Exponential decay of a difference between a global solution to a reaction-diffusion system and its spatial average (Analytical Studies for Singularities to the Nonlinear Evolution Equation Appearing in Mathematical Physics)

Author(s)
Hoshino, Hiroki

Citation
数理解析研究所講究録 (2000), 1123: 70-75

Issue Date
2000-01

URL
http://hdl.handle.net/2433/63553

Type
Departmental Bulletin Paper

Textversion
publisher

Kyoto University
Exponential decay of a difference between a global solution to
a reaction-diffusion system and its spatial average

Hiroki HOSHINO *
Fujita Health University College, Toyoake, Aichi 470-1192, Japan

(星野 弘喜・藤田保健衛生大学短期大学)

§1. Introduction.
This report is based on Hoshino [9].
We are concerned with asymptotic behavior of a unique nonnegative global solution \((u, v)(t, x)\) to the following system of reaction-diffusion equations with homogeneous Neumann boundary conditions:

\[
\begin{align*}
  &u_t = d_1 \Delta u + f(u)v^n, &\text{in} &\quad (0, \infty) \times \Omega, \\
  &v_t = d_2 \Delta v - f(u)v^n, &\text{in} &\quad (0, \infty) \times \Omega, \\
  &\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, &\text{on} &\quad (0, \infty) \times \partial \Omega, \\
  &(u, v)(0, x) = (u_0, v_0)(x), &\text{in} &\quad \Omega.
\end{align*}
\]  

(1.1)

Here \(\Omega\) is a bounded domain in \(\mathbb{R}^N (N \geq 1)\) with smooth boundary \(\partial \Omega\), and \(\partial/\partial \nu\) stands for the outward normal derivative to \(\partial \Omega\). We assume

Assumption 1. (i) \(d_1\) and \(d_2\) are positive constants.
(ii) \(u_0\) and \(v_0\) are bounded, \(u_0 \geq 0, v_0 \geq 0\) and \(\overline{u}_0 > 0, \overline{v}_0 > 0\), where \(\overline{w} = |\Omega|^{-1} \int_{\Omega} w(x)dx\), and \(|\Omega|\) is the volume of \(\Omega\).
(iii) \(f\) is smooth in \(u \geq 0\) and \(f(u) > 0\) if \(u > 0\). Moreover, either

\[
\lim_{u \to \infty} u^{-1} \log(1 + f(u)) = 0
\]

(cf. [6]) or

\[
f(u) \leq e^{\alpha u} \quad \text{with} \quad d_1 \neq d_2 \quad \text{and} \quad \sup_{x \in \Omega} v_0(x) < \frac{8d_1d_2}{\alpha N(d_1 - d_2)^2}
\]

(cf. [2]) holds.

Assumption 2. \(n > 1\).

Assumption 1 with \(n \geq 1\) assures the existence of a unique nonnegative global solution \((u, v)(t, x)\) to (1.1) – (1.3). In fact, we have Alikakos [1], Masuda [11], Haraux and Kirane [5], Haraux and Youkana [6], Hollis, Martin and Pierre [7], Pao [12], Barabanova [2], Hoshino [8], and so on. Especially in [8], under Assumptions 1 and 2, Hoshino has shown a uniform convergence property of \((u, v)(t, x)\) to \((u_\infty, 0)\) with a polynomial rate, that is to say,

\[
\|(u - u_\infty, v)(t)\|_\infty \leq Kt^{-1/(n-1)} \quad \text{as} \quad t \to \infty,
\]

where

\[
u_\infty = \overline{u}_0 + \overline{v}_0
\]

*The research was partially supported by Grant-in-aid for Encouragement of Young Scientists, The Ministry of Education, Science, Sports and Culture, Japan.
and has also proved that
\[ \|(u - \overline{u}, v - \overline{v})(t)\|_{\infty} \leq K t^\mu e^{-d_0 \lambda t} \]  
(1.4)
as \( t \to \infty \). Here, \( \overline{w}(t) = |\Omega|^{-1} \int_{\Omega} w(t, x) d_{X} \), \( \mu = (\sqrt{2} - 1)n/(2(n - 1)) > 0 \), \( \lambda \) is the smallest positive eigenvalue of \(-\Delta\) with homogeneous Neumann boundary condition on \( \partial \Omega \), and
\[ d_0 = \min\{d_1, d_2\}. \]

