{u(t_n, \cdot)\}\) becomes relatively compact in \(X\) so that there exists an element \(u_\infty \in E\) such that \(u(t_n, \cdot) \to u_\infty\) in \(X\) as \(n \to +\infty\) along a subsequence.

**Remark 1.4** In Theorem 1.3, if, in particular, \(C_0 > 0\), then one has \(u_\infty \in E \setminus \{0\}\). Furthermore, the construction of such a sequence \(\{t_n\}\) is in the same way as in Theorem 1.2.

## 2 Palais-Smale sequence

In this section, reviewing some results concerning Theorem 1.1 due to [7] we shall construct some Palais-Smale sequences of a global solution to the problem (1)-(3).

First, suppose \(1 < p \leq \frac{N+2}{N-2}\) and \(T_m = +\infty\) in Proposition 1.1. Since its solution satisfies the energy identity:

\[
J(u(t, \cdot)) + \int_0^t ||u_t(s, \cdot)||_2^2 ds = J(u_0) \tag{5}
\]

for all \(t \geq 0\), this implies that the function \(t \mapsto J(u(t, \cdot))\) is monotone decreasing so that \(C_0 \geq 0\) (see (4)) is meaningful. Letting \(t \to +\infty\) in (5), the improper integral \(\int_0^\infty ||u_t(s, \cdot)||_2^2 ds\) is finite determined. Therefore, there exists a sequence \(\{t_n\}\) with \(t_n \to +\infty\) as \(n \to +\infty\) such that

\[
\lim_{n \to +\infty} ||u_t(t_n, \cdot)||_2^2 = 0.
\]

Note that this sequence \(\{t_n\}\) coincides with the one in Theorem 1.1.

Next, multiplying the both sides of (1) by \(u(t, x)\) and integrating it over \(\Omega\), we have

\[
(u_t(t, \cdot), u(t, \cdot)) = -I(u(t, \cdot)), \tag{6}
\]

where \((f, g) = \int_\Omega f(x)g(x)dx\). Because of [2], it holds true that \(\|u(t, \cdot)\|_2 \leq C\) for all \(t \geq 0\) with some constant \(C > 0\). Therefore, one has

\[
|I(u(t_n, \cdot))| \leq C||u_t(t_n, \cdot)||_2
\]

for all \(n \in N\). Letting \(n \to +\infty\), it follows that

\[
\lim_{n \to +\infty} I(u(t_n, \cdot)) = 0. \tag{7}
\]

On the other hand, the identity holds good:

\[
J(u) = \frac{p-1}{2(p+1)}\|\nabla u\|_2^2 + \frac{1}{p+1}I(u). \tag{8}
\]

So, from (8) with \(u = u(t_n, \cdot)\) and (6)-(7) we find that
Lemma 2.1 Let \( u(t,\cdot) \) be as in Proposition 1.1. If \( T_{m} = +\infty \), then there exists a sequence \( \{t_{n}\} \) with \( t_{n} \to +\infty \) as \( n \to +\infty \) such that

\[
\lim_{n \to +\infty} \|u(t_{n}, \cdot)\|_{2} = 0,
\]

\[
\lim_{n \to +\infty} \|\nabla u(t_{n}, \cdot)\|_{2}^{2} = C_{0},
\]

\[
\lim_{n \to +\infty} \|u(t_{n}, \cdot)\|_{p+1}^{p+1} = C_{0}.
\]

From this lemma, one obtains the next ones:

Lemma 2.2 Let \( u(t, x) \) be a local solution constructed in Proposition 1.1. If \( T_{m} = +\infty \), then there exists a Palais-Smale sequence to the problem (1)-(3).

Proof. Let \( \{t_{n}\} \) be as in Lemma 2.1. Then, it follows that

\[
J(u_{0}) \geq J(u(t_{n}, \cdot)) \to \frac{p-1}{2(p+1)}C_{0} \geq 0 \quad \text{as} \quad n \to +\infty. \tag{9}
\]

Furthermore, for such sequence, since \( J \in C^{1}(X, \mathbb{R}) \), by equation (1) we have

\[
J'(u(t_{n}, \cdot))[v] = -(u_{t}(t_{n}, \cdot), v)
\]

for each \( v \in X \), where \( J'(u) \in X^{*} \) means the usual Fréchet-derivative of \( J \) at \( u \in X \). By this equality and the Schwarz inequality together with the Poincaré inequality one gets:

\[
|J'(u(t_{n}, \cdot))[v]| \leq C_{1}\|u_{t}(t_{n}, \cdot)\|_{2}\|\nabla v\|_{2}
\]

which implies

\[
\|J'(u(t_{n}, \cdot))\|_{H^{-1}(\Omega)} \to 0(n \to +\infty), \tag{10}
\]

where \( C_{1} > 0 \) is a Poincaré constant. We find that \( \{u(t_{n}, \cdot)\} \) becomes a Palais-Smale sequence because of (9) and (10).

In particular, in the case when \( p \in (1, \frac{N+2}{N-2}) \) one gets the following compactness result. For the detailed proof, see the forthcoming paper [8].

Lemma 2.3 Suppose \( p \in (1, \frac{N+2}{N-2}) \). Let \( u(t, x) \) be a global (i.e., \( T_{m} = +\infty \)) solution to (1)-(3) as in Proposition 1.1. Then, the sequence \( \{u(t_{n}, \cdot)\} \) constructed in Lemma 2.1 becomes relatively compact in \( X \).

