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K-APPROXIMATIONS AND INFINITE DIMENSIONAL SPACES

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1. Introduction

Throughout the present note, by the dimension we mean the covering dimension dim. We shall consider characterizations of classes of infinite dimensional spaces in terms of $K$-approximations and discuss some questions related to the characterizations. In [DMS], Dydak-Mishra-Shukla introduced a concept of a $K$-approximation of a mapping to a metric simplicial complex and characterized $n$-dimensional spaces and finitistic spaces in terms of $K$-approximations. Let $X$ be a space, $K$ a metric simplicial complex and $f : X \to K$ a continuous mapping. A mapping $g : X \to K$ is said to be a $K$-approximation of $f$ if for each simplex $\sigma \in K$ and each $x \in X$, $f(x) \in \sigma$ implies $g(x) \in \sigma$. A $K$-approximation $g : X \to K$ of $f$ is called an $n$-dimensional $K$-approximation if $g(X) \subset K^{(n)}$ and a finite dimensional $K$-approximation if $g(X) \subset K^{(m)}$ for some natural number $m$, where $K^{(m)}$ denotes the $m$-skelton of $K$.

The concept of finitistic spaces was introduced by Swan [Sw] for working in fixed point theory and is applied to the theory of transformation groups by using the cohomological structures (cf. [AP]). For a family $\mathcal{U}$ of a space $X$ the order ord$\mathcal{U}$ of $\mathcal{U}$ is defined as follows: ord$_x\mathcal{U} = |\{U \in \mathcal{U} : x \in U\}|$ for each $x \in X$ and ord$\mathcal{U} = \sup\{\text{ord}_x\mathcal{U} : x \in X\}$. We say a family $\mathcal{U}$ has finite order if ord$\mathcal{U} = n$ for some natural number $n$. A space $X$ is said to be finitistic if every open cover of $X$ has an open refinement with finite order. We notice that finitistic spaces are also called boundedly metacompact spaces (cf. [FMS]). It is obvious that all compact spaces and all finite dimensional paracompact spaces are finitistic spaces. More precisely, we have a useful characterization of finitistic spaces.

Proposition A [H2], [DMS]. A paracompact space $X$ is finitistic if and only if there is a compact subspace $C$ of $X$ such that dim $F < \infty$ for every closed subspace $F$ with $F \cap C = \emptyset$.

The dimension-theoretic properties of finitistic spaces are investigated by several authors (cf. [DP], [DS], [DT], [DMS] [H2] and [H6]). In particular, Dydak-Mishra-Shukla ([DMS]) proved the following.

Theorem A [DMS]. For a paracompact space $X$ the following are equivalent.

(a) dim $X \leq n$.

(b) For every metric simplicial complex $K$ and every continuous mapping $f : X \to K$ there is an $n$-dimensional $K$-approximation $g$ of $f$.

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(c) For every metric simplicial complex $K$ and every continuous mapping $f : X \to K$ there is an $n$-dimensional $K$-approximation $g$ of $f$ such that $g|f^{-1}(K^{(n)}) = f|f^{-1}(K^{(n)})$.

**Theorem B** [DMS]. For a paracompact space $X$ the following are equivalent.

(a) $X$ is a finitistic space.

(b) For every metric simplicial complex $K$ and every continuous mapping $f : X \to K$ there is a finite dimensional $K$-approximation $g$ of $f$.

(c) For every integer $m \geq -1$, every metric simplicial complex $K$ and every continuous mapping $f : X \to K$ there is a finite dimensional $K$-approximation $g$ of $f$ such that $g|f^{-1}(K^{(m)}) = f|f^{-1}(K^{(m)})$.

In §2, we extend Theorem A to a class of metrizable spaces that have strong large transfinite dimension. In §3, we shall discuss some questions related to strongly countable-dimensional spaces and finitistic spaces. We denote the set of natural numbers by $\mathbb{N}$. We refer the reader to [E] and [N] for basic results in dimension theory.