Here and hereafter, we make use of the notations
\[ \|w\|_p = \|w\|_{L^p(\Omega)}, \quad \|(w_1, w_2)\|_p = (\|w_1\|_p^2 + \|w_2\|_p^2)^{1/2}. \]

For the details of the previous results, see Theorem 1 in Section 2.

In this report, we will obtain a sharper decay rate of the difference between \((u, v)(t, x)\) and \((\overline{u}, \overline{v})(t)\) than (1.4). In fact, we can show
\[ (u, v)(t, x) = (\overline{u}, \overline{v})(t) + O(e^{-d_0 \lambda t}) \]  
(1.5)
uniformly in \( x \in \Omega \) as \( t \to \infty \), and furthermore in the case \( d_1 > d_2 \) or \( d_1 < d_2 \),
\[ u(t, x) = \overline{u}(t) + O(t^{-\min\{n, 2n-2\}/(n-1)} e^{-d_0 \lambda t}), \]  
(1.6)
or
\[ v(t, x) = \overline{v}(t) + O(t^{-\min\{n, 2n-2\}/(n-1)} e^{-d_0 \lambda t}) \]  
(1.7)
uniformly in \( x \in \Omega \) as \( t \to \infty \), respectively. For the details of our results, see Theorems 2 – 4 in Section 2 below.

Our results are related to those obtained by Conway, Hoff and Smoller [3] or Hale [4]. However, if we restrict ourselves to the case where we have a balance law in a reaction-diffusion system under homogeneous Neumann boundary conditions, then in comparison with previous results we confirm that we can improve the description of the approximation of \((u, v)(t, x)\) by \((\overline{u}, \overline{v})(t)\) in the sense that we can sharpen the estimate of \( \|(u - \overline{u}, v - \overline{v})(t)\|_{\infty} \) such as (1.5), (1.6) and (1.7). Actually, we have
\[ \int_{\Omega} u(t, x) dx + \int_{\Omega} v(t, x) dx = \int_{\Omega} u_0(x) dx + \int_{\Omega} v_0(x) dx, \quad t \geq 0 \]
in our system (1.1) – (1.3).

Our idea for the analysis to get the results is that we make use of \((\phi, \psi)(t, x)\) which is defined by
\[ \begin{cases} 
  u(t, x) - u_\infty = (U(t) - u_\infty)(1 + \phi(t, x)) = -V(t)(1 + \phi(t, x)), \\
  v(t, x) = V(t)(1 + \psi(t, x)),
\end{cases} \]  
(1.8)
where \((U, V)(t)\) is a unique global solution to
\[ \begin{cases} 
  U' = f(U)V^n, \\
  V' = -f(U)V^n,
\end{cases} \quad t > 0, \]  
(1.9)
with \( n > 1 \), where \( ' = d/dt \). Obviously, \( V(t) \) verifies
\[ V' = -f(u_\infty - V)V^n, \quad t > 0 \]
and we see that there is a positive constant \( C_0 \) such that
\[ C_0^{-1}(1 + t)^{-1/(n-1)} \leq u_\infty - U(t) = V(t) \leq C_0(1 + t)^{-1/(n-1)} \]  
(1.11)
for \( t \geq 0 \).
§2. Results.

First, let us recall some preliminary results on our system (1.1) – (1.3).