Now, we are in a position to prove Theorems 1.2 and 1.3.

Proof of Theorem 1.2. This result is a direct consequence of [14] (Theorem 3.1, p.184) and Lemma 2.2 and so, we shall omit the details. But, since \( \Omega = \text{ball} \), note that \( E = \{0\} \) holds true in the present case.

Proof of Theorem 1.3. The first half is a direct consequence of Lemma 2.3. In order to prove \( u_{\infty} \in E \), note that the following estimates are proven:

\[
\|f(u) - f(v)\|_{p+1} \leq p\|u\|_{p+1} + \|v\|_{p+1})^{p-1}\|u - v\|_{p+1}
\]
for all $u, v \in L^{p+1}(\Omega)$, and
\[
|f(u(t_{n}, \cdot)) - f(u_{\infty}), \phi)| \leq \|f(u(t_{n}, \cdot)) - f(u_{\infty})\|_{1, p} \|\phi\|_{p+1}
\]
for each $\phi \in C_{0}^{\infty}(\Omega)$, where $\{u(t_{n}, \cdot)\}$ is a sequence constructed in the first half. By combining these estimates with Lemma 2.1 and the Sobolev embedding $X \hookrightarrow L^{p+1}(\Omega)$, one obtains the desired statement.

From Lemma 2.1 one has a result reviewed from the view point of the Palais-Smale condition.

**Corollary 2.1** Let $1 < p \leq \frac{N+2}{N-2}$ and $u(t, x)$ be a global solution constructed in Proposition 1.1, i.e., $T_{m} = +\infty$. If $C_{0} = 0$, then the sequence $\{u(t_{n}, \cdot)\}$ stated in Lemma 2.1 becomes relatively compact, and in fact, $u(t, \cdot) \to 0$ in $X$ as $t \to +\infty$.

From Theorem 1.1 and Corollary 2.1 with $p = \frac{N+2}{N-2}$, one can say that it depends on the least energy level $\frac{p-1}{2(p+1)}C_{0}$ whether the Palais-Smale condition holds good or not to the sequence $\{u(t_{n}, \cdot)\}$ as in Lemma 2.1.

Finally in this section, we shall apply Theorem 1.3 and Lemma 2.2 for the finite time blowup problem concerning (1)-(3). First, as a consequence of [14] one obtains the following lemma.

**Lemma 2.4** Let $\Omega$ be a bounded smooth domain and $p = \frac{N+2}{N-2}$. Then, for all $v \in E$, one has $J(v) \in \{0\} \cup (d^{*}, +\infty)$, and also, for each $w \in E^{*} \setminus \{0\}$, one has $J_{*}(w) \in \{d^{*}\} \cup (2d^{*}, +\infty)$.


**Proposition 2.1** Let $1 < p \leq \frac{N+2}{N-2}$ and $u(t, x)$ be a local solution of (1)-(3) on $[0, T_{m})$ constructed in Proposition 1.1. If $u(t_{0}, \cdot) \in V$ for some $t_{0} \in [0, T_{m})$, then $T_{m} < +\infty$.

**Proof.** First, we shall deal with the case when $1 < p < \frac{N+2}{N-2}$. Suppose $T_{m} = +\infty$. Then, it follows from Theorem 1.3 that there exist a Palais-Smale sequence $\{u(t_{n}, \cdot)\}$ to the problem (1)-(3) and $u_{\infty} \in E$ such that $u(t_{n}, \cdot) \to u_{\infty}$ in $X$ along a subsequence. On the other hand, it is well-known (see [6]) that $u(t, \cdot) \in V$ for all $t \in [t_{0}, \infty)$. Since $W$ is a neighborhood of 0 in $X$, if $u_{\infty} = 0$, then $u(t_{m}, \cdot) \in W$ holds with some $t_{m}$ and this contradicts the fact that $W \cap V = \emptyset$. Thus, $u_{\infty} \in E \setminus \{0\}$. Because of the monotone decreasingness of a function $t \mapsto J(u(t, \cdot))$, one obtains $J(u(t_{m}, \cdot)) \geq J(u_{\infty}) \geq d_{p}$ which contradicts $u(t_{m}, \cdot) \in V$ with large $t_{n}$.

Next, we are concerned with the critical case $p = \frac{N+2}{N-2}$. Suppose $T_{m} = +\infty$. Obviously, $C_{0} > 0$ holds true. Then, from Lemma 2.2 and Theorem 3.1 of [14], p.184 that there exist a Palais-Smale sequence $\{u(t_{n}, \cdot)\}$, $k \in N$, $u^{0} \in E$, and $u^{i} \in E^{*} \setminus \{0\}$ ($1 \leq i \leq k$) such that
\[
\lim_{n \to +\infty} J(u(t_{n}, \cdot)) = \lim_{t \to +\infty} J(u(t, \cdot)) = J(u^{0}) + \sum_{i=1}^{k} J_{*}(u^{i}).
\]
By Lemma 2.4 and the monotone decreasingness of a function $t \mapsto J(u(t, \cdot))$, one finds that
\[
J(u(t, \cdot)) \geq d^{*}
\]
for all $t \geq 0$. This contradicts also $u(t, \cdot) \in V$ for all $t \geq t_{0}$.
References


