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K-APPROXIMATIONS AND INFINITE DIMENSIONAL SPACES

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1. Introduction

Throughout the present note, by the dimension we mean the covering dimension dim. We shall consider characterizations of classes of infinite dimensional spaces in terms of K-approximations and discuss some questions related to the characterizations. In [DMS], Dydak-Mishra-Shukla introduced a concept of a K-approximation of a mapping to a metric simplicial complex and characterized n-dimensional spaces and finitistic spaces in terms of K-approximations. Let X be a space, K a metric simplicial complex and $f : X \to K$ a continuous mapping. A mapping $g : X \to K$ is said to be a *K-approximation of f* if for each simplex $\sigma \in K$ and each $x \in X$, $f(x) \in \sigma$ implies $g(x) \in \sigma$. A K-approximation $g : X \to K$ of f is called an *n-dimensional K-approximation* if $g(X) \subset K^{(n)}$ and a *finite dimensional K-approximation* if $g(X) \subset K^{(m)}$ for some natural number m, where $K^{(m)}$ denotes the m-skelton of K.

The concept of finitistic spaces was introduced by Swan [Sw] for working in fixed point theory and is applied to the theory of transformation groups by using the cohomological structures (cf. [AP]). For a family $\mathcal{U}$ of a space X the order $\operatorname{ord}\mathcal{U}$ of $\mathcal{U}$ is defined as follows: $\operatorname{ord}_x\mathcal{U} = |\{U \in \mathcal{U} : x \in U\}|$ for each $x \in X$ and $\operatorname{ord}\mathcal{U} = \sup\{\operatorname{ord}_x\mathcal{U} : x \in X\}$. We say a family $\mathcal{U}$ has *finite order* if $\operatorname{ord}\mathcal{U} = n$ for some natural number n. A space X is said to be *finitistic* if every open cover of X has an open refinement with finite order. We notice that finitistic spaces are also called *boundedly metacompact spaces* (cf. [FMS]). It is obvious that all compact spaces and all finite dimensional paracompact spaces are finitistic spaces. More precisely, we have a useful characterization of finitistic spaces.

Proposition A [H2], [DMS]. *A paracompact space X is finitistic if and only if there is a compact subspace C of X such that* $\dim F < \infty$ *for every closed subspace F with $F \cap C = \emptyset$.*

The dimension-theoretic properties of finitistic spaces are investigated by several authors (cf. [DP], [DS], [DT], [DMS] [H2] and [H6]). In particular, Dydak-Mishra-Shukla ([DMS]) proved the following.

Theorem A [DMS]. *For a paracompact space X the following are equivalent.*

(a) $\dim X \leq n$.
(b) *For every metric simplicial complex K and every continuous mapping $f : X \to K$ there is an n-dimensional K-approximation g of f.*

(c) *For every metric simplicial complex K and every continuous mapping $f : X \to K$ there is an n-dimensional K-approximation g of f such that $g|f^{-1}(K^{(n)}) = f|f^{-1}(K^{(n)})$.*

Theorem B [DMS]. *For a paracompact space X the following are equivalent.*

(a) *X is a finitistic space.*

(b) *For every metric simplicial complex K and every continuous mapping $f : X \to K$ there is a finite dimensional K-approximation g of f.*

(c) *For every integer $m \geq -1$, every metric simplicial complex K and every continuous mapping $f : X \to K$ there is a finite dimensional K-approximation g of f such that $g|f^{-1}(K^{(m)}) = f|f^{-1}(K^{(m)})$.*

In §2, we extend Theorem A to a class of metrizable spaces that have strong large transfinite dimension. In §3, we shall discuss some questions related to strongly countable-dimensional spaces and finitistic spaces. We denote the set of natural numbers by $\mathbb{N}$. We refer the reader to [E] and [N] for basic results in dimension theory.

2. Characterizations of inifinite-dimensional spaces by means of K-approximations

We begin with the definition of strong small transfinite dimension introduced by Borst [B]. For each ordinal number α, we write $\alpha = \lambda(\alpha) + n(\alpha)$, where $\lambda(\alpha)$ is a limit ordinal and $n(\alpha)$ is a finite ordinal. For a normal space X and a non-negative integer n, we put

$$P_n(X) = \bigcup\{U : U \text{ is an open set of } X \text{ such that } \dim \overline{U} \leq n\}.$$

Let X be a normal space and α be either an ordinal number or the integer -1. The *strong small transfinite dimension* sind of X is defined as follows ([B]):

(i) $\operatorname{sind} X = -1$ if and only if $X = \emptyset$.

(ii) $\operatorname{sind} X \leq \alpha$ if X is expresed in the form $X = \bigcup\{P_\xi : \xi < \alpha\}$, where $P_\xi = P_{n(\xi)}(X \setminus \bigcup\{P_\eta : \eta < \lambda(\xi)\})$.

