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On the cohomology of Coxeter groups

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§1 はじめに

本研究では，有限生成な Coxeter group との cohomology を調べることを目的としている。まず Coxeter group の定義を与える。集合 $S$ と写像 $m : S \times S \rightarrow \mathbb{N} \cup \{\infty\}$ で次の条件をみたすものを考える。

(1) $m(s, t) = m(t, s)$ for all $s, t \in S$,
(2) $m(s, s) = 1$ for all $s \in S$,
(3) $m(s, t) \geq 2$ for all $s \neq t \in S$.

このような $S$ と $m$ によって

$$W = \langle S \mid (st)^{m(s,t)} = 1 \text{ for } s, t \in S \rangle,$$

と表現される群 $W$ を Coxeter group とよぶ。そして $(W, S)$ の組みを Coxeter system とよぶ。

Coxeter group の歴史は古く，その由来は，鏡映によって生成される有限群 (有限鏡映群) が上記のような表現をもつ有限群として特徴付けられることを H. S. M. Coxeter が証明したことによる。現在では有限無限を問わず，上記のような表現をもつ群は Coxeter group とよばれる。有限な Coxeter group については [B] にみられるように，完全に分類が与えられるなど，ある程度のことがわかっているのだが，無限の場合についてはほとんど何も分かっていない状況にある。本研究では，直接扱うことの難しい無限の Coxeter group に対して，Coxeter system から定義される幾何的な対象を扱うことによって，もとの Coxeter group に関する情報を得ることを目的としている。特にここでは，最近 M. W. Davis によって [D3] の中で与
えられた Coxeter group の cohomology に関する公式を改良し、Coxeter group の cohomology について考察する。

§2 Davis の定理

まず、いくつかの定義を与える。

**Definition.** Let \((W, S)\) be a Coxeter system. For a subset \(T \subset S\), \(W_T\) is defined as the subgroup of \(W\) generated by \(T\), and called a *parabolic subgroup*. It is known that the pair \((W_T, T)\) is also a Coxeter system ([B]). If \(T\) is the empty set, then \(W_T\) is the trivial group.

Let \(\mathcal{S}^f(W, S)\) be the family of subsets \(T\) of \(S\) such that \(W_T\) is finite. We note that the empty set is a member of \(\mathcal{S}^f(W, S)\). We define a simplicial complex \(L(W, S)\) by the following conditions:

1. the vertex set of \(L(W, S)\) is \(S\), and
2. for each nonempty subset \(T\) of \(S\), \(T\) spans a simplex of \(L(W, S)\) if and only if \(T \in \mathcal{S}^f(W, S)\).

For each nonempty subset \(T\) of \(S\), \(L(W_T, T)\) is a subcomplex of \(L(W, S)\).

**Remark.** If \((W_1, S_1)\) and \((W_2, S_2)\) are Coxeter systems, then \((W_1 \times W_2, S_1 \cup S_2)\) and \((W_1 * W_2, S_1 \cup S_2)\) are also Coxeter system. Indeed, if

\[
W_i = \langle S_i \mid (st)^{m_i(s,t)} = 1 \text{ for } s, t \in S_i \rangle
\]

for each \(i = 1, 2\), then we define \(m, m' : S_1 \cup S_2 \to \mathbb{N} \cup \{\infty\}\) as

\[
m(s, t) := \begin{cases} m_1(s, t) & \text{if } s, t \in S_1 \\ m_2(s, t) & \text{if } s, t \in S_2 \\ 2 & \text{otherwise,} \end{cases}
\]

\[
m'(s, t) := \begin{cases} m_1(s, t) & \text{if } s, t \in S_1 \\ m_2(s, t) & \text{if } s, t \in S_2 \\ \infty & \text{otherwise.} \end{cases}
\]