Theorem 1. (i) Under Assumptions 1 and 2, (1.1) – (1.3) has a unique global solution \((u, v)(t, x)\). It holds true that

\[
0 \leq v(t, x) \leq \|v_0\|_\infty, \quad t > 0, \quad x \in \overline{\Omega},
\]

and there exists a constant \(M > 0\) such that

\[
0 \leq u(t, x) \leq M, \quad t > 0, \quad x \in \overline{\Omega}.
\]

(ii) There are positive constants \(T\) and \(K\) such that

\[
\begin{align*}
\|(u - u_\infty, v)(t)\|_\infty & \leq K(1 + t - T)^{-1/(n-1)}, \\
\|(u - \overline{u}, v - \overline{v})(t)\|_\infty & \leq K(1 + t - T)^Ke^{-d_0\lambda t},
\end{align*}
\]

where \(u_\infty = \overline{u}_0 + \overline{v}_0\), \(d_0 = \min\{d_1, d_2\}\), and \(\lambda\) is the smallest positive eigenvalue of \(-\Delta\) with the homogeneous Neumann boundary condition on \(\partial\Omega\).

(iii) Let \((U, V)(t)\) be the solution to (1.9), (1.10). Then, \((U, V)(t)\) plays a role of an asymptotic solution to (1.1) – (1.3) and

\[
(u, v)(t, x) = (U, V)(t) + O(t^{-1/(n-1)}
\]

uniformly in \(x \in \Omega\) as \(t \to \infty\).

(iv) Moreover, \((\overline{u}, \overline{v})(t)\) approximates \((u, v)(t, x)\) as follows:

\[
(u, v)(t, x) = (\overline{u}, \overline{v})(t) + O(t^\mu e^{-d_0\lambda t})
\]

uniformly in \(x \in \Omega\) as \(t \to \infty\), where \(\mu = (\sqrt{2} - 1)n/(2(n-1)) > 0\).

Next, we state our main results, that is to say, we can sharpen the approximation of the global solution \((u, v)(t, x)\) to (1.1) – (1.3) by its spatial average \((\overline{u}, \overline{v})(t)\) than (iv) of Theorem 1.

Theorem 2. The following asymptotic approximation of \((u, v)(t, x)\) by its spatial average holds true:

\[
(u, v)(t, x) = (\overline{u}, \overline{v})(t) + O(e^{-d_0\lambda t})
\]

uniformly in \(x \in \Omega\) as \(t \to \infty\).

In the case \(d_1 \neq d_2\), we can obtain stronger asymptotic relations.

Theorem 3. When \(d_1 > d_2\),

\[
u(t, x) = \overline{u}(t) + O(t^{-1}e^{-d_0\lambda t})
\]

uniformly in \(x \in \Omega\) as \(t \to \infty\).

Theorem 4. When \(d_1 < d_2\),

\[
v(t, x) = \overline{v}(t) + O(t^{-\min\{2n-2, 2\}/(n-1)}e^{-d_0\lambda t})
\]

uniformly in \(x \in \Omega\) as \(t \to \infty\).

§3. Deformation of the problem.

Substituting (1.8) into (1.1) – (1.3), easy calculations give

\[
\begin{align*}
\phi_t &= d_1 \Delta \phi - V^{n-1} \{ -f(u_\infty - V)\phi - Vf_u(u_\infty - V)\phi + nf(u_\infty - V)\phi + h \}, \\
\psi_t &= d_2 \Delta \psi - V^{n-1} \{ -Vf_u(u_\infty - V)\phi + (n-1)f(u_\infty - V)\psi + h \},
\end{align*}
\]

in \((0, \infty) \times \Omega\), (3.1)
\[ \frac{\partial \phi}{\partial \nu} = \frac{\partial \psi}{\partial \nu} = 0, \quad \text{on} \quad (0, \infty) \times \partial \Omega, \quad (3.2) \]

\[ \left\{ \begin{array}{l}
\phi(0, x) = \phi_0(x) = \frac{u_0(x) - \overline{u}_0}{\overline{v}_0}, \\
\psi(0, x) = \psi_0(x) = \frac{v_0(x) - \overline{v}_0}{\overline{v}_0},
\end{array} \right. \quad \text{in} \quad \Omega, \quad (3.3) \]

where \( f_u = df/du \) and \( h = h(\phi, \psi) \) satisfies

\[ -f(u_\infty - V)(1 + \phi) + f(u_\infty - V(1 + \phi))(1 + \psi)^n = -f(u_\infty - V)\phi - Vf_u(u_\infty - V)\phi + nf(u_\infty - V)\psi + h. \]

