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On the cohomology of Coxeter groups

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§1 はじめに

本研究では，有限生成な Coxeter group の cohomology を調べることを目的としている。まず Coxeter group の定義を与える。集合 S と写像 $m: S\times S\to \mathbf{N}\cup\{\infty\}$ で次の条件をみたすものを考える。

(1) $m(s,t)=m(t,s)$ for all $s,t\in S$,

(2) $m(s,s)=1$ for all $s\in S$,

(3) $m(s,t)\geq 2$ for all $s\neq t\in S$.

このような S と m によって

$$W=\langle\, S\,|\,(st)^{m(s,t)}=1 \text{ for } s,t\in S\,\rangle,$$

と表現される群 W を *Coxeter group* とよぶ。そして (W,S) の組みを *Coxeter system* とよぶ。

Coxeter group の歴史は古く，その由来は，鏡映によって生成される有限群 (有限鏡映群) が上記のような表現をもつ有限群として特徴付けられることを H. S. M. Coxeter が証明したことによる。現在では有限無限を問わず，上記のような表現をもつ群は Coxeter group とよばれる。有限な Coxeter group については [B] にみられるように，完全に分類が与えられるなど，ある程度のことがわかっているのだが，無限な場合についてはほとんど何も分かっていない状況にある。本研究では，直接扱うことの難しい無限の Coxeter group に対して，Coxeter system から定義される幾何的な対象を扱うことによって，もとの Coxeter group に関する情報を得ることを目的としている。特にここでは，最近 M. W. Davis によって [D3] の中で与

えられた Coxeter group の cohomology に関する公式を改良し，Coxeter group の cohomology について考察する。

§2 Davis の定理

まず，いくつかの定義を与える。

Definition. Let (W, S) be a Coxeter system. For a subset $T \subset S$, W_T is defined as the subgroup of W generated by T, and called a *parabolic subgroup*. It is known that the pair (W_T, T) is also a Coxeter system ([B]). If T is the empty set, then W_T is the trivial group.

Let $\mathcal{S}^f(W, S)$ be the family of subsets T of S such that W_T is finite. We note that the empty set is a member of $\mathcal{S}^f(W, S)$. We define a simplicial complex $L(W, S)$ by the following conditions:

(1) the vertex set of $L(W, S)$ is S, and

(2) for each nonempty subset T of S, T spans a simplex of $L(W, S)$ if and only if $T \in \mathcal{S}^f(W, S)$.

For each nonempty subset T of S, $L(W_T, T)$ is a subcomplex of $L(W, S)$.

Remark. If (W_1, S_1) and (W_2, S_2) are Coxeter systems, then $(W_1 \times W_2, S_1 \cup S_2)$ and $(W_1 * W_2, S_1 \cup S_2)$ are also Coxeter system. Indeed, if

$$W_i = \langle\, S_i \,|\, (st)^{m_i(s,t)} = 1 \text{ for } s, t \in S_i \,\rangle$$

for each $i = 1, 2$, then we define $m, m' : S_1 \cup S_2 \to \mathbf{N} \cup \{\infty\}$ as

$$m(s,t) := \begin{cases} m_1(s,t) & \text{if } s, t \in S_1 \\ m_2(s,t) & \text{if } s, t \in S_2 \\ 2 & \text{otherwise,} \end{cases} \qquad m'(s,t) := \begin{cases} m_1(s,t) & \text{if } s, t \in S_1 \\ m_2(s,t) & \text{if } s, t \in S_2 \\ \infty & \text{otherwise.} \end{cases}$$

Then we have

$$W_1 \times W_2 = \langle\, S_1 \cup S_2 \,|\, (st)^{m(s,t)} = 1 \text{ for } s, t \in S_1 \cup S_2 \,\rangle,$$
$$W_1 * W_2 = \langle\, S_1 \cup S_2 \,|\, (st)^{m'(s,t)} = 1 \text{ for } s, t \in S_1 \cup S_2 \,\rangle.$$

By the definition of $L(W,S)$, we also have

$$L(W_1 \times W_2, S_1 \cup S_2) = L(W_1, S_1) * L(W_2, S_2) \quad \text{(simplicial join)}$$
$$L(W_1 * W_2, S_1 \cup S_2) = L(W_1, S_1) \cup L(W_2, S_2) \quad \text{(disjoint union)}.$$

本論文では，Coxeter system (W,S) について，S は常に有限集合であるものとする。また，簡単のため

$$\mathcal{S}^f := \mathcal{S}^f(W,S)$$
$$L := L(W,S)$$
$$L_T := L(W_T, T)$$

とあらわすことにする。

Definition. Let (W,S) be a Coxeter system. We define K as the simplicial cone over the barycentric subdivision sd L of $L = L(W,S)$. For each $s \in S$, the closed star of s in sd L is denoted by K_s. The closed star K_s is a subcomplex of K. For each nonempty subset T of S, we set

$$K^T := \bigcup_{s \in T} K_s.$$

We note that K^T has the same homotopy type as L_T.