Then we have

\[
W_1 \times W_2 = \langle S_1 \cup S_2 \mid (st)^{m(s,t)} = 1 \text{ for } s, t \in S_1 \cup S_2 \rangle,
\]

\[
W_1 * W_2 = \langle S_1 \cup S_2 \mid (st)^{m'(s,t)} = 1 \text{ for } s, t \in S_1 \cup S_2 \rangle.
\]
By the definition of $L(W, S)$, we also have
\[ L(W_1 \times W_2, S_1 \cup S_2) = L(W_1, S_1) \ast L(W_2, S_2) \quad \text{(simplicial join)} \]
\[ L(W_1 \ast W_2, S_1 \cup S_2) = L(W_1, S_1) \cup L(W_2, S_2) \quad \text{(disjoint union)} \]

Definition. Let $(W, S)$ be a Coxeter system. We define $K$ as the simplicial cone over the barycentric subdivision $sdL$ of $L = L(W, S)$. For each $s \in S$, the closed star of $s$ in $sdL$ is denoted by $K_s$. The closed star $K_s$ is a subcomplex of $K$. For each nonempty subset $T$ of $S$, we set
\[ K^T := \bigcup_{s \in T} K_s. \]
We note that $K^T$ has the same homotopy type as $L_T$.

Definition. For each $w \in W$, we define a subset $S(w)$ of $S$ as
\[ S(w) := \{ s \in S \mid \ell(ws) < \ell(w) \}, \]
where $\ell(w)$ is the minimum length of word in $S$ which represents $w$. For each subset $T$ of $S$, we define a subset $W^T$ of $W$ as
\[ W^T := \{ w \in W \mid S(w) = T \}. \]

Theorem 1 (Davis [D3]). Let $(W, S)$ be a Coxeter system and let $\Gamma$ be a torsion-free subgroup of finite index in $W$. Then there exists the following isomorphism:
\[ H^*(\Gamma; \mathbb{Z}\Gamma) \cong \bigoplus_{T \in S^f} \left( \mathbb{Z}(W^T) \otimes H^*(K, K^S\backslash T) \right), \]
where $\mathbb{Z}(W^T)$ is the free abelian group on $W^T$. 
Remark. It is known that there exists a torsion-free subgroup $\Gamma$ of finite index in a Coxeter group $W$, and $H^*(W;\mathbb{Z}W) \cong H^*(\Gamma;\mathbb{Z}\Gamma)$ (cf. [D1], [D3]).

им, $K$ является контрактными, так что $H^*(K^{S\setminus T})$ и $H^*(L_{S\setminus T})$ изоморфны, следовательно, обе группы связаны с $W$.
Idea. We give an idea of the proof. Suppose that $W^T$ is finite and not a singleton. Then $W$ does not decompose as the direct product of $W_{S\setminus T}$ and $W_T$ by Lemma 2. Hence there exist $s_0 \in S \setminus T$ and $t_0 \in T$ such that $m(s_0, t_0) \neq 2$. Then we show that $L_{S\setminus T} = s_0 * L_{S\setminus \{s_0\}\cup T}$.

Definition. A Coxeter system $(W, S)$ is said to be irreducible if, for any nonempty and proper subset $T$ of $S$, $W$ does not decompose into the direct product of $W_T$ and $W_{S\setminus T}$.

Let $(W, S)$ be a Coxeter system. Then there exists a unique decomposition \{S_1, \ldots, S_r\} of $S$ such that $W$ is the direct product of the parabolic subgroups $W_{S_1}, \ldots, W_{S_r}$ and each Coxeter system $(W_{S_i}, S_i)$ is irreducible (cf. [B], [H, p.30]). Here we enumerate \{S_i\} so that $S_1, \ldots, S_q \in S'$ and $S_{q+1}, \ldots, S_r \not\in S'$. Let $\tilde{T} := \bigcup_{i=1}^{q} S_i$ and $\tilde{S} := S \setminus \tilde{T}$. We say that $W_{\tilde{S}}$ is the essential parabolic subgroup in $W$. We note that $W_{\tilde{T}}$ is finite and $W$ is the direct product of $W_{\tilde{S}}$ and $W_{\tilde{T}}$.

