<table>
<thead>
<tr>
<th>Title</th>
<th>STRONG $n$-SHAPE THEORY (Research in General and Geometric)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>Iwamoto, Yutaka; Sakai, Katsuro</td>
</tr>
<tr>
<td>Citation</td>
<td>数理解析研究所講究録 1126: 19-27</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2000-01</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/63599">http://hdl.handle.net/2433/63599</a></td>
</tr>
<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
</tr>
</tbody>
</table>

Kyoto University
STRONG $n$-SHAPE THEORY

YUTAKA IWAMOTO AND KATSUMORO SAKAI

INTRODUCTION

Let $\mu^{n+1}$ be the $(n+1)$-dimensional universal Menger compactum. In [Chi$_1$], A. Chigogidze introduced the concept of $n$-shape and established the $(n+1)$-dimensional analogue of Chapman's complement theorem [Cha, Theorem 2], that is, two $Z$-sets $X$ and $Y$ in $\mu^{n+1}$ have the same $n$-shape type if and only if their complements $\mu^{n+1} \setminus X$ and $\mu^{n+1} \setminus Y$ are homeomorphic ($\simeq$), where $X \subset M$ is a $Z$-set in $M$ if there are maps $f: M \rightarrow M \setminus X$ arbitrarily close to $\text{id}_M$. The $n$-shape category of compacta was discussed in [Chi$_2$] (cf. [Chi$_3$]). Later, corresponding to [Cha, Theorem 1], Y. Akaike [Aka] defined the weak proper $n$-homotopy category of complements of $Z$-sets in $\mu^{n+1}$ which is isomorphic to the $n$-shape category of $Z$-sets in $\mu^{n+1}$. Then, as Strong Shape Theory ([EH], [DS], [KO], etc.), it is a natural attempt to define the strong $n$-shape category which corresponds to the proper $n$-homotopy category of complements of $Z$-sets in $\mu^{n+1}$. Properly, one require this category to factorize the natural functor (called the $n$-shape functor) from the $n$-homotopy category to the $n$-shape category into two functors through it. In this paper, we introduce the $(n+1)$-skeletal conic telescope to define the strong $n$-shape category of compacta.

Throughout the paper, spaces are separable metrizable and maps are continuous. It is said that two (proper) maps $f, g: X \rightarrow Y$ are (properly) $n$-homotopic relative to $A \subset X$ and denoted by $f \simeq^n g$ rel. $A$ ($f \simeq^n_p g$ rel. $A$) if, for any (proper) map $\varphi: Z \rightarrow X$, there is a (proper) homotopy $h: Z \times I \rightarrow Y$ such that $h_0 = f\varphi$, $h_1 = g\varphi$ $h_t|\varphi^{-1}(A) = f\varphi|\varphi^{-1}(A)$ for each $t \in I$. When $A = \emptyset$, we say that $f$ and $g$ are (properly) $n$-homotopic and denote $f \simeq g$ ($f \simeq^n_p g$).

A map $\varphi: M \rightarrow X$ is said to be $n$-invertible if any map $\psi: Z \rightarrow X$ of a space $Z$ with $\dim Z \leq n$ lifts to $M$, that is, there exists a map $\tilde{\psi}: Z \rightarrow M$ such that $\varphi\tilde{\psi} = \psi$. In case $\varphi$ is a proper map, if $\psi$ is proper then $\tilde{\psi}$ is also proper. For an $n$-invertible map $\varphi: M \rightarrow X$ and $A \subset X$, $\varphi|\varphi^{-1}(A): \varphi^{-1}(A) \rightarrow A$ is also $n$-invertible. By the result of Dranishnikov [Dra, Theorem 1], for any compactum $X$, there exists an $n$-invertible map $\varphi: M \rightarrow X$ of a compactum $M$ with $\dim M \leq n$. Then, for two (proper) maps $f, g: X \rightarrow Y$, $f \simeq^n g$ rel. $A$ ($f \simeq^n_p g$ rel. $A$) if and only if $f\varphi \simeq g\varphi$ rel. $\varphi^{-1}(A)$ ($f\varphi \simeq^n_p g\varphi$ rel. $\varphi^{-1}(A)$) for an invertible (proper) map $\varphi: M \rightarrow X$.

1991 Mathematics Subject Classification. 54C56, 57N25.

Key words and phrases. The universal Menger compactum, $Z$-sets, $n$-homotopy, the proper $n$-homotopy category, the strong $n$-shape category.

This research was supported by Grant-in-Aid for Scientific Research (No. 10640060), Ministry of Education, Science and Culture, Japan.

Typeset by A4\$-\LaTeX\$
1. The Polyhedral Telescope

The $n$-skeleton of a simplicial complex $K$ is denoted by $K^{(n)}$, whence $K^{(0)}$ is the set of vertices of $K$. The polyhedron of $K$ is denoted by $|K|$ (i.e., $|K| = \bigcup_{\sigma \in K} \sigma$).