**2. Characterizations of Infinite-Dimensional Spaces by Means of $K$-Approximations**

We begin with the definition of strong small transfinite dimension introduced by Borst [B]. For each ordinal number $\alpha$, we write $\alpha = \lambda(\alpha) + n(\alpha)$, where $\lambda(\alpha)$ is a limit ordinal and $n(\alpha)$ is a finite ordinal. For a normal space $X$ and a non-negative integer $n$, we put

$$P_n(X) = \bigcup \{U : U \text{ is an open set of } X \text{ such that } \dim U \leq n\}.$$ 

Let $X$ be a normal space and $\alpha$ be either an ordinal number or the integer $-1$. The strong small transfinite dimension $\text{sind}$ of $X$ is defined as follows ([B]):

(i) $\text{sind} X = -1$ if and only if $X = \emptyset$.

(ii) $\text{sind} X \leq \alpha$ if $X$ is expressed in the form $X = \bigcup \{P_\xi : \xi < \alpha\}$, where $P_\xi = P_{n(\xi)}(X \setminus \bigcup \{P_\eta : \eta < \lambda(\xi)\})$.

If $\text{sind} X \leq \alpha$ for some $\alpha$, we say that $X$ has strong small transfinite dimension.

Recall from [H4] that a normal space $X$ has strong large transfinite dimension if $X$ has both large transfinite dimension and strong small transfinite dimension. (See [E] and [N] for the definition of large transfinite dimension.) We use the following characterization of spaces that have strong large transfinite dimension. A normal space $X$ is said to be strongly countable-dimensional if $X$ is a union of countably many finite dimensional closed subsets.

**Lemma 1** [H3, Propositions 2.2 and 2.3]. Let $X$ be a metrizable space. Then $X$ has strong large transfinite dimension if and only if $X$ is finitistic and strongly countable-dimensional.
The following is a main result. For a space $X$ we denote $\mathcal{D}(X) = \{ D : D \text{ is a closed discrete subset of } X \}$.

**Theorem 1.** For a metrizable space $X$ the following are equivalent.

(a) $X$ has strong large transfinite dimension.

(b) There is a function $\varphi : D(X) \to \omega$ such that for every metric simplicial complex $K$ and every continuous mapping $f : X \to K$ there is a $K$-approximation $g$ of $f$ such that $g(D) \subset K^{(\varphi(D))}$ for each $D \in \mathcal{D}(X)$.

(c) For every integer $m \geq 1$ there is a function $\psi : D(X) \to \omega$ such that for every metric simplicial complex $K$ and every continuous mapping $f : X \to K$ there is a finite dimensional $K$-approximation $g$ of $f$ such that $g(D) \subset K^{(\psi(D))}$ for each $D \in \mathcal{D}(X)$ and $g|f^{-1}(K^{(m)}) = f|f^{-1}(K^{(m)})$.

**Corollary 1.** For a paracompact space $X$ the following are equivalent.

(a) $X$ is a strongly countable-dimensional space.

(b) There is a function $\varphi : X \to \omega$ such that for every metric simplicial complex $K$ and every continuous mapping $f : X \to K$ there is a $K$-approximation $g$ of $f$ such that $g(x) \in K^{(\varphi(x))}$ for each $x \in X$.

(c) For every integer $m \geq 1$ there is a function $\psi : X \to \omega$ such that for every metric simplicial complex $K$ and every continuous mapping $f : X \to K$ there is a $K$-approximation $g$ of $f$ such that $g(x) \in K^{(\psi(x))}$ for each $x \in X$ and $g|f^{-1}(K^{(m)}) = f|f^{-1}(K^{(m)})$.

See [H5] for the proof of the theorem.

### 3. Questions Related to Theorem 1

Concerning the theorem in the previous section, we can ask the following.