Remark. The essential parabolic subgroup $W_{\tilde{S}}$ has a finite index in $W$. Hence a torsion-free subgroup $\Gamma$ of finite index in $W_{\tilde{S}}$ has a finite index in $W$ as well, and $H^*(W; \mathbb{Z}W) \cong H^*(\Gamma; \mathbb{Z}W) \cong H^*(W_{\tilde{S}}; \mathbb{Z}W_{\tilde{S}})$.

If $W$ is finite, then $\tilde{T} = S$ and $\tilde{S}$ is empty, hence the essential parabolic subgroup is the trivial subgroup.

ここで定義された $\tilde{T}$ は次のような性質をもつ。

Lemma 4. Let $T$ be a subset of $S$. If $\tilde{T}\setminus T$ is nonempty, then $L_{S\setminus T}$ is contractible.

Proof. Suppose that $\tilde{T}\setminus T$ is nonempty. By definition, $W$ is the direct product of $W_{\tilde{S}}$ and $W_{\tilde{T}}$. Hence

$$W_{S\setminus T} = W_{\tilde{S}\setminus T} \times W_{\tilde{T}\setminus T}$$

and

$$L_{S\setminus T} = L_{\tilde{S}\setminus T} * L_{\tilde{T}\setminus T}.$$
Since \( W_{\overline{T}} \) is finite, \( W_{\tilde{T}\backslash T} \) is finite. Hence \( L_{\overline{T}\backslash T} \) is a simplex. Thus \( L_{S\backslash T} \) is contractible.

**Lemma 5.** Suppose that \( T \in S^{f} \) and \( L_{S\backslash T} \) is not contractible. Then \( W^{T} \) is finite if and only if \( T = \tilde{T} \).

**Proof.** Since \( W \) is the direct product of \( W_{S\backslash T} \) and \( W_{T} \), \( W^{T} \) is a singleton by Lemma 2. Thus \( W^{T} \) is finite if \( T = \tilde{T} \).

Suppose that \( W^{T} \) is finite and \( L_{S\backslash T} \) is not contractible. Since \( L_{S\backslash T} \) is not contractible, \( \tilde{T} \backslash T \) is empty by Lemma 4. Hence \( \tilde{T} \subset T \). Since \( W^{T} \) is finite and \( L_{S\backslash T} \) is not contractible, \( W^{T} \) is a singleton by Lemma 3. Hence \( W \) is the direct product of \( W_{S\backslash T} \) and \( W_{T} \) by Lemma 2. Then

\[
W = W_{S\backslash T} \times W_{T} = W_{S\tilde{T}} \times W_{\tilde{T}}.
\]

Since \( W_{T} \) is finite and \( \tilde{T} \subset T \), we have \( T = \tilde{T} \) by the definition of \( \tilde{T} \).

\[\text{§4 Coxeter group の cohomology について}\]

Lemma 5 を用いることにより，Theorem 1 は次のように書き換えることができる。

**Theorem 6.** Let \((W, S)\) be a Coxeter system and \( \Gamma \) a torsion-free subgroup of finite index in \( W \). Then

\[
H^{*}(\Gamma; \mathbb{Z}\Gamma) \cong \tilde{H}^{*-1}(-L) \bigoplus_{\tilde{T} \neq \tilde{T} \in S^{f}} \bigoplus_{T \in S^{f}} \tilde{H}^{*-1}(L_{S\backslash T})
\]

\[
\cong \tilde{H}^{*-1}(\tilde{L}) \bigoplus_{\emptyset \neq \tilde{T} \in S^{f}} \bigoplus_{\tilde{T} \neq \tilde{T} \in S^{f}} \tilde{H}^{*-1}(\tilde{L}_{S\backslash T}),
\]