By $<v_1, \ldots, v_n>$, we denote the simplex with vertices $v_1, \ldots, v_n$. A subdivision $\delta K$ of $K$ induces the subdivision $\delta K^{(n)}$ of $K^{(n)}$. It should be remarked that $\delta K^{(n)} \subset (\delta K)^{(n)}$ but $\delta K^{(n)} \neq (\delta K)^{(n)}$ in general. The following is well known:

**Fact 1.** Let $L$ be a subcomplex of $K$ and $Z$ a space with $\dim Z \leq n$. Then, for any map $\varphi: Z \to |K|$, there is a map $\psi: Z \to |K^{(n)} \cup L|$ such that $\varphi \simeq \psi$ rel. $\varphi^{-1}(|K^{(n)} \cup L|)$.

An ordered simplicial complex is a simplicial complex with an order of vertices such that the set of vertices of each simplex is totally ordered. The barycentric subdivision $\text{Sd} K$ of a simplicial complex $K$ is an ordered simplicial complex with the following order:

$$\hat{\sigma} \preceq \hat{\tau} \iff \text{def} \quad \sigma \text{ is a face of } \tau,$$

where $\hat{\sigma}$ is the barycenter of $\sigma$.

Let $I = \{0,1, \mathbb{I}\}$ be the natural triangulation of the unit interval $\mathbb{I} = [0,1]$. Then, $I$ is an ordered simplicial complex with the natural order $0 < 1$. For an ordered simplicial complex $K$, the product simplicial complex $K \times I$ is defined as follows:

$$K \times I = \{ \sigma \times \{0\}, \sigma \times \{1\} \mid \sigma \in K \} \cup \{ \langle (v_1, 0), \ldots, (v_k, 0), (v_j, 1), \ldots, (v_k, 1) \rangle \mid \langle v_1, \ldots, v_k \rangle \in K, \quad v_1 < \cdots < v_k \in K^{(0)}, \quad 1 \leq i \leq j \leq k \}.$$ 

Then $K \times I$ is an ordered simplicial complex with the following order on $(K \times I)^{(0)} = K^{(0)} \times \{0,1\}$:

$$(v, i) \preceq (v', i') \iff \text{def} \quad v \leq v' \text{ and } i \leq i'.$$

Let $K$ and $L$ be ordered simplicial complexes and $f: K \to L$ a simplicial map. The simplicial mapping cylinder $M(f)$ is defined as follows:

$$M(f) = K \cup L \cup \{ \langle f(v_1), \ldots, f(v_i), v_j, \ldots, v_k \rangle \mid \langle v_1, \ldots, v_k \rangle \in K, \quad v_1 < \cdots < v_k, \quad 1 \leq i \leq j \leq k \}.$$ 

When $L$ is degenerate (i.e., a singleton), $M(f)$ is the simplicial cone $C(K)$ over $K$.

We have the natural simplicial map $q_f: K \times I \to M(f)$ which is naturally defined by $q_f(v,0) = f(v)$ and $q_f(v,1) = v$ for $v \in K^{(0)}$. The simplicial collapsing map $c_f: M(f) \to L$ is defined by $c_f(v) = f(v)$ for $v \in K^{(0)}$ and $c_f(u) = u$ for $u \in L^{(0)}$. Then $c_f q_f = f \mid K$ and $c_f \simeq \text{id rel. } |L| \text{ in } |M(f)|$. Extending the orders on $K^{(0)}$ and $L^{(0)}$ to $M(f)^{(0)} = K^{(0)} \cup L^{(0)}$ so that $u \prec v$ for each $u \in L^{(0)}$ and $v \in K^{(0)}$, $M(f)$ is an ordered simplicial complex. Let $f^{(n)} = f \mid K^{(n)}: K^{(n)} \to L^{(n)}$ be the restriction of $f$. Observe that

$$M(f)^{(n)} \subset M(f^{(n)}) \subset M(f^{(n+1)}) \subset M(f^{(n)}) \cup K \cup L$$

and $c_f M(f^{(n)}) = c_{f^{(n)}} \simeq \text{id rel. } |L^{(n)}| \text{ in } |M(f^{(n)})|$. 

Fact 2. For a simplicial map \( f : K \to L \), \( c_f\|M(f)^{(n+1)} \cup K \cup L\) \( \cong \) id rel. \( |L| \) in \( |M(f)^{(n+1)} \cup K \cup L| \), hence \( f = c_f|K \cong \) id\(_K\) in \( |M(f)^{(n+1)} \cup K \cup L| \).

Since \( K \times I \) can be regarded as \( M(\text{id}_K) \), we have the following:

Fact 3. Let \( p : |(K^{(n)} \times I) \cup (K \times \{0, 1\})| \to |K \times \{0\}| \) be the retraction defined by \( p(x, t) = (x, 0) \). Then, \( p \cong \) id rel. \( |K \times \{0\}| \) in \( |(K^{(n)} \times I) \cup (K \times \{0, 1\})| \), where we identify \( K = K \times \{0\} \).