Definition. For each $w \in W$, we define a subset $S(w)$ of S as

$$S(w) := \{s \in S \mid \ell(ws) < \ell(w)\},$$

where $\ell(w)$ is the minimum length of word in S which represents w. For each subset T of S, we define a subset W^T of W as

$$W^T := \{w \in W \mid S(w) = T\}.$$

上記の定義のもと，M. W. Davis によって次の定理が証明された。

Theorem 1 (Davis [D3]). *Let (W,S) be a Coxeter system and let Γ be a torsion-free subgroup of finite index in W. Then there exists the following isomorphism:*

$$H^*(\Gamma; \mathbf{Z}\Gamma) \cong \bigoplus_{T \in \mathcal{S}^f} \left(\mathbf{Z}(W^T) \otimes H^*(K, K^{S \setminus T}) \right),$$

where $\mathbf{Z}(W^T)$ is the free abelian group on W^T.

Remark. It is known that there exists a torsion-free subgroup Γ of finite index in a Coxeter group W, and $H^*(W;\mathbf{Z}W) \cong H^*(\Gamma;\mathbf{Z}\Gamma)$ (cf. [D1], [D3]).

いま，K は contractible で，$H^*(K^{S\setminus T})$ と $H^*(L_{S\setminus T})$ は同型となるため，上の定理は次のように書き換えることができる。

$$H^*(\Gamma;\mathbf{Z}\Gamma) \cong \bigoplus_{T\in\mathcal{S}^f} \left(\mathbf{Z}(W^T)\otimes \tilde{H}^{*-1}(L_{S\setminus T})\right),$$

where $\tilde{H}^*$ denotes the reduced cohomology.

この式からわかるように，$H^i(\Gamma;\mathbf{Z}\Gamma)$ が (アーベル群として) 無限生成となる必要十分条件は，ある $T\in\mathcal{S}^f$ で，W^T が無限集合となり，$\tilde{H}^{i-1}(L_{S\setminus T})\neq 0$ となるものが存在することである。

このように，W^T の元の個数は，$H^i(\Gamma;\mathbf{Z}\Gamma)$ が有限生成となるか無限生成となるかに関わる係数であるのだが，定義からもわかるように，実際に直接求めることは困難である。ここで，$\tilde{H}^*(L_{S\setminus T})$ が自明でない場合に，W^T の元の個数がどのようになるのかを調べ，上で述べた Davis の定理をより簡単にすることを考える。

§3 W^T について

次の補題が成り立つ。この補題の中の 'only if' については，[D3] の中で Davis によって証明されている。その逆の 'if' の部分についても，自然な議論によって証明することができる。

Lemma 2 (cf. [D3, Lemma 1.10]). *Suppose that $T\in\mathcal{S}^f$. Then W^T is a singleton if and only if W decomposes as the direct product: $W = W_{S\setminus T}\times W_T$.*

上の補題を用いて，次の補題を示すことに成功した。証明には様々な準備が必要となるため，ここでは証明は省略して証明の方針についてのみ記す。

Lemma 3. *Suppose that $T\in\mathcal{S}^f$. If W^T is finite and not a singleton, then $L_{S\setminus T}$ is contractible.*

Idea. We give an idea of the proof. Suppose that W^T is finite and not a singleton. Then W does not decompose as the direct product of $W_{S\setminus T}$ and W_T by Lemma 2. Hence there exist $s_0 \in S \setminus T$ and $t_0 \in T$ such that $m(s_0, t_0) \neq 2$. Then we show that $L_{S\setminus T} = s_0 * L_{S\setminus(\{s_0\}\cup T)}$. ■

この補題から，$\tilde{H}^*(L_{S\setminus T})$ が自明でない場合には，W^T は一点からなる集合か，もしくは無限集合となることがわかる。次に，T がどのような場合に W^T が一点集合となるのかについて考える。その準備として，定義を与える。

Definition. A Coxeter system (W, S) is said to be *irreducible* if, for any nonempty and proper subset T of S, W does not decompose into the direct product of W_T and $W_{S\setminus T}$.