where \( \tilde{S} \) is the subset of \( S \) such that \( W_{\tilde{S}} \) is the essential parabolic subgroup in \( W \), \( \tilde{T} = S \setminus \tilde{S}, \tilde{L} = L(W_{\tilde{S}}, \tilde{S}) \) and \( \tilde{S}^{f} = S^{f}(W_{\tilde{S}}, \tilde{S}) = S^{f} \cap \tilde{S} \).
Proof. We note that $W^\tilde{T}$ is a singleton by Lemma 2. By Theorem 1 and Lemma 5, we have that

$$H^*(\Gamma; \mathbb{Z}\Gamma) \cong \tilde{H}^{*-1}(L_{S\backslash \tilde{T}}) \oplus \bigoplus_{\tilde{T} \neq T \in S^f} \bigoplus_{\mathbb{Z}} \tilde{H}^{*-1}(L_{S\backslash T}).$$

If $\tilde{T} \not\subset T$ (i.e., $\tilde{T} \setminus T$ is nonempty), then $L_{S\backslash T}$ is contractible by Lemma 4. Hence,

$$H^*(\Gamma; \mathbb{Z}\Gamma) \cong \tilde{H}^{*-1}(L_{S\backslash \tilde{T}}) \oplus \bigoplus_{\tilde{T} \not\subset T \in S^f} \bigoplus_{\mathbb{Z}} \tilde{H}^{*-1}(L_{S\backslash T}).$$

The parabolic subgroup $W_{\tilde{S}}$ has a finite index in $W$, and $W_{\tilde{S}}$ is the essential parabolic subgroup in the Coxeter system $(W_{\tilde{S}}, \tilde{S})$. Therefore,

$$H^*(\Gamma; \mathbb{Z}\Gamma) \cong \tilde{H}^{*-1}(L_{S\backslash \tilde{T}}) \oplus \bigoplus_{\tilde{T} \subsetneqq T \in S^f} \bigoplus_{\mathbb{Z}} \tilde{H}^{*-1}(L_{S\backslash T}).$$

by Theorem 1 and Lemma 5. ■

この Theorem 6 により、直ちに次の系を得ることができる。

Corollary 7. Let $(W, S)$ be a Coxeter system, $\Gamma$ a torsion-free subgroup of finite index in $W$, $\tilde{S}$ the subset of $S$ such that $W_{\tilde{S}}$ is the essential parabolic subgroup in $W$, and $\tilde{T} = S \setminus \tilde{S}$. Then the following statements are equivalent:

1. $H^i(\Gamma; \mathbb{Z}\Gamma)$ is finitely generated;
2. $H^i(\Gamma; \mathbb{Z}\Gamma)$ is isomorphic to $\tilde{H}^{i-1}(L_{\tilde{S}})$;
3. $\tilde{H}^{i-1}(L_{S\backslash \tilde{T}}) = 0$ for each $\tilde{T} \subsetneqq T \in S^f$.

ここで例を一つ与える。

Example. It is known that, for every finite simplicial complex $M$, there exists a Coxeter system $(W, S)$ such that $L(W, S)$ is equal to the barycentric subdivision of $M$ ([D1, Lemma 11.3]).

Let $(W, S)$ be a Coxeter system such that $L = L(W, S)$ is the barycentric subdivision of a triangulation of the projective plane. In [Dr], Dranishnikov showed
that \( \text{vcd}_Z W = 3 \) and \( \text{vcd}_Q W = 2 \), where \( \text{vcd}_R W \) is the virtual cohomological dimension of \( W \) over \( R \). Now, using Theorem 6, we calculate the cohomology of a torsion-free subgroup \( \Gamma \) of finite index in \( W \).