Let \( K = (|K_i|, q_{i,i+1})_{i \in \mathbb{N}} \) be an inverse sequence of ordered simplicial complexes such that each \( q_{i,i+1} : K_{i+1} \to \delta K_i \) is simplicial, where \( \delta K_i \) is some subdivision of \( K_i \). Let \( q_i : \lim K \to |K_i| \) be the projection of the inverse limit of \( K \) to \( |K_i| \) and denote
\[
q_{i,j} = q_{i,i+1} \circ \cdots \circ q_{j-1,j} : |K_j| \to |K_i|, \quad i < j.
\]

We define
\[
\text{Tel}_{[0,\infty)}(K) = \bigcup_{i=0}^{\infty} |M(q_i)| \quad \text{and} \quad \text{Tel}_{[j,k]}(K) = \bigcup_{i=j}^{k} |M(q_{i,j})|,
\]
for \( j < k \). The polyhedron \( \text{Tel}_{[1,\infty)}(K) \) is called the polyhedral telescope for \( K \). One should note that \( \bigcup_{i=1}^{\infty} M(q_i) \) is not a simplicial complex unless \( \delta K_i = K_i \) for every \( i \in \mathbb{N} \). Let
\[
\text{Tel}_{[0,\infty)}(K) = |C(K_1)| \cup \text{Tel}_{[1,\infty)}(K) \quad \text{and} \quad \text{Tel}_{[0,k]}(K) = |C(K_1)| \cup \text{Tel}_{[1,k]}(K),
\]
where \( |C(K_1)| \cap \text{Tel}_{[1,\infty)}(K) = |K_1| \). We call \( \text{Tel}_{[0,\infty)}(K) \) the polyhedral conic telescope.

The simplicial collapsing map \( c_{q_{i,i+1}} : M(q_{i,i+1}) \to \delta K_i \) extends to the deformation retraction
\[
c_{i,i+1}^K : \text{Tel}_{[0,i+1]}(K) = \text{Tel}_{[0,i]}(K) \cup |M(q_{i,i+1})| \to T_{[0,i]}(K).
\]

The following diagram is commutative:
\[
\begin{array}{cccccc}
\text{Tel}_{[0,1]}(K) & \xrightarrow{c_{1,2}} & \text{Tel}_{[0,2]}(K) & \xrightarrow{c_{2,3}} & \text{Tel}_{[0,3]}(K) & \xrightarrow{c_{3,4}} & \cdots \\
& \cup & & \cup & & \cup & \\
|K_1| & \xleftarrow{q_{1,2}} & |K_2| & \xleftarrow{q_{2,3}} & |K_3| & \xleftarrow{q_{3,4}} & \cdots
\end{array}
\]

The inverse limit of the upper sequence is denoted by \( \text{Tel}_{[0,\infty)}(K) \) with the projection \( c_i^K : \text{Tel}_{[0,\infty)}(K) \to \text{Tel}_{[0,i]}(K) \). We denote
\[
c_{i,j}^K = c_{i,i+1} \circ \cdots \circ c_{j-1,j} : \text{Tel}_{[0,j]}(K) \to \text{Tel}_{[0,i]}(K), \quad i < j.
\]

Regarding \( \text{Tel}_{[0,\infty)}(K) \) as an open subspace of \( \text{Tel}_{[0,\infty)}(K) \), we have
\[
\text{Tel}_{[0,\infty)}(K) \setminus \text{Tel}_{[0,\infty)}(K) = \lim_{\arrow} K \quad \text{and} \quad c_i^K|_{\lim K} = q_i, \quad i \in \mathbb{N}.
\]
It is easy to see that each $c^K_i$ is a strong deformation retraction. Hence, it follows that $\text{Tel}_{[0,\infty)}(K)$ is homotopy dense in $\text{Tel}_{[0,\infty)}(K)$, that is, there is a homotopy $h: \text{Tel}_{[0,\infty)}(K) \times I \to \text{Tel}_{[0,\infty)}(K)$ such that $h_0 = \text{id}$ and $h_t(\text{Tel}_{[0,\infty)}(K)) \subset \text{Tel}_{[0,\infty)}(K)$ for $t > 0$. Since $\text{Tel}_{[0,\infty)}(K)$ is a polyhedron, $\text{Tel}_{[0,\infty)}(K)$ is an ANR by Hanner's characterization of ANR's (cf. [Hu]). Since $\text{Tel}_{[0,\infty)}(K)$ is contractible, it is an AR. The above construction was founded in [Ko, Theorem 1 and Corollary 1]. For each $j \in \mathbb{N}$, we can similarly define $\text{Tel}_{[j,\infty)}(K)$, which is an ANR and a closed subspace of $\text{Tel}_{[0,\infty)}(K)$. Clearly,

$$\text{Tel}_{[j,\infty)}(K) \setminus \text{Tel}_{[j,\infty)}(K) = \text{Tel}_{[0,\infty)}(K) \setminus \text{Tel}_{[0,\infty)}(K) = \lim \downarrow K.$$ 

Each $d^K_j = c^K_j|\text{Tel}_{[j,\infty)}(K)$: $\text{Tel}_{[j,\infty)}(K) \to |K_j|$ is a strong deformation retraction and $q_{i,j}d^K_j = d^K_i|\text{Tel}_{[j,\infty)}(K)$.