If $\operatorname{sind} X \leq \alpha$ for some α, we say that X *has strong small transfinite dimension.*

Recall from [H4] that a normal space X *has strong large transfinite dimension* if X has both large transfinite dimension and strong small transfinite dimension. (See [E] and [N] for the definition of large transfinite dimension.) We use the following characterization of spaces that have strong large transfinite dimension. A normal space X is said to be *strongly countable-dimensional* if X is a union of countably many finite dimensional closed subsets.

Lemma 1 [H3, Propositions 2.2 and 2.3]. *Let X be a metrizable space. Then X has strong large transfinite dimension if and only if X is finitistic and strongly countable-dimensional.*

The following is a main result. For a space X we denote

$$\mathcal{D}(X) = \{D : D \text{ is a closed discrete subset of } X\}.$$

Theorem 1. *For a metrizable space X the following are equivalent.*

(a) *X has strong large transfinite dimension.*

(b) *There is a function $\varphi : \mathcal{D}(X) \to \omega$ such that for every metric simplicial complex K and every continuous mapping $f : X \to K$ there is a K-approximation g of f such that $g(D) \subset K^{(\varphi(D))}$ for each $D \in \mathcal{D}(X)$.*

(c) *For every integer $m \geq -1$ there is a function $\psi : \mathcal{D}(X) \to \omega$ such that for every metric simplicial complex K and every continuous mapping $f : X \to K$ there is a finite dimensional K-approximation g of f such that $g(D) \subset K^{(\psi(D))}$ for each $D \in \mathcal{D}(X)$ and $g|f^{-1}(K^{(m)}) = f|f^{-1}(K^{(m)})$.*

Corollary 1. *For a paracompact space X the following are equivalent.*

(a) *X is a strongly countable-dimensional space.*

(b) *There is a function $\varphi : X \to \omega$ such that for every metric simplicial complex K and every continuous mapping $f : X \to K$ there is a K-approximation g of f such that $g(x) \in K^{(\varphi(x))}$ for each $x \in X$.*

(c) *For every integer $m \geq -1$ there is a function $\psi : X \to \omega$ such that for every metric simplicial complex K and every continuous mapping $f : X \to K$ there is a K-approximation g of f such that $g(x) \in K^{(\psi(x))}$ for each $x \in X$ and $g|f^{-1}(K^{(m)}) = f|f^{-1}(K^{(m)})$.*

See [H5] for the proof of the theorem.

3. Questions related to Theorem 1

Concerning the theorem in the previous section, we can ask the following.

Question 1. Are the conditions (a) and (b) in the theorem equivalent for paracompact spaces?

We have an easy answer the question, i.e., the implication (b) $\Rightarrow$ (a) does not hold. In fact, for each $m, n \in \mathbb{N}$ with $m \leq n$, Vopěnka [Vo] constructed a compact space $X_{m,n}$ such that $\dim X_{m,n} = m$ and $\operatorname{Ind} X_{m,n} = n$. Let X be the topological sum $\bigoplus_{n=1}^{\infty} X_{1,n}$ of $X_{1,n}$, $n \in \mathbb{N}$. Then X does not have large transfinite dimension (and hence X does not satisfy (a)). Since $\dim X = 1$, it follows from Theorem A that for every metric simplicial complex K and every continuous mapping $f : X \to K$ there is a 1-dimensional K-approximation g of f. Hence X satisfies the condition (b).

Now, we consider the following condition which is weaker than (a).

(a') X is a strongly countable-dimensional space satisfying the following condition (K) (cf. [P]):

(K) There is a compact subspace C of X such that $\dim F < \infty$ for every closed subspace F of X with $F \cap C = \emptyset$.

We consider the relations between (a), (b) in Theorem 1 and (a') for normal (paracompact) spaces.

In [E, §7.3], Engelking reformulated the class of spaces that have strong small transfinite dimension by use of a new dimension function transfinite dimensional kernel trker. He called a space that has transfinite dimensional kernel as a *shallow space.* One should notice that a normal space X is a shallow space if and only if X has strong small trandfinite dimension and $\operatorname{sind} X = \operatorname{trker} X$ if $\operatorname{sind} X$ is a limit ordinal and $\operatorname{sind} X = \operatorname{trker} X + 1$ otherwise.

We shall consider four implications separately.

I. (a) ⇒ (a').

Fact 1 ([E, Theorem 7.1.23]). *If a weakly paracompact, strongly hereditarily normal space X has large transfinite dimension* $\operatorname{Ind} X$*, then X satisfies the condition (K).*

Fact 2 ([E, Theorem 7.3.13]; [H1, Theorem 1.2] for metrizable spaces). *If a weakly paracompact perfectly normal shallow space X, then X is a strongly countable-dimensional space.*

We can ask the following.

Question 2. Can we drop the perfectness in Fact 2?

We have a partial answer the question.