Let (W, S) be a Coxeter system. Then there exists a unique decomposition $\{S_1, \ldots, S_r\}$ of S such that W is the direct product of the parabolic subgroups $W_{S_1}, \ldots, W_{S_r}$ and each Coxeter system (W_{S_i}, S_i) is irreducible (cf. [B], [H, p.30]). Here we enumerate $\{S_i\}$ so that $S_1, \ldots, S_q \in \mathcal{S}^f$ and $S_{q+1}, \ldots, S_r \notin \mathcal{S}^f$. Let $\tilde{T} := \bigcup_{i=1}^q S_i$ and $\tilde{S} := S \setminus \tilde{T}$. We say that $W_{\tilde{S}}$ is the *essential parabolic subgroup* in W. We note that $W_{\tilde{T}}$ is finite and W is the direct product of $W_{\tilde{S}}$ and $W_{\tilde{T}}$.

Remark. The essential parabolic subgroup $W_{\tilde{S}}$ has a finite index in W. Hence a torsion-free subgroup Γ of finite index in $W_{\tilde{S}}$ has a finite index in W as well, and $H^*(W; \mathbf{Z}W) \cong H^*(\Gamma; \mathbf{Z}\Gamma) \cong H^*(W_{\tilde{S}}; \mathbf{Z}W_{\tilde{S}})$.

If W is finite, then $\tilde{T} = S$ and $\tilde{S}$ is empty, hence the essential parabolic subgroup is the trivial subgroup.

ここで定義された $\tilde{T}$ は次のような性質をもつ。

Lemma 4. *Let T be a subset of S. If $\tilde{T}\setminus T$ is nonempty, then $L_{S\setminus T}$ is contractible.*

Proof. Suppose that $\tilde{T} \setminus T$ is nonempty. By definition, W is the direct product of $W_{\tilde{S}}$ and $W_{\tilde{T}}$. Hence

$$W_{S\setminus T} = W_{\tilde{S}\setminus T} \times W_{\tilde{T}\setminus T} \text{ and}$$
$$L_{S\setminus T} = L_{\tilde{S}\setminus T} * L_{\tilde{T}\setminus T}.$$

Since $W_{\tilde{T}}$ is finite, $W_{\tilde{T}\setminus T}$ is finite. Hence $L_{\tilde{T}\setminus T}$ is a simplex. Thus $L_{S\setminus T}$ is contractible. ∎

Lemma 3 と Lemma 4 を用いることにより，次の補題を証明することができる。この補題によって $H^*(L_{S\setminus T})$ が自明でない場合の W^T の元の個数が決定される。

Lemma 5. *Suppose that $T \in \mathcal{S}^f$ and $L_{S\setminus T}$ is not contractible. Then W^T is finite if and only if $T = \tilde{T}$.*

Proof. Since W is the direct product of $W_{S\setminus\tilde{T}}$ and $W_{\tilde{T}}$, $W^{\tilde{T}}$ is a singleton by Lemma 2. Thus W^T is finite if $T = \tilde{T}$.

Suppose that W^T is finite and $L_{S\setminus T}$ is not contractible. Since $L_{S\setminus T}$ is not contractible, $\tilde{T}\setminus T$ is empty by Lemma 4. Hence $\tilde{T} \subset T$. Since W^T is finite and $L_{S\setminus T}$ is not contractible, W^T is a singleton by Lemma 3. Hence W is the direct product of $W_{S\setminus T}$ and W_T by Lemma 2. Then

$$W = W_{S\setminus T} \times W_T = W_{S\setminus\tilde{T}} \times W_{\tilde{T}}.$$

Since W_T is finite and $\tilde{T} \subset T$, we have $T = \tilde{T}$ by the definition of $\tilde{T}$. ∎