Now, we define

$$\text{Tel}^{n+1}_{[j,\infty)}(K) = \bigcup_{i=j}^{\infty} |K_i| \cup \bigcup_{i=j}^{\infty} |M(q_{i,i+1})^{(n+1)}|$$

and

$$\text{Tel}^{n+1}_{[j,k]}(K) = \bigcup_{i=j}^{k} |K_i| \cup \bigcup_{i=j}^{k-1} |M(q_{i,i+1})^{(n+1)}|, \ j < k.$$ 

These are subpolyhedra of $\text{Tel}_{[1,\infty)}(K)$. Recall that $\bigcup_{i=1}^{\infty} M(q_i)$ is not a simplicial complex in general. We call $\text{Tel}^{n+1}_{[1,\infty)}(K)$ the $(n+1)$-skeletal telescope for $K$. Let

$$\text{Tel}^{n+1}_{[0,\infty)}(K) = |C(K_1)^{(n+1)}| \cup \text{Tel}^{n+1}_{[1,\infty)}(K) \quad \text{and}$$

$$\text{Tel}^{n+1}_{[0,k]}(K) = |C(K_1)^{(n+1)}| \cup \text{Tel}^{n+1}_{[1,k]}(K).$$

These are $n$-connected. The polyhedron $\text{Tel}^{n+1}_{[0,\infty)}(K)$ is called the $(n+1)$-skeletal conic telescope for $K$.

Observe that $c^K_i(\text{Tel}^{n+1}_{[0,i+1]}(K)) = \text{Tel}^{n+1}_{[0,i]}(K)$. The following diagram is commutative:

$$\begin{array}{cccccc}
\text{Tel}^{n+1}_{[0,1]}(K) & \xrightarrow{c^K_{1,2}} & \text{Tel}^{n+1}_{[0,2]}(K) & \xrightarrow{c^K_{2,3}} & \text{Tel}^{n+1}_{[0,3]}(K) & \xrightarrow{c^K_{3,4}} & \cdots \\
\cup & & \cup & & \cup & & \cdots \\
|K_1| & \xleftarrow{q_{1,2}} & |K_2| & \xleftarrow{q_{2,3}} & |K_3| & \xleftarrow{q_{3,4}} & \cdots
\end{array}$$

Then the inverse limit of the upper sequence is the closed subspace

$$\text{Tel}^{n+1}_{[0,\infty)}(K) = \text{Tel}^{n+1}_{[0,\infty)}(K) \cup \lim \downarrow K \subset \text{Tel}_{[0,\infty)}(K).$$

For each $j \in \mathbb{N}$, let $\text{Tel}^{n+1}_{[j,\infty)}(K) = \text{Tel}^{n+1}_{[j,\infty)}(K) \cup \lim \downarrow K$. 

Fact 4. For each $j \in \mathbb{N} \cup \{0\}$, $\text{Tel}_{[j,\infty)}^{n+1}(K) \setminus \text{Tel}_{[j,\infty)}^{n+1}(K) = \lim_{\rightarrow} K$ is a $Z$-set in $\text{Tel}_{[j,\infty)}^{n+1}(K)$.

Let $\psi: Z \to \text{Tel}_{[j,\infty)}^{n+1}(K)$ be a map of a space $Z$ with $\dim Z \leq n$. Then it is easy to construct a homotopy $h: Z \times I \to \text{Tel}_{[i,j]}^{n+1}(K)$ such that $h_0 = \psi$ and $h_t(Z) \subset \text{Tel}_{[i,j]}^{n+1}(K)$ for $t > 0$. In general, $\text{Tel}_{[0,\infty)}^{n+1}(K)$ is not an ANR, but we have the following:

Fact 5. Each $\text{Tel}_{[j,\infty)}^{n+1}(K)$ is $LC^m$, hence it is an $\text{ANE}(n+1)$. Moreover, the space $\text{Tel}_{[0,\infty)}^{n+1}(K)$ is $n$-connected, so it is an $\text{AE}(n+1)$.\footnote{A space $Y$ is an $\text{AE}(n+1)$ (or an $\text{ANE}(n+1)$) if every map of any closed set $A$ in an arbitrary metrizable space $X$ with $\dim X \leq n + 1$ extends over $X$ (or a neighborhood of $A$). A space $Y$ is an $\text{AE}(n+1)$ if and only if $Y$ is an $n$-connected $\text{ANE}(n)$, and $Y$ is an $\text{ANE}(n+1)$ if and only if $Y$ is $\text{LC}^m$.}

The following follows from Fact 2:

Fact 6. For $i < j \in \mathbb{N} \cup \{0\}$, $d_{i,j}^{K}|\text{Tel}_{[i,j]}^{n+1}(K) \simeq^n \text{id}$ in $\text{Tel}_{[i,j]}^{n+1}(K)$, hence $q_{i,j} \simeq^n \text{id}_{K_j}$ in $\text{Tel}_{[i,j]}^{n+1}(K)$. Moreover, $d_{i}^{K}|\text{Tel}_{[i,\infty)}^{n+1}(K) \simeq^n \text{id}$ in $\text{Tel}_{[i,\infty)}^{n+1}(K)$, so $q_{i} \simeq^n \text{id}_{K_j}$ in $\text{Tel}_{[i,\infty)}^{n+1}(K)$.

