THE COMPLETIONS OF METRIC ANR’S AND UNIFORM ANR’S

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A subset $Y$ of a space $X$ is said to be homotopy dense in $X$ if there exists a homotopy $h: X \times [0,1] \to X$ such that $h_0 = \text{id}$ and $h_t(X) \subset Y$ for $t > 0$. This concept is very important in ANR Theory and Infinite-Dimensional Topology. When $X$ is an ANR, the concept of the homotopy denseness is dual to the one of local homotopy negligibility introduced by Toruńczyk in [To3]; that is, $Y \subset X$ is homotopy dense in $X$ if and only if the complement $X \setminus Y$ is locally homotopy negligible in $X$ (cf. [To3, Theorem 2.4]). The following fact is well-known:

Fact. Every homotopy dense subset of an ANR is also an ANR and a metrizable space is an ANR if it contains an ANR as a homotopy dense subset.

The lack of the homotopy denseness of a metric ANR in its completion often destroys the ANR property of the completion. For instance, the $\sin \frac{1}{x}$-curve in the plane $\mathbb{R}^2$ is an ANR but the completion of this curve (= the closure in $\mathbb{R}^2$) is not an ANR. Moreover, even if the completion is an ANR, it is very different from the original ANR. The circle $S^1$ is the completion of the space $S^1 \setminus \{\text{pt}\}$ and the both spaces are ANR but they are topologically very different from each other. It should be remarked that $S^1 \setminus \{\text{pt}\}$ is not homotopy dense in $S^1$. In this note,\footnote{The results mentioned in this note were obtained in [Sa]. Then, for details, one can refer to the paper [Sa].} we consider the following interesting problem:

Problem. When is a metric ANR homotopy dense in the metric completion?

1. A CHARACTERIZATION OF METRIC ANR’S

The nerve of an open cover $\mathcal{V}$ of a space $X$ is denoted by $N(\mathcal{V})$. A sequence $\mathcal{U} = (\mathcal{U}_n)_{n \in \mathbb{N}}$ of open covers of a metric space $X$ is called a zero-sequence if $\lim_{n \to \infty} \text{mesh} \mathcal{U}_n = 0$. For such a sequence, we define the simplicial complex

$$ TN(\mathcal{U}) = \bigcup_{n \in \mathbb{N}} N(\mathcal{U}_n \cup \mathcal{U}_{n+1}), $$

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where we regard $U_n \cap U_m = \emptyset$ ($n \neq m$) as sets of vertices of $TN(U)$ even if $U_n \cap U_m \neq \emptyset$ as collections of open sets, \(^2\) whence
\[
N(U_n \cup U_{n+1}) \cap N(U_{n+1} \cup U_{n+2}) = N(U_{n+1}).
\]

**Theorem 1.** A metric space $X = (X, d)$ is an ANR if and only if $X$ has a zero-sequence $U = (U_n)_{n \in \mathbb{N}}$ of open covers with a map $f : |TN(U)| \to X$ satisfying the following conditions:

(i) $f(U) \subseteq U$ for each $U \in TN(U)^{(0)} = \bigcup_{n \in \mathbb{N}} U_n$, and

(ii) $\lim_{n \to \infty} \text{mesh}\{f(\sigma) \mid \sigma \in N(U_n \cup U_{n+1})\} = 0$. 

Under the above circumstances, if the image $f(|TN(U)|)$ is always contained in $Y \subset X$, then $Y$ is homotopy dense in $X$.

This characterization of ANR’s is due to Nguyen To Nhu [N] (cf. [NS]). By the alternative proof given in [Sa], the additional assertion was obtained. As a corollary, we have the following:

**Corollary 1.** Let $X$ be an ANR (resp. an AR) contained in a metric space $M$. Then, there exists a $G_\delta$-set $Z \subset M$ such that $Z$ is an ANR (resp. an AR) and $X$ is homotopy dense in $Z$.

We can also apply Theorem 1 to find conditions such that the metric completion of a metric space $X$ is an ANR with $X$ a homotopy dense subset. A subset $D$ of a metric space $X$ is said to be \(\delta\)-dense in $X$ if $\text{dist}(x, D) < \delta$ for every $x \in X$.