§4 Coxeter group の cohomology について

Lemma 5 を用いることにより，Theorem 1 は次のように書き換えることができる。

Theorem 6. *Let (W, S) be a Coxeter system and Γ a torsion-free subgroup of finite index in W. Then*

$$\begin{aligned} H^*(\Gamma; \mathbf{Z}\Gamma) &\cong \tilde{H}^{*-1}(L_{\tilde{S}}) \oplus \left(\bigoplus_{\tilde{T} \subsetneqq T \in \mathcal{S}^f} \bigoplus_{\mathbf{Z}} \tilde{H}^{*-1}(L_{S\setminus T}) \right) \\ &\cong \tilde{H}^{*-1}(\tilde{L}) \oplus \left(\bigoplus_{\emptyset \neq T \in \tilde{\mathcal{S}}^f} \bigoplus_{\mathbf{Z}} \tilde{H}^{*-1}(\tilde{L}_{\tilde{S}\setminus T}) \right), \end{aligned}$$

where $\tilde{S}$ is the subset of S such that $W_{\tilde{S}}$ is the essential parabolic subgroup in W, $\tilde{T} = S \setminus \tilde{S}$, $\tilde{L} = L(W_{\tilde{S}}, \tilde{S})$ and $\tilde{\mathcal{S}}^f = \mathcal{S}^f(W_{\tilde{S}}, \tilde{S}) = \mathcal{S}^f \cap \tilde{S}$.

Proof. We note that $W^{\tilde{T}}$ is a singleton by Lemma 2. By Theorem 1 and Lemma 5, we have that

$$H^*(\Gamma;\mathbf{Z}\Gamma) \cong \tilde{H}^{*-1}(L_{S\setminus\tilde{T}}) \oplus \left(\bigoplus_{\tilde{T}\neq T\in\mathcal{S}^f} \bigoplus_{\mathbf{Z}} \tilde{H}^{*-1}(L_{S\setminus T}) \right).$$

If $\tilde{T} \not\subset T$ (i.e., $\tilde{T}\setminus T$ is nonempty), then $L_{S\setminus T}$ is contractible by Lemma 4. Hence,

$$H^*(\Gamma;\mathbf{Z}\Gamma) \cong \tilde{H}^{*-1}(L_{S\setminus\tilde{T}}) \oplus \left(\bigoplus_{\tilde{T}\subsetneq T\in\mathcal{S}^f} \bigoplus_{\mathbf{Z}} \tilde{H}^{*-1}(L_{S\setminus T}) \right).$$

The parabolic subgroup $W_{\tilde{S}}$ has a finite index in W, and $W_{\tilde{S}}$ is the essential parabolic subgroup in the Coxeter system $(W_{\tilde{S}}, \tilde{S})$. Therefore,

$$H^*(\Gamma;\mathbf{Z}\Gamma) \cong \tilde{H}^{*-1}(\tilde{L}) \oplus \left(\bigoplus_{\emptyset\neq T\in\tilde{\mathcal{S}}^f} \bigoplus_{\mathbf{Z}} \tilde{H}^{*-1}(\tilde{L}_{\tilde{S}\setminus T}) \right)$$

by Theorem 1 and Lemma 5. ∎

この Theorem 6 により，直ちに次の系を得ることができる。

Corollary 7. *Let (W, S) be a Coxeter system, Γ a torsion-free subgroup of finite index in W, $\tilde{S}$ the subset of S such that $W_{\tilde{S}}$ is the essential parabolic subgroup in W, and $\tilde{T} = S\setminus\tilde{S}$. Then the following statements are equivalent:*

(1) $H^i(\Gamma;\mathbf{Z}\Gamma)$ *is finitely generated;*

(2) $H^i(\Gamma;\mathbf{Z}\Gamma)$ *is isomorphic to* $\tilde{H}^{i-1}(L_{\tilde{S}})$*;*

(3) $\tilde{H}^{i-1}(L_{S\setminus T}) = 0$ *for each* $\tilde{T} \subsetneq T \in \mathcal{S}^f$.

ここで例を一つ与える。

Example. It is known that, for every finite simplicial complex M, there exists a Coxeter system (W, S) such that $L(W, S)$ is equal to the barycentric subdivision of M ([D1, Lemma 11.3]).

Let (W, S) be a Coxeter system such that $L = L(W, S)$ is the barycentric subdivision of a triangulation of the projective plane. In [Dr], Dranishnikov showed that $\mathrm{vcd}_{\mathbf{Z}} W = 3$ and $\mathrm{vcd}_{\mathbf{Q}} W = 2$, where $\mathrm{vcd}_R W$ is the virtual cohomological dimension of W over R. Now, using Theorem 6, we calculate the cohomology of a torsion-free subgroup Γ of finite index in W.