2. The Strong n-shape Category $\text{Sh}^n_S$

Let $\mathcal{H}^n$ be the $n$-homotopy category of compacta and $\text{Sh}^n$ the $n$-shape category of compacta. In this section, we define the strong $n$-shape category $\text{Sh}^n_S$ of compacta and show that the $n$-shape functor from $\mathcal{H}^n$ to $\text{Sh}^n$ is factorized into two functors through the category $\text{Sh}^n_S$.

Every compactum $X$ is the limit of an inverse sequence $K = (K_i, q_i)_{i \in \mathbb{N}}$ of finite simplicial complexes such that each $q_{i,i+1}: K_{i+1} \to \text{Sd} K_i$ is simplicial for the barycentric subdivision $\text{Sd} K_i$ of $K_i$ and $\dim K_i \leq \dim X$ for all $i \in \mathbb{N}$ [Ish, Lemma 33] (cf. Proof of [Ko2, Theorem 1]). We call $K$ a barycentric sequence associated with $X$. It should be noted that $q_{i,i+1}: K_{i+1} \to K_i$ is not simplicial in general. In fact, there exists a 1-dimensional compact AR which is not the limit of any inverse sequence of simplicial complexes and simplicial maps [Ko1, Theorem 1(2)] (cf. [Ko2, p.536]). It should be also noted that a barycentric sequence associated with $X$ is an $\text{LC}^m(n+1)$-sequence associated with $X$ (cf. [Ch1]).

Theorem 1. Let $X$ and $Y$ be compacta and $K$, $L$ be barycentric sequences associated with $X$ and $Y$, respectively.

1. Every map $f: X \to Y$ extends to a map $\bar{f}: \text{Tel}_{[0,\infty)}(K) \to \text{Tel}_{[0,\infty)}(L)$ such that $\bar{f}(|\text{Tel}_{[0,\infty)}^{k}(K)|) \subset \text{Tel}_{[0,\infty)}^{k}(L)$ for each $k \in \mathbb{N}$.

2. For two maps $f, g: \text{Tel}_{[0,\infty)}^{n+1}(K) \to \text{Tel}_{[0,\infty)}^{n+1}(L)$ with $f^{-1}(Y) = g^{-1}(Y) = X$, if $f|X \simeq^n g|X$ in $Y$ then $f|\text{Tel}_{[0,\infty)}^{n+1}(K) \simeq^n g|\text{Tel}_{[0,\infty)}^{n+1}(K)$ in $\text{Tel}_{[0,\infty)}^{n+1}(L)$.

In Theorem 1(1) above, a proper map $\bar{f}|\text{Tel}_{[0,\infty)}^{n+1}(K): \text{Tel}_{[0,\infty)}^{n+1}(K) \to \text{Tel}_{[0,\infty)}^{n+1}(L)$ is said to be induced by $f$. By Theorem 1(2), the proper homotopy class of such a map is unique. The following is a direct consequence of Theorem 1.
Corollary 1. Let $K$ and $L$ be barycentric sequences associated with the same compactum $X$. Then a proper map $h: \text{Tel}^{n+1}_{[0,\infty)}(K) \to \text{Tel}^{n+1}_{[0,\infty)}(L)$ induced by $\text{id}_X$ is a proper $n$-homotopy equivalence.

Definition of $\text{Sh}_S^n$. Let $X$ and $Y$ be compacta. Let $K, K'$ be barycentric sequences associated with $X$ and $L, L'$ barycentric sequences associated with $Y$. Two proper maps $F: \text{Tel}^{n+1}_{[0,\infty)}(K) \to \text{Tel}^{n+1}_{[0,\infty)}(L)$ and $F': \text{Tel}^{n+1}_{[0,\infty)}(K') \to \text{Tel}^{n+1}_{[0,\infty)}(L')$ are $n$-fundamentally equivalent (written by $F \simeq^n_p F'$) if $h'F \simeq^n_p F'h$ for some proper $n$-homotopy equivalences $h: \text{Tel}^{n+1}_{[0,\infty)}(K) \to \text{Tel}^{n+1}_{[0,\infty)}(K')$ and $h': \text{Tel}^{n+1}_{[0,\infty)}(L') \to \text{Tel}^{n+1}_{[0,\infty)}(L)$ induced by $\text{id}_X$ and $\text{id}_Y$, respectively. A strong $n$-shape morphism from $X$ to $Y$ is the $n$-fundamentally equivalence class of a proper map $F: \text{Tel}^{n+1}_{[0,\infty)}(K) \to \text{Tel}^{n+1}_{[0,\infty)}(L)$, where $K$ and $L$ are barycentric sequences associated with $X$ and $Y$ respectively. Thus, the strong $n$-shape category $\text{Sh}_S^n$ of compacta can be defined.