Note that we also have

\[ -f(u_\infty - V)(1 + \psi) + f(u_\infty - V(1 + \psi))(1 + \psi)^n = -Vf_u(u_\infty - V)\phi + (n-1)f(u_\infty - V)\psi + h \]

at the same time and that \( h = O(|\phi|^2 + |\psi|^2) \) as \((\phi, \psi) \to (0, 0)\).

We will investigate the decay rate of \((\phi, \psi)(t, x)\) in order that we show Theorems 2-4 (cf. [10]). We use the following projection operators.

**Definition 3.1.** \( P_0 w = \overline{w} = |\Omega|^{-1} \int_\Omega w(x) dx \), \( P_+ w = w - P_0 w. \)

The following lemma is important for us.

**Lemma 3.1.** There exists a nondecreasing function \( L(r) \) on \([0, \infty)\) such that if

\[ K(t) \equiv L\left( \sup_{0 \leq \tau \leq t} \| (\phi, \psi)(\tau) \|_\infty \right), \]

then for every \( p \in [1, \infty] \),

\[ \| h(t) \|_p \leq K(t) \| (\phi, \psi)(t) \|_{2p}, \]

\[ \| (P_+ h)(t) \|_p \leq K(t) \| (\phi, \psi)(t) \|_\infty \| (V^{1/2} P_+ \phi, P_+ \psi)(t) \|_p, \]

\[ \| (P_+ h)(t) \|_p \leq K(t) \| (\phi, \psi)(t) \|_\infty \| (P_+ \phi, P_+ \psi)(t) \|_p. \]

When \( C \) is a constant, we will identify \( CK(t) \) with \( K(t) \) in the following sections. Finally, we give the equations and the boundary and initial conditions which \((\phi^+, \psi^+)(t, x)\) satisfies:

\[ \left\{ \begin{array}{l}
\phi^+_t = d_1 \Delta \phi^+ - V^{n-1} \{-f(u_\infty - V)\phi^+ - Vf_u(u_\infty - V)\phi^+ + nf(u_\infty - V)\psi^+ + h^+ \}, \\
\psi^+_t = d_2 \Delta \psi^+ - V^{n-1} \{-Vf_u(u_\infty - V)\phi^+ + (n-1)f(u_\infty - V)\psi^+ + h^+ \},
\end{array} \right. \quad \text{in} \quad (0, \infty) \times \Omega, \quad (3.4) \]

\[ \frac{\partial \phi^+}{\partial \nu} = \frac{\partial \psi^+}{\partial \nu} = 0, \quad \text{on} \quad (0, \infty) \times \partial \Omega, \quad (3.5) \]

\[ (\phi^+, \psi^+)(0, x) = (\phi_0, \psi_0)(x), \quad \text{in} \quad \Omega. \quad (3.6) \]

Note that \( P_0 \phi_0 = P_0 \psi_0 = 0 \), in other words, \((P_+ \phi_0)(x) = \phi_0(x), (P_+ \psi_0)(x) = \psi_0(x)\). Here and hereafter, we use the notation

\[ w^+ = P_+ w \]

for simplicity.

§4. The case of small initial perturbation.

We will restrict ourselves to the case where \( \| (\phi_0, \psi_0) \|_\infty \) is small and we will obtain the following theorem in terms of \((\phi, \psi)(t, x)\) instead of Theorem 2. We can reduce the case where the size of \( \| (\phi_0, \psi_0) \|_\infty \) is large to the small case by virtue of Theorem 1 (ii) (see [8]).
Theorem 4.1. There exists a constant \( \delta_0 > 0 \) such that if \( \| (\phi_0, \psi_0) \|_{\infty} \leq \delta_0 \), then

\[
\| (\phi^+, \psi^+) (t) \|_{\infty} \leq C \| (\phi_0, \psi_0) \|_{\infty} V(t)^{-1} e^{-\delta_0 \lambda t}
\]

for \( t \geq 0 \), where \( C \) is a positive constant.