Since L is the projective plane,

$$\tilde{H}^i(L) \cong \begin{cases} \mathbf{Z}_2, & i = 2, \\ 0, & i \neq 2. \end{cases}$$

Since $L = L_{S \setminus \emptyset}$ is not contractible and $W^{\emptyset}$ is a singleton, $\tilde{T}$ is the empty set (i.e., $W = W_S$ is the essential parabolic subgroup) by Lemma 5. For each $T \in \mathcal{S}^f \setminus \{\emptyset\}$, $L_{S \setminus T}$ has the same homotopy type as a circle. Hence,

$$\tilde{H}^i(L_{S \setminus T}) \cong \begin{cases} \mathbf{Z}, & i = 1, \\ 0, & i \neq 1. \end{cases}$$

Therefore, by Theorem 6, we have

$$\begin{aligned} H^i(\Gamma; \mathbf{Z}\Gamma) &\cong \tilde{H}^{i-1}(L) \oplus \left(\bigoplus_{\emptyset \neq T \in \mathcal{S}^f} \bigoplus_{\mathbf{Z}} \tilde{H}^{i-1}(L_{S \setminus T}) \right) \\ &\cong \begin{cases} \mathbf{Z}_2, & i = 3, \\ \mathbf{Z} \oplus \mathbf{Z} \oplus \cdots, & i = 2, \\ 0, & \text{otherwise}. \end{cases} \end{aligned}$$

上の例からもわかるように，Coxeter group W で，$H^i(\Gamma; \mathbf{Z}\Gamma)$ は有限生成となり，$H^j(\Gamma; \mathbf{Z}\Gamma)$ は無限生成となるもの $(i \neq j)$ が存在する。特に，各 i について $H^i(\Gamma; \mathbf{Z}\Gamma)$ が有限生成となる場合については，Theorem 6 の系として次が得られる。

Corollary 8. *Let (W, S) be a Coxeter system, Γ a torsion-free subgroup of finite index in W, and $\tilde{S}$ the subset of S such that $W_{\tilde{S}}$ is the essential parabolic subgroup in W. Then the following statements are equivalent:*

(1) $H^i(\Gamma; \mathbf{Z}\Gamma)$ *is finitely generated for each i.*

(2) $H^i(\Gamma; \mathbf{Z}\Gamma)$ *is isomorphic to $\tilde{H}^{i-1}(L_{\tilde{S}})$ for each i.*

(3) Γ *is a Poincaré duality group.*

Coxeter system に対して，空間 Σ が次のように定義される。

Definition. Let (W, S) be a Coxeter system and let $W\mathcal{S}^f$ be the set of all cosets of the form wW_T, with $w \in W$ and $T \in \mathcal{S}^f$. The set $W\mathcal{S}^f$ is partially ordered by inclusion. The contractible simplicial complex Σ is defined as the geometric realization of the partially ordered set $W\mathcal{S}^f$ ([D3, §3], [D1]). If W is infinite, then Σ is noncompact.

Σ は次のような性質をもつことが知られている。

Remark. It is known that Σ can be cellulated so that the link of each vertex is L ([D2, §9, §10], [M]). In [M], G. Moussong proved that a natural metric on Σ satisfies the CAT(0) condition. Hence, if W is infinite, Σ can be compactified by adding its ideal boundary $\partial\Sigma$ ([D2, §4]). It is known that

$$H^*(\Gamma; \mathbf{Z}\Gamma) \cong H^*(W; \mathbf{Z}W) \cong H_c^*(\Sigma) \cong \check{H}^{*-1}(\partial\Sigma),$$

where Γ is a torsion-free subgroup of finite index in W. Here H_c^* and $\check{H}^*$ denote the compactly supported cohomology and the Čech reduced cohomology, respectively.

実際，Davis によって証明された Theorem 1 は，この Σ の cohomology $H_c^*(\Sigma)$ を計算することによって得られている。

上の Remark と Corollary 8 から直ちに次を得る。

Corollary 9. *Let W be a Coxeter group.*

(1) *$H^i(W; \mathbf{Z}W)$ is finitely generated for each i if and only if W is a virtual Poincaré duality group.*

(2) *Suppose that W is infinite. Then $\check{H}^i(\partial\Sigma)$ is finitely generated for each i if and only if the Čech cohomology of $\partial\Sigma$ is isomorphic to the cohomology of an n-sphere for some n.*

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