The following follows immediately from Theorem 1 and the definition above.

Corollary 2. There exists a functor $\Xi: \mathcal{H}^n \to \text{Sh}_S^n$ which maps objects identically.

For simplicity, let us assign each compactum $X$ to a barycentric sequence $K^X = (K^X_i, q^X_{i,i+1})_{i \in \mathbb{N}}$ associated with $X$ and denote as follows:

$$
\text{Tel}^{n+1}_{[0,\infty)}(X) = \text{Tel}^{n+1}_{[0,\infty)}(K^X), \quad \text{Tel}^{n+1}_{[j,k]}(X) = \text{Tel}^{n+1}_{[j,k]}(K^X),
$$

$$
c^X_{i,i+1} = c^X_{i,i+1} | \text{Tel}^{n+1}_{[0,i+1]}(K^X), \quad c^X_{i} = c^X_{i} | \text{Tel}^{n+1}_{[0,\infty)}(K^X),
$$

$$
d^X_{i} = d^X_{i} | \text{Tel}^{n+1}_{[i,\infty)}(K^X), \text{ etc.}
$$

Thus, $X$ is assigned to the following commutative diagram of inverse sequences:

$$
\begin{array}{cccccc}
\text{Tel}^{n+1}_{[0,1]}(X) & \xleftarrow{c^X_{1,2}} & \text{Tel}^{n+1}_{[0,2]}(X) & \xleftarrow{c^X_{2,3}} & \text{Tel}^{n+1}_{[0,3]}(X) & \xleftarrow{c^X_{3,4}} & \cdots \\
\cup & & \cup & & \cup & & \\
|K^X_1| & \xleftarrow{q^X_{1,2}} & |K^X_2| & \xleftarrow{q^X_{2,3}} & |K^X_3| & \xleftarrow{q^X_{3,4}} & \cdots \\
\end{array}
$$

Now, we prove the following:

Theorem 2. There exists a full\footnote{The functor is full if the induced maps of the sets of morphisms are surjective.} functor $\Theta: \text{Sh}_S^n \to \text{Sh}^n$ such that $\Theta \circ \Xi: \mathcal{H}^n \to \text{Sh}^n$ is the $n$-shape functor.

Remarks. The following proposition can be proved similarly to Theorem 1(1).

Proposition. Let $K$ and $L$ be barycentric sequences associated with compacta $X$ and $Y$, respectively. Every proper map $f: \text{Tel}^{n+1}_{[0,\infty)}(K) \to \text{Tel}^{n+1}_{[0,\infty)}(L)$ is properly homotopic to a proper map $\bar{f}: \text{Tel}^{n+1}_{[0,\infty)}(K) \to \text{Tel}^{n+1}_{[0,\infty)}(L)$ such that $\bar{f}(\text{Tel}^{n+1}_{[0,\infty)}(K)) \subset \text{Tel}^{n+1}_{[0,\infty)}(L)$ for each $k \in \mathbb{N}$.

By the same proof, Theorem 1(2) is valid even if $\text{Tel}^{n+1}_{[0,\infty)}$ is replaced with $\text{Tel}_{[0,\infty]}$. Then, in the definition of $\text{Sh}_S^n$, replacing $\text{Tel}^{n+1}_{[0,\infty)}$ by $\text{Tel}_{[0,\infty]}$, we can define the
category \( \overline{\mathrm{Sh}}_S^n \) which factorizes the \( n \)-shape functor into two functors through \( \overline{\mathrm{Sh}}_S^n \). In fact, the functor \( \Xi \) in Corollary 2 is factorized into two natural functors through \( \overline{\mathrm{Sh}}_S^n \), where the natural functor from \( \overline{\mathrm{Sh}}_S^n \) to \( \mathrm{Sh}_S^n \) can be obtained by the proposition above. As is easily observed, the functor from \( \mathrm{Sh}_S^n \) to \( \mathrm{Sh}_S^n \) is injective, but it is a problem whether it is surjective or not.

\[
\begin{array}{ccc}
\mathcal{H}^n & \longrightarrow & \mathrm{Sh}^n \\
\downarrow & & \downarrow \\
\overline{\mathrm{Sh}}_S^n & \longrightarrow & \mathrm{Sh}_S^n
\end{array}
\]