**Corollary 2.** Let $X$ be a metric space which has a zero-sequence $U = (U_n)_{n \in \mathbb{N}}$ of open covers with a map $f : |TN(U)| \to X$ satisfying the conditions (i) and (ii) of Theorem 1, where suppose $U_n = \{B_X(x, \gamma_n) \mid x \in D_n\}$ for some $\delta_n$-dense subset $D_n \subset X$ and $0 < \delta_n < \gamma_n$. Then, any metric space $Z$ containing $X$ isometrically as a dense subset is an ANR and $X$ is homotopy dense in $Z$. In particular, the metric completion $\tilde{X}$ of $X$ is an ANR and $X$ is homotopy dense in $\tilde{X}$.

In the above, note that the $\gamma_n$-dense subset $D_n$ of $X$ may not be $\delta_n$-dense in $Z$. For example, $D_n = \{i/n \mid 1 \leq i < n\}$ is 1/n-dense in $(0, 1)$ but it is not 1/n-dense in $[0, 1]$.

Now, we consider the following extension property:

\((e)_k\) There exist constants $\alpha > 0$ and $\beta > 1$ such that every map $f : |K^{(k)}| \to X$ of the $k$-skeleton of an arbitrary simplicial complex $K$ with $\text{mesh}\{f(\sigma^{(k)}) \mid \sigma \in K\} < \alpha$ extends to a map $\tilde{f} : |K| \to X$ such that $\text{diam} \tilde{f}(\sigma) \leq \beta \text{diam} f(\sigma^{(k)})$ for each $\sigma \in K$.

\(^2\)In [NS], we did not regard like this. Considering the set $\bigcup_{n \in \mathbb{N}} U_n \times \{n\}$ as the set of vertices of $NT(U)$, this is reasonable.
The following corollary is motivated by the proof of AR property of the hyperspaces (cf. [vM, §5.3]).

**Corollary 3.** Every $LC^{k-1}$ metric space $X$ with the property $(e)_k$ is an ANR. In particular, a metric space $X$ with $(e)_0$ is an ANR (cf. Theorem 3).

**Remark.** In Theorem 1, $X$ is an AR when $\mathcal{U}_1 = \{X\}$. Every $C^{k-1}$ and $LC^{k-1}$ metric space $X$ is an AR if it has the following:

$(\varepsilon)_k$ there exists a constant $\beta > 1$ such that every map $f : |K^{(k)}| \to X$ of the $k$-skeleton of an arbitrary simplicial complex $K$ extends to a map $\tilde{f} : |K| \to X$ such that $\text{diam} \tilde{f}(\sigma) \leq \beta \text{diam} f(\sigma^{(k)})$ for each $\sigma \in K$.

2. Uniform ANR's

In [Mi$_2$], E. Michael introduced uniform AR's and uniform ANR's, and studied them. Let $X = (X, d_X)$ and $Y = (Y, d_Y)$ be metric spaces and $A \subset X$. A map $f : X \to Y$ is said to be **uniformly continuous** at $A$ if, for any $\epsilon > 0$, there exists $\delta > 0$ such that if $a \in A$, $x \in X$ and $d_X(a, x) < \delta$ then $d_Y(f(a), f(x)) < \epsilon$. A neighborhood $U$ of $A$ in $X$ is called a **uniform neighborhood** if $\bigcup_{a \in A} B_X(a, \delta) \subset U$ for some $\delta > 0$. A metric space $Y$ is called a **uniform ANR** if, for an arbitrary metric space $X$ and a closed set $A \subset X$, every uniformly continuous map $f : A \to Y$ extends to a map $\tilde{f} : U \to Y$ from some uniform neighborhood $U$ of $A$ in $X$ which is uniformly continuous at $A$. When $f$ always extends over $X$ (i.e., $U = X$), $Y$ is a **uniform AR**. By virtue of [Mi$_2$, Theorem 1.2], a metric space $Y$ is a uniform ANR (resp. a uniform AR) if and only if, for an arbitrary metric space $Z$ which contains $Y$ isometrically as a closed subset, there exists a retraction $r : U \to Y$ for some uniform neighborhood $U$ in $Y$ in $Z$ (resp. $r : Z \to Y$) which is uniformly continuous at $Y$.\footnote{Such a retraction is called a **regular retraction** by H. Toruńczyk in [To$_2$].} The concept of uniform ANR's is useful since the metric completion of every uniform ANR is also a uniform ANR.