We introduce some quantities as follows:

Definition 4.1. For \( 1 \leq p \leq \infty \),

\[
I_p = \| (\phi_0, \psi_0) \|_p,
\]

\[
M_p(t) = \sup_{0 \leq \tau \leq t} V(\tau)^{-(n-1)} \| (\phi, \psi)(\tau) \|_p,
\]

\[
M_\infty^0(t) = \sup_{0 \leq \tau \leq t} V(\tau)^{-(n-1)} \| (P_0 \phi, P_0 \psi)(\tau) \|,
\]

\[
M_{p,V}^+(t) = \sup_{0 \leq \tau \leq t} V(\tau) e^{d_1/2} \| (\phi^+, \psi^+)(\tau) \|_p,
\]

\[
M_p^+(t) = \sup_{0 \leq \tau \leq t} V(\tau) e^{d_1/2} \| (\phi^+, \psi^+)(\tau) \|_p,
\]

where \( V(t) \) is the solution for (1.9) and (1.10) satisfying (1.11), \( d_0 = \min\{d_1, d_2\} \), and \( \lambda \) is the smallest positive eigenvalue of \(-\Delta\) with the homogeneous Neumann boundary conditions on \( \partial \Omega \).

According to the following scheme, we can show Theorem 4.1.

1. \( M_\infty^0(t) \leq K(t) M_\infty(t)^2 \).
2. \( M_{p,V}^+(t) \leq CI_\infty + K(t) M_\infty(t) M_{p}^+(t) \) for \( p \in [1, 2] \).
3. \( M_{\infty,V}^+(t) \leq CI_\infty + K(t) M_\infty(t) M_{\infty}^+(t) \).
4. \( M_p^+(t) \leq CI_\infty + K(t) M_\infty(t) M_{\infty}^+(t) \) for \( p \in [1, 2] \).
5. \( M_{\infty}^+(t) \leq CI_\infty + K(t) M_\infty(t) M_{\infty}^+(t) \).

In the Steps 2 and 4, we investigate \( L^2(\Omega) \)-energy of \( (\phi^+, \psi^+)(t, x) \) with use of (3.4) – (3.6). On the other hand, in Steps 3 and 5 we treat (3.7) and (3.8) by means of \( L^p(\Omega) - L^q(\Omega) \) estimate of an analytic semigroup \( \{ e^{-tA} \}_{t \geq 0} \), where \( A \) means \(-\Delta\) with the homogeneous Neumann boundary condition on \( \partial \Omega \).

The following Theorem 4.2 (resp. 4.3) corresponds to Theorem 3 (resp. 4) in the case \( I_\infty \) is small.

Theorem 4.2. Suppose that \( d_1 > d_2 \). If \( \| (\phi_0, \psi_0) \|_{\infty} \leq \delta_0 \), then

\[
V(t) e^{d_0 \lambda t} \| \phi^+(t) \|_{\infty} \leq C \| (\phi_0, \psi_0) \|_{\infty} V(t)^{n-1}
\]

for \( t \geq 0 \), where \( C \) is a positive constant.

Theorem 4.3. Suppose that \( d_1 < d_2 \). If \( \| (\phi_0, \psi_0) \|_{\infty} \leq \delta_0 \), then

\[
V(t) e^{d_0 \lambda t} \| \psi^+(t) \|_{\infty} \leq C \| (\phi_0, \psi_0) \|_{\infty} V(t)^{\min\{n,2n-2\}}
\]

for \( t \geq 0 \), where \( C \) is a positive constant.

For the details of the proofs of our results in this report, refer to [9].
References


Fujita Health University College, Toyoake, Aichi 470-1192, Japan

470-1192
愛知県豊明市沓掛町田楽ケ窪 1-98
藤田保健衛生大学短期大学

e-mail: hhoshino@fujita-hu.ac.jp