In the definition of \( \overline{\mathrm{Sh}}_S^n \), replacing \( \mathrm{Tel}_{[0,\infty)}^n \) and \( \simeq_p^n \) by \( \mathrm{Tel}_{[0,\infty)} \) and \( \simeq_p \), we can obtain the strong shape category \( \mathrm{Sh}_S \) (cf. [DS]). Then, we can easily obtain the natural functor from \( \mathrm{Sh}_S \) to \( \overline{\mathrm{Sh}}_S^n \). Let \( \mathcal{H} \) be the homotopy category of compacta. We have the following diagram of categories and functors:

\[
\begin{array}{ccc}
\mathcal{H} & \longrightarrow & \mathrm{Sh}_S \\
\downarrow & & \downarrow \\
\mathcal{H}^n & \longrightarrow & \overline{\mathrm{Sh}}_S^n
\end{array}
\]

Restricting the objects to compacta with \( \dim \leq k \), we have the subcategories \( \mathrm{Sh}(k), \mathrm{Sh}^n(k), \mathrm{Sh}_S(k), \mathrm{Sh}_S^n(k), \mathrm{Sh}_S^n(k) \) of \( \mathrm{Sh}, \mathrm{Sh}^n, \mathrm{Sh}_S, \mathrm{Sh}_S^n \), respectively. Then, \( \mathrm{Sh}_S^n(n) = \overline{\mathrm{Sh}}_S^n(n) \) because \( \mathrm{Tel}_{[0,\infty)}^n(X) = \mathrm{Tel}_{[0,\infty)}(X) \) if \( \dim X \leq n \). Moreover, \( \mathrm{Sh}_S^n(n - 1) = \overline{\mathrm{Sh}}_S^n(n - 1) = \mathrm{Sh}_S(n - 1) \) because \( \dim \mathrm{Tel}_{[0,\infty)}(X) \leq n \) if \( \dim X \leq n - 1 \). Although \( \mathrm{Sh}^n(n) = \mathrm{Sh}(n) \), it is not known whether \( \overline{\mathrm{Sh}}_S^n(n) = \mathrm{Sh}_S(n) \) or not.

3. An isomorphism between \( \overline{\mathrm{Sh}}_S^n(Z(\mu^{n+1})) \) and \( \mathcal{H}^n_p(M_{n+1}) \)

Let \( Z(\mu^{n+1}) \) be the class of \( Z \)-sets in \( \mu^{n+1} \) and \( M_{n+1} \) the class of \( \mu^{n+1} \)-manifolds. In this section, we prove that the strong \( n \)-shape category \( \overline{\mathrm{Sh}}_S^n(Z(\mu^{n+1})) \) of \( Z(\mu^{n+1}) \) is categorically isomorphic to the proper \( n \)-homotopy category \( \mathcal{H}^n_p(M_{n+1}) \) of \( M_{n+1} \).

Lemma 1. Let \( f : X \to Y \) be a map from a locally compact separable metrizable space \( X \) with \( \dim X \leq n + 1 \) to a completely metrizable ANE\((n + 1)\) \( Y \). For any closed set \( A \subset X \) and a \( Z \)-set \( B \subset Y \), \( f \) is approximated by maps \( g : X \to Y \) such that \( g|A = f|A \) and \( g(X \setminus A) \subset Y \setminus B \).

As in §2, we assign each \( X \in Z(\mu^{n+1}) \) to the following diagram:

\[
\begin{array}{ccccccc}
\mathrm{Tel}_{[0,\infty)}^n(X) & \overset{c_{X,2}}{\leftarrow} & \mathrm{Tel}_{[0,2]}^n(X) & \overset{c_{X,3}}{\leftarrow} & \mathrm{Tel}_{[0,3]}^n(X) & \overset{c_{X,4}}{\leftarrow} & \cdots \\
\mathrm{Tel}_{[0,\infty)}^n(X) & \overset{c_{X,2}}{\leftarrow} & \mathrm{Tel}_{[0,2]}^n(X) & \overset{c_{X,3}}{\leftarrow} & \mathrm{Tel}_{[0,3]}^n(X) & \overset{c_{X,4}}{\leftarrow} & \cdots \\
|K_1^X| & \overset{q_{1,2}}{\leftarrow} & |K_2^X| & \overset{q_{2,3}}{\leftarrow} & |K_3^X| & \overset{q_{3,4}}{\leftarrow} & \cdots ,
\end{array}
\]
where the lower sequence is a barycentric sequence associated with \( X \). To prove Theorem 3, we apply the construction in [Sa] to this diagram.