**Question 1.** Are the conditions (a) and (b) in the theorem equivalent for paracompact spaces?

We have an easy answer to the question, i.e., the implication (b) $\Rightarrow$ (a) does not hold. In fact, for each $m, n \in \mathbb{N}$ with $m \leq n$, Vopěnka [Vo] constructed a compact space $X_{m,n}$ such that $\dim X_{m,n} = m$ and $\text{Ind} X_{m,n} = n$. Let $X$ be the topological sum $\bigoplus_{n=1}^{\infty} X_{1,n}$ of $X_{1,n}$, $n \in \mathbb{N}$. Then $X$ does not have large transfinite dimension (and hence $X$ does not satisfy (a)). Since $\dim X = 1$, it follows from Theorem A that for every metric simplicial complex $K$ and every continuous mapping $f : X \to K$ there is a 1-dimensional $K$-approximation $g$ of $f$. Hence $X$ satisfies the condition (b).

Now, we consider the following condition which is weaker than (a).

(a') $X$ is a strongly countable-dimensional space satisfying the following condition (K) (cf. [P]):

(K) There is a compact subspace $C$ of $X$ such that $\dim F < \infty$ for every closed subspace $F$ of $X$ with $F \cap C = \emptyset$. 

We consider the relations between (a), (b) in Theorem 1 and (a') for normal (paracompact) spaces.

In [E, §7.3], Engelking reformulated the class of spaces that have strong small transfinite dimension by use of a new dimension function transfinite dimensional kernel trker. He called a space that has transfinite dimensional kernel as a shallow space. One should notice that a normal space $X$ is a shallow space if and only if $X$ has strong small transfinite dimension and $\dim X = \text{trker} X$ if $\dim X$ is a limit ordinal and $\dim X = \text{trker} X + 1$ otherwise.

We shall consider four implications separately.

I. (a) $\Rightarrow$ (a').

Fact 1 ([E, Theorem 7.1.23]). If a weakly paracompact, strongly hereditarily normal space $X$ has large transfinite dimension $\text{Ind} X$, then $X$ satisfies the condition (K).

Fact 2 ([E, Theorem 7.3.13]; [H1, Theorem 1.2] for metrizable spaces). If a weakly paracompact perfectly normal shallow space $X$, then $X$ is a strongly countable-dimensional space.

We can ask the following.

Question 2. Can we drop the perfectness in Fact 2?

We have a partial answer the question.

Theorem 2. Let $X$ be a hereditarily weakly paracompact and hereditarily normal space. If $X$ is a shallow space, then $X$ is a strongly countable-dimensional space.

Proof. We show by the transfinite induction on $\text{sind} X = \alpha$.

Case 1. Suppose that $\alpha$ is a limit ordinal number. We notice that $X$ is expressed in the form $X = \bigcup \{P_\xi : \xi < \alpha\}$, where $P_\xi = P_{n(\xi)}(X \setminus \bigcup \{P_\eta : \eta < \lambda(\xi)\})$. We put $G_\xi = \bigcup \{P_\eta : \eta < \xi\}$ for $\xi < \alpha$. Then $\{G_\xi : \xi < \alpha\}$ is an open covering of $X$ and $\text{sind} G_\xi \leq \xi < \alpha$. By the inductive assumption, $G_\xi$ is strongly countable-dimensional for each $\xi < \alpha$. Hence it follows from [E, Theorem 5.2.17] that $X$ is strongly countable-dimensional.

Case 2. Suppose that $\alpha = \beta + 1$. Let $Y = X \setminus \bigcup \{P_\xi : \xi < \lambda(\alpha)\}$. Then $\text{sind} Y \leq \beta < \alpha$. By the inductive assumption, $Y$ is strongly countable-dimensional. Hence there is a countable cover $\{F_1, F_2, \ldots\}$ of $Y$ by finite dimensional closed sets. Since $P_\alpha$ is a closed set of $X$ such that $X = Y \cup P_\alpha$ and $\dim P_\alpha = n(\alpha) < \infty$. We put $E_i = F_i \cup P_\alpha$ for each $i \in \mathbb{N}$. Then it follows that $E_i$ closed in $X$ and $\dim E_i \leq \max\{\dim F_i, \dim P_\alpha\}$. Hence $X$ is strongly countable-dimensional.