Theorem 2. *Let X be a hereditarily weakly paracompact and hereditarily normal space. If X is a shallow space, then X is a strongly countable-dimensional space.*

Proof. We show by the transfinite induction on $\operatorname{sind} X = \alpha$.
Case 1. Suppose that α is a limit ordinal number. We notice that X is expresed in the form $X = \bigcup\{P_\xi : \xi < \alpha\}$, where $P_\xi = P_{n(\xi)}(X \setminus \bigcup\{P_\eta : \eta < \lambda(\xi)\})$. We put $G_\xi = \bigcup\{P_\eta : \eta < \xi\}$ for $\xi < \alpha$. Then $\{G_\xi : \xi < \alpha\}$ is an open covering of X and $\operatorname{sind} G_\xi \leq \xi < \alpha$. By the inductive assumption, G_ξ is strongly countable-dimensional for each $\xi < \alpha$. Hence it follows from [E, Theorem 5.2.17] that X is strongly countable-dimensional.

Case 2. Suppose that $\alpha = \beta + 1$. Let $Y = X \setminus \bigcup\{P_\xi : \xi < \lambda(\alpha)\})$. Then $\operatorname{sind} Y \leq \beta < \alpha$. By the inductive assumption, Y is strongly countable-dimensional. Hence there is a countable cover $\{F_1, F_2, \dots\}$ of Y by finite dimensional closed sets. Since P_α is a closed set of X such that $X = Y \cup P_\alpha$ and $\dim P_\alpha = n(\alpha) < \infty$. We put $E_i = F_i \cup P_\alpha$ for each $i \in \mathbb{N}$. Then it follows that E_i closed in X and $\dim E_i \leq \max\{\dim F_i, \dim P_\alpha\}$. Hence X is strongly countable-dimensional. □

Corollary 2. *Let X be a hereditarily weakly paracompact, strongly hereditarily normal space. Then the implication (a) ⇒ (a') holds.*

Question 2'. Do Theorem 2 and the corollary hold for weakly paracompact normal spaces?

II. (a') ⇒ (a).

It is known that a normal space X is a shallow space if and only if every non-empty closed subspace F of X contains a non-empty normal open subspace U of F such that $\dim U < \infty$ ([E, Problem 7.3.A]). Hence, by the Baire category theorem, it follows that every normal Čech-complete, strongly countable-dimensional space is a shallow space. This implies the following.

Proposition 1. *Let X be a normal space satisfying the condition (K). If X is a strongly countable-dimensional space, then X is a shallow space.*

Proof. Let C be a compact subspace of X such that $\dim F < \infty$ for every closed subspace F of X with $F \cap C = \emptyset$. Then C is a compact strongly countable-dimensional space. Hence C is shallow and hence X is a shallow space by [E, Problem 7.3.H]. □

As we mentioned above, there is a paracompact space X such that $\dim X = 1$, but X does not have large transfinite dimension. This example leads the Ind-version of the condition (a'). A normal space X is *strongly contable-dimensional with respect to* Ind (shortly *s.c.d.*-Ind) if X is a union of countably many closed subspaces X_n, $n \in \mathbb{N}$ such that $\operatorname{Ind} X_n < \infty$ for each $n \in \mathbb{N}$. Further, we introduce a notion similar to the condition (K).

(K-Ind) There is a compact subspace C of X such that $\operatorname{Ind} F < \infty$ for every closed subspace F of X with $F \cap C = \emptyset$.

We consider the following.

(a") X is an s.c.d.-Ind space satisfying the condition (K-Ind).

Then we have

Proposition 2. *Let X be a hereditarily normal space. If X is an s.c.d.*-Ind *space satisfying the condition (K-*Ind*), then X has large transfinite dimension* $\operatorname{Ind} X$.

Proof. Let C be a compact subspace of X such that $\operatorname{Ind} F < \infty$ for every closed subspace F of X with $F \cap C = \emptyset$. Since C is an s.c.d.-Ind compact space, by [F, Theroems 1, 3], C has large transfinite dimension. Then it follows from [E, Lemma 7.1.24] that X has large transfinite dimension and $\operatorname{Ind} X \leq \omega_0 + \operatorname{Ind} C$. □

Question 3. Does Proposition 2 hold for normal spaces?

Question 4 [F, Problem 3]**.** Does every compact space which can be represented as the union of countably many subspaces which all have large transfinite dimension have itrself large transfinite dimension?

If Question 4 has an affirmative answer, then Question 3 does.

III. (a') ⇒ (b).

By the proof of Theorem 1 (see [H5]), we have the following.

Proposition 3. *Let X be a strongly countable-dimensional paracompact space. If there is a compact subspace C of X such that C has a countable character and* $\operatorname{Ind} F < \infty$ *for every closed subspace F of X with $F \cap C = \emptyset$, then X satisfies the condition (b).*

We do not know the proposition above holds for every s.c.d. paracompact space satisfying the condition (K).

IV. (b) $\Rightarrow$ (a').

The proof of (b) $\Rightarrow$ (a) of Theorem 1 works well for paracompact spaces and it shows that the condition (b) implies the condition (a') ([H5]). (The metrizability is used for the eqivalence between (a) and (a') in Theroem 1). Hence the implication (b) $\Rightarrow$ (a') holds for every paracompact space X.

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