Let \( M_i^X = C(K_i^X)(n+1) \). Then \( |M_i^X| = \text{Tel}_{[0,1]}^{n+1}(X) \). We inductively define a simplicial complex

\[
M_{i+1}^X = (\text{Sd} M_i^X \times I)^{(n+1)} \cup M(q_{i,i+1}^X)^{(n+1)},
\]

where we identify \( \text{Sd} M_i^X = M_i^X \times \{0\} \). So we have

\[
M(q_{i,i+1}^X)^{(n+1)} \cap (\text{Sd} M_i^X \times I) = M(q_{i,i+1}^X)^{(n+1)} \cap \text{Sd} M_i^X = \text{Sd} K_i.
\]

Observe that \( \text{Tel}_{[0,i]}^{n+1}(X) = \text{Tel}_{[0,i]}^{n+1}(X) \cup M(q_{i,i+1}^X)^{(n+1)} \subset |M_i^X| \). The simplicial collapsing map \( c_{q_{i,i+1}^X} : M(q_{i,i+1}^X) \rightarrow \text{Sd} K_i^X \) extends to the simplicial retraction

\[
\tilde{c}_{i,i+1} : M_i^X = (\text{Sd} M_{i-1}^X \times I)^{(n+1)} \cup M(q_{i,i+1}^X)^{(n+1)} \rightarrow (\text{Sd} M_{i-1}^X \times I)^{(n+1)}.
\]

We define \( r_{i,i+1}^X = \text{pr}_i \tilde{c}_{i,i+1} : M_{i+1}^X \rightarrow M_i^X \), where \( \text{pr}_i : (\text{Sd} M_i^X \times I) \rightarrow M_i^X \) is the projection. Let \( \pi_i^X = \text{id} : |M_i^X| \rightarrow \text{Tel}_{[0,i]}^{n+1}(X) (= |M_i^X|) \) and inductively define the retraction \( \pi_{i+1}^X : |M_{i+1}^X| \rightarrow \text{Tel}_{[0,i+1]}^{n+1}(X) \) by \( \pi_i^X |M(q_{i,i+1}^X)^{(n+1)}| = \text{id} \) and \( \pi_{i+1}^X |(\text{Sd} M_i^X \times I)^{(n+1)}| = \pi_{i+1}^X \text{pr}_i \). Thus, we obtain the following commutative diagram of the inverse sequences:

\[
\begin{array}{cccccccc}
|M_1^X| & \overset{\pi_1^X}{\leftarrow} & |M_2^X| & \overset{\pi_2^X}{\leftarrow} & |M_3^X| & \overset{\pi_3^X}{\leftarrow} & \cdots \\
\pi_i^X & \downarrow & \pi_i^X & \downarrow & \pi_i^X & \downarrow & \cdots \\
|K_1^X| & \leftarrow & |K_2^X| & \leftarrow & |K_3^X| & \leftarrow & \cdots \\
\end{array}
\]

Recall that \( \text{Tel}_{[0,1]}^{n+1}(X) = \bigcup_{i \in \mathbb{N}} \text{Tel}_{[0,i]}^{n+1}(X) \), \( \text{Tel}_{[0,\infty]}^{n+1}(X) = \text{Tel}_{[0,\infty]}^{n+1}(X) \cup X \) is the inverse limit of the middle sequence and \( X \) is the inverse limit of the bottom sequence. Let \( M^X \) be the inverse limit of the upper sequence. Then \( X \subset \text{Tel}_{[0,\infty]}^{n+1}(X) \subset M^X \) but \( M^X \neq X \cup \bigcup_{i \in \mathbb{N}} |M_i^X| \). Applying Bestvina's characterization of \( \mu^{n+1} \) [Be], one can see that \( M^X \approx \mu^{n+1} \) (cf. [Sa] and [Iwa, Proposition 2.1]). It is easily seen that \( X \) is a \( Z \)-set in \( M^X \) (it is also a \( Z \)-set in \( \text{Tel}_{[0,\infty]}^{n+1}(X) \) [Sa]). Since \( (M^X, X) \approx (\mu^{n+1}, X) \) by the \( Z \)-set unknotting theorem [Be], we have a homeomorphism \( h_X : M^X \setminus X \rightarrow \mu^{n+1} \setminus X \). On the other hand, we have the retraction \( \pi_X : M^X \rightarrow \text{Tel}_{[0,\infty]}^{n+1}(X) \) induced by \( \pi_i^X \). Observe that \( \pi_X |X = \text{id} \) and \( \pi_X (M^X \setminus X) = \text{Tel}_{[0,\infty]}^{n+1}(X) \).

**Lemma 2.** \( \pi_X |M^X \setminus X \approx_p \text{id} \) in \( M^X \setminus X \).

Now we have the following:

**Theorem 3.** There is a categorical isomorphism \( \Phi : \text{Sh}^n_S(Z(\mu^{n+1})) \rightarrow \mathcal{H}^n_{p}(M_{n+1}) \) such that \( \Phi(X) = \mu^{n+1} \setminus X \) for \( X \in Z(\mu^{n+1}) \).
REFERENCES


[Iwa] Iwamoto, Y., Infinite deficiency in Menger manifolds, Glasnik Mat. 30(50) (1995), 311–322.


Y. Iwamoto: YUGE NATIONAL COLLEGE OF MARITIME TECHNOLOGY, YUGE 794-2593, JAPAN
E-mail address: iwamoto@gen.yuge.ac.jp

K. Sakai: INSTITUTE OF MATHEMATICS, UNIVERSITY OF TSUKUBA, TSUKUBA 305-8571, JAPAN
E-mail address: sakaiktr@sakura.cc.tsukuba.ac.jp