Since \( L \) is the projective plane,

\[
\tilde{H}^i(L) \cong \begin{cases} 
\mathbb{Z}_2, & i = 2, \\
0, & i \neq 2.
\end{cases}
\]

Since \( L = L_{S \setminus \emptyset} \) is not contractible and \( W^\emptyset \) is a singleton, \( \hat{T} \) is the empty set (i.e., \( W = W_S \) is the essential parabolic subgroup) by Lemma 5. For each \( T \in S^f \setminus \{\emptyset\} \), \( L_{S \setminus T} \) has the same homotopy type as a circle. Hence,

\[
\tilde{H}^i(L_{S \setminus T}) \cong \begin{cases} 
\mathbb{Z}, & i = 1, \\
0, & i \neq 1.
\end{cases}
\]

Therefore, by Theorem 6, we have

\[
H^i(\Gamma; \mathbb{Z}\Gamma) \cong \tilde{H}^{i-1}(L) \oplus (\bigoplus_{\emptyset \neq T \in S^f} \tilde{H}^{-1}(L_{S \setminus T}))
\]

\[
\cong \begin{cases} 
\mathbb{Z}_2, & i = 3, \\
\mathbb{Z} \oplus \mathbb{Z} \oplus \cdots, & i = 2, \\
0, & \text{otherwise}.
\end{cases}
\]

Corollary 8. Let \((W, S)\) be a Coxeter system, \( \Gamma \) a torsion-free subgroup of finite index in \( W \), and \( \tilde{S} \) the subset of \( S \) such that \( W_{\tilde{S}} \) is the essential parabolic subgroup in \( W \). Then the following statements are equivalent:

1. \( H^i(\Gamma; \mathbb{Z}\Gamma) \) is finitely generated for each \( i \).
2. \( H^i(\Gamma; \mathbb{Z}\Gamma) \) is isomorphic to \( \tilde{H}^{i-1}(L_{\tilde{S}}) \) for each \( i \).
3. \( \Gamma \) is a Poincaré duality group.
Definition. Let \((W, S)\) be a Coxeter system and let \(WS^j\) be the set of all cosets of the form \(wW_T\), with \(w \in W\) and \(T \in S^f\). The set \(WS^j\) is partially ordered by inclusion. The contractible simplicial complex \(\Sigma\) is defined as the geometric realization of the partially ordered set \(WS^f\) ([D3, §3], [D1]). If \(W\) is infinite, then \(\Sigma\) is noncompact.

\(\Sigma\) は次のような性質をもつことが知られている。

Remark. It is known that \(\Sigma\) can be cellulated so that the link of each vertex is \(L\) ([D2, §9, §10], [M]). In [M], G. Moussong proved that a natural metric on \(\Sigma\) satisfies the CAT(0) condition. Hence, if \(W\) is infinite, \(\Sigma\) can be compactified by adding its ideal boundary \(\partial \Sigma\) ([D2, §4]). It is known that

\[
H^*(\Gamma; \mathbb{Z} \Gamma) \cong H^*(W; \mathbb{Z} W) \cong H^*_c(\Sigma) \cong \check{H}^{*-1}(\partial \Sigma),
\]

where \(\Gamma\) is a torsion-free subgroup of finite index in \(W\). Here \(H^*_c\) and \(\check{H}^*\) denote the compactly supported cohomology and the Čech reduced cohomology, respectively.

実際, Davis によって証明された Theorem 1 は、この \(\Sigma\) の cohomology \(H^*_c(\Sigma)\) を計算することによって得られている。

上の Remark と Corollary 8 から直ちに次を得る。

Corollary 9. Let \(W\) be a Coxeter group.

1. \(H^i(W; \mathbb{Z}W)\) is finitely generated for each \(i\) if and only if \(W\) is a virtual Poincaré duality group.

2. Suppose that \(W\) is infinite. Then \(\check{H}^i(\partial \Sigma)\) is finitely generated for each \(i\) if and only if the Čech cohomology of \(\partial \Sigma\) is isomorphic to the cohomology of an \(n\)-sphere for some \(n\).
References