Corollary 2. Let $X$ be a hereditarily weakly paracompact, strongly hereditarily normal space. Then the implication (a) $\Rightarrow$ (a') holds.

Question 2'. Do Theorem 2 and the corollary hold for weakly paracompact normal spaces?

II. (a') $\Rightarrow$ (a).
It is known that a normal space $X$ is a shallow space if and only if every non-empty closed subspace $F$ of $X$ contains a non-empty normal open subspace $U$ of $F$ such that $\dim U < \infty$ ([E, Problem 7.3.A]). Hence, by the Baire category theorem, it follows that every normal \v{C}ech-complete, strongly countable-dimensional space is a shallow space. This implies the following.

**Proposition 1.** Let $X$ be a normal space satisfying the condition $(K)$. If $X$ is a strongly countable-dimensional space, then $X$ is a shallow space.

**Proof.** Let $C$ be a compact subspace of $X$ such that $\dim F < \infty$ for every closed subspace $F$ of $X$ with $F \cap C = \emptyset$. Then $C$ is a compact strongly countable-dimensional space. Hence $C$ is shallow and hence $X$ is a shallow space by [E, Problem 7.3.H]. □

As we mentioned above, there is a paracompact space $X$ such that $\dim X = 1$, but $X$ does not have large transfinite dimension. This example leads the Ind-version of the condition $(a')$. A normal space $X$ is **strongly countable-dimensional with respect to Ind** (shortly s.c.d.-Ind) if $X$ is a union of countably many closed subspaces $X_n$, $n \in \mathbb{N}$ such that $\text{Ind } X_n < \infty$ for each $n \in \mathbb{N}$. Further, we introduce a notion similar to the condition $(K)$.

(K-Ind) There is a compact subspace $C$ of $X$ such that $\text{Ind } F < \infty$ for every closed subspace $F$ of $X$ with $F \cap C = \emptyset$.

We consider the following.

(a'') $X$ is an s.c.d.-Ind space satisfying the condition (K-Ind).

Then we have

**Proposition 2.** Let $X$ be a hereditarily normal space. If $X$ is an s.c.d.-Ind space satisfying the condition (K-Ind), then $X$ has large transfinite dimension $\text{Ind } X$.

**Proof.** Let $C$ be a compact subspace of $X$ such that $\text{Ind } F < \infty$ for every closed subspace $F$ of $X$ with $F \cap C = \emptyset$. Since $C$ is an s.c.d.-Ind compact space, by [F, Theorems 1, 3], $C$ has large transfinite dimension. Then it follows from [E, Lemma 7.1.24] that $X$ has large transfinite dimension and $\text{Ind } X \leq \omega_0 + \text{Ind } C$. □

**Question 3.** Does Proposition 2 hold for normal spaces?

**Question 4** [F, Problem 3]. Does every compact space which can be represented as the union of countably many subspaces which all have large transfinite dimension have itself large transfinite dimension?

If Question 4 has an affirmative answer, then Question 3 does.

III. (a') $\Rightarrow$ (b).

By the proof of Theorem 1 (see [H5]), we have the following.
Proposition 3. Let $X$ be a strongly countable-dimensional paracompact space. If there is a compact subspace $C$ of $X$ such that $C$ has a countable character and $\text{Ind} F < \infty$ for every closed subspace $F$ of $X$ with $F \cap C = \emptyset$, then $X$ satisfies the condition (b).

We do not know the proposition above holds for every s.c.d. paracompact space satisfying the condition (K).

IV. (b) $\Rightarrow$ (a').

The proof of (b) $\Rightarrow$ (a) of Theorem 1 works well for paracompact spaces and it shows that the condition (b) implies the condition (a') ([H5]). (The metrizability is used for the equivalence between (a) and (a') in Theorem 1). Hence the implication (b) $\Rightarrow$ (a') holds for every paracompact space $X$.

REFERENCES


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