By using a zero-sequence of open covers in §1, we can prove the following version of Proposition 1.4 in [Mi$_2$]:

**Theorem 2.** For an arbitrary metric space $X$, the following are equivalent:

(a) $X$ is a uniform ANR;

(b) Every metric space $Z$ containing $X$ isometrically as a dense subset is a uniform ANR and $X$ is homotopy dense in $Z$;

(c) $X$ is isometrically embedded in some uniform ANR $Z$ as a homotopy dense subset.
Theorem 2 above means that a metric space $X$ is a uniform ANR if and only if
the metric completion of $X$ is a uniform ANR with $X$ a homotopy dense subset.
However, in order that the metric completion of a metric ANR $X$ is an ANR with
$X$ a homotopy dense subset, it is not necessary that $X$ is a uniform ANR. In case
$X$ is totally bounded, $X$ is a uniform ANR if and only if the metric completion of
$X$ is an ANR with $X$ a homotopy dense subset.

A metric space $Y$ is said to be uniformly $LC^k$ if, for each $\varepsilon > 0$, there exists $\delta > 0$
such that any map $f: S^i \to Y$ with $\text{diam} f(S^i) < \delta$ extends to a map $\tilde{f}: B^{i+1} \to Y$
with $\text{diam} \tilde{f}(B^{i+1}) < \varepsilon$ for every $i \leq k$. In stead of “uniformly $LC^0$”, we also say
“uniformly locally path-connected”. The subspace of $\mathbb{R}^2$ in the example above is
not uniformly locally path-connected.

**Theorem 3.** Every uniformly $LC^{k-1}$ metric space $Y$ with the property $(e)_k$ is a
uniform ANR. In particular, a metric space $X$ with $(e)_0$ is a uniform ANR.

By Theorems 2 and 3, we have the following variation of Corollary 3 (cf. [SU, Lemma 2]):

**Corollary 4.** Let $X$ be a metric space and $Y$ a dense subset of $X$. If $Y$ is uniformly
$LC^{k-1}$ and has the property $(e)_k$, then $X$ and $Y$ are uniformly ANR’s and $Y$ is
homotopy dense in $X$.

**Remark.** In Theorem 3 and Corollary 4, by replacing the property $(e)_k$ with $(\bar{e})_k$
and adding the condition that $Y$ is $C^{k-1}$, “uniform ANR” can be “uniform AR”.
In particular, a metric space $X$ with $(\bar{e})_0$ is a uniform AR.

3. Dense (or uniform) local hyper-connectedness

By using the notion of (local) hyper-connectedness, C.R. Borges [Bo] and R.
Cauty [Ca] characterized AR’s and ANR’s, respectively. Here is introduced a little
weaker notion. By $\Delta^{n-1}$, we denote the standard $(n-1)$-simplex in $\mathbb{R}^n$, that is,

$$\Delta^{n-1} = \{ (t_1, \ldots, t_n) \in \mathbb{R}^n \mid t_i \geq 0, \sum_{i=1}^{n+1} t_i = 1 \}.$$

For an open cover $\mathcal{U}$ of a space $X$ and $Y \subset X$, we denote

$$Y^n(\mathcal{U}) = \{ (y_1, \ldots, y_n) \in Y^n \mid \exists U \in \mathcal{U} \text{ such that } \{y_1, \ldots, y_n\} \subset U \}.$$

It is said that a space $X$ is densely locally hyper-connected if $X$ has an open cover
$\mathcal{W}$, a dense subset $D$ and functions $h_n: D^n(\mathcal{W}) \times \Delta^{n-1} \to X, n \in \mathbb{N}$, which satisfy
the following conditions:

(i) if $t_i = 0$ then

$$h_n(y_1, \ldots, y_n; t_1, \ldots, t_n) = h_{n-1}(y_1, \ldots, y_i-1, y_i+1, \ldots, y_n; t_1, \ldots, t_{i-1}, t_{i+1}, \ldots, t_n);$$
(ii) $\Delta^{n-1} \ni (t_1, \ldots, t_n) \mapsto h_n(y_1, \ldots, y_n; t_1, \ldots, t_n) \in X$ is continuous for each $(y_1, \ldots, y_n) \in D^n(\mathcal{W})$;

(iii) every open cover $\mathcal{U}$ of $X$ has an open refinement $\mathcal{V}$ such that $\mathcal{V} \prec \mathcal{W}$ (hence $D^n(\mathcal{V}) \subset D^n(\mathcal{W})$) and

$$\{h_n((D \cap V)^n \times \Delta^{n-1}) \mid V \in \mathcal{V}\} \prec \mathcal{U} \quad \text{for each } n \in \mathbb{N}.$$  

It should be noticed that each $h_n$ need not be continuous. If $\mathcal{W}$ can be taken as $\mathcal{W} = \{X\}$ (i.e., $D^n(\mathcal{W}) = D^n$), we say that $X$ is densely hyper-connected. In case $D = X$ (resp. $D = X$ and $\mathcal{W} = \{X\}$), $X$ is locally hyper-connected (resp. hyper-connected). This concept is very similar to Michael's convex structure in [Mi$_1$]. In [Bo] and [Ca], AR's and ANR’s are characterized by the hyper-connectedness and the local hyper-connectedness, respectively. A similar characterization was obtained by Himmelberg [Hi] (cf. Curtis [Cu]). These characterizations can be generalized in terms of the dense hyper-connectedness as follows:

**Theorem 4.** A metrizable space $X$ is an ANR if and only if $X$ is densely locally hyper-connected. Moreover, $X$ is an AR if and only if $X$ is densely hyper-connected.

**Remark.** In the definition of densely local hyper-connectedness, if the images of functions $h_n$ are contained in $Y$, then $Y$ is homotopy dense in $X$. In fact, if the images of functions $h_n$ are contained in $Y$, then $f(|TN(\mathcal{U})|) \subset Y$, hence $Y$ is homotopy dense in $X$ by the additional statement of Theorem 1.

For a metric space $X$ and $\eta > 0$, we denote

$$X^n(\eta) = \{(x_1, \ldots, x_n) \in X^n \mid \text{diam}\{x_1, \ldots, x_n\} < \eta\}.$$  

A metric space $X$ is said to be uniformly locally hyper-connected if there are $\eta > 0$ and functions $h_n: X^n(\eta) \times \Delta^{n-1} \to X$, $n \in \mathbb{N}$, which satisfy the same conditions as (i) and (ii) above, and the following (iii') instead of (iii):

(iii') for each $\varepsilon > 0$, there is $0 < \delta < \varepsilon$ such that

$$\text{diam } h_n(\{x\} \times \Delta^{n-1}) < \varepsilon \quad \text{for every } n \in \mathbb{N} \text{ and } x \in X^n(\delta).$$  

When every $h_n$ is defined on the whole space $X^n \times \Delta^{n-1}$, it is said that $X$ is uniformly hyper-connected.

Now, we give a characterization of uniform ANR's and uniform AR’s.

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$^4$The local hyper-connectedness is in the sense of [Ca] but not in the sense of [Bo].
Theorem 5. A metric space $X = (X, d)$ is a uniform ANR if and only if $X$ is uniformly locally hyper-connected. Moreover, $X$ is a uniform AR if and only if $X$ is uniformly hyper-connected.

The following is a combination of Theorems 2 and 5:

Corollary 5. Let $X$ be a uniformly (locally) hyper-connected metric space and $Z$ a metric space which contains $X$ isometrically as a dense subset. Then, $X$ and $Z$ are uniform AR's (uniform ANR's) and $X$ is homotopy dense in $Z$. In particular, the metric completion $\tilde{X}$ of $X$ is a uniform AR (uniform ANR) and $X$ is homotopy dense in $\tilde{X}$.

References


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