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Kyoto University
STRONG $n$-SHAPE THEORY

YUTAKA IWAMOTO AND KATSURO SAKAI

INTRODUCTION

Let $\mu^{n+1}$ be the $(n+1)$-dimensional universal Menger compactum. In [Chi$_1$], A. Chigogidze introduced the concept of $n$-shape and established the $(n+1)$-dimensional analogue of Chapman’s complement theorem [Cha, Theorem 2], that is, two $Z$-sets $X$ and $Y$ in $\mu^{n+1}$ have the same $n$-shape type if and only if their complements $\mu^{n+1}\setminus X$ and $\mu^{n+1}\setminus Y$ are homeomorphic ($\cong$), where $X \subset M$ is a $Z$-set in $M$ if there are maps $f : M \rightarrow M \setminus X$ arbitrarily close to $\text{id}_M$. The $n$-shape category of compacta was discussed in [Chi$_2$] (cf. [Chi$_3$]). Later, corresponding to [Cha, Theorem 1], Y. Akaike [Aka] defined the weak proper $n$-homotopy category of complements of $Z$-sets in $\mu^{n+1}$ which is isomorphic to the $n$-shape category of $Z$-sets in $\mu^{n+1}$. Then, as Strong Shape Theory ([EH], [DS], [KO], etc.), it is a natural attempt to define the strong $n$-shape category which corresponds to the proper $n$-homotopy category of complements of $Z$-sets in $\mu^{n+1}$. Properly, one require this category to factorize the natural functor (called the $n$-shape functor) from the $n$-homotopy category to the $n$-shape category into two functors through it. In this paper, we introduce the $(n+1)$-skeletal conic telescope to define the strong $n$-shape category of compacta.

Throughout the paper, spaces are separable metrizable and maps are continuous. It is said that two (proper) maps $f, g : X \rightarrow Y$ are (properly) $n$-homotopic relative to $A \subset X$ and denoted by $f \simeq^n_n g$ rel. $A$ ($f \simeq^n_n g$ rel. $A$) if, for any (proper) map $\varphi : Z \rightarrow X$, there is a (proper) homotopy $h : Z \times I \rightarrow Y$ such that $h_0 = f \varphi$, $h_1 = g \varphi$ for each $t \in I$. When $A = \emptyset$, we say that $f$ and $g$ are (properly) $n$-homotopic and denote $f \simeq g$ ($f \simeq^n_n g$).

A map $\varphi : M \rightarrow X$ is said to be $n$-invertible if any map $\psi : Z \rightarrow X$ of a space $Z$ with $\dim Z \leq n$ lifts to $M$, that is, there exists a map $\tilde{\psi} : Z \rightarrow M$ such that $\varphi \tilde{\psi} = \psi$. In case $\varphi$ is a proper map, if $\psi$ is proper then $\tilde{\psi}$ is also proper. For an $n$-invertible map $\varphi : M \rightarrow X$ and $A \subset X$, $\varphi|\varphi^{-1}(A) : \varphi^{-1}(A) \rightarrow A$ is also $n$-invertible. By the result of Dranishnikov [Dra, Theorem 1], for any compactum $X$, there exists an $n$-invertible map $\varphi : M \rightarrow X$ of a compactum $M$ with $\dim M \leq n$. Then, for two (proper) maps $f, g : X \rightarrow Y$, $f \simeq^n_n g$ rel. $A$ ($f \simeq^n_n g$ rel. $A$) if and only if $f \varphi \simeq g \varphi$ rel. $\varphi^{-1}(A)$ ($f \varphi \simeq^n_n g \varphi$ rel. $\varphi^{-1}(A)$) for an invertible (proper) map $\varphi : M \rightarrow X$.

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1. The Polyhedral Telescope

The $n$-skeleton of a simplicial complex $K$ is denoted by $K^{(n)}$, whence $K^{(0)}$ is the set of vertices of $K$. The polyhedron of $K$ is denoted by $|K|$ (i.e., $|K| = \bigcup_{\sigma \in K} \sigma$). By $(v_1, \ldots, v_n)$, we denote the simplex with vertices $v_1, \ldots, v_n$. A subdivision $\delta K$ of $K$ induces the subdivision $\delta K^{(n)}$ of $K^{(n)}$. It should be remarked that $\delta K^{(n)} \subset (\delta K)^{(n)}$ but $\delta K^{(n)} \neq (\delta K)^{(n)}$ in general. The following is well known:

**Fact 1.** Let $L$ be a subcomplex of $K$ and $Z$ a space with $\dim Z \leq n$. Then, for any map $\varphi: Z \to |K|$, there is a map $\psi: Z \to |K^{(n)} \cup L|$ such that $\varphi \simeq \psi$ rel. $\varphi^{-1}(|K^{(n)} \cup L|)$.

An ordered simplicial complex is a simplicial complex with an order of vertices such that the set of vertices of each simplex is totally ordered. The barycentric subdivision $\text{Sd} K$ of a simplicial complex $K$ is an ordered simplicial complex with the following order:

\[ \hat{\sigma} \leq \hat{\tau} \iff \sigma \text{ is a face of } \tau, \]

where $\hat{\sigma}$ is the barycenter of $\sigma$.

Let $I = \{0, 1\}$ be the natural triangulation of the unit interval $I = [0, 1]$. Then, $I$ is an ordered simplicial complex with the natural order $0 < 1$. For an ordered simplicial complex $K$, the product simplicial complex $K \times I$ is defined as follows:

\[ K \times I = \{ \sigma \times \{0\}, \sigma \times \{1\} \mid \sigma \in K \} \]

\[ \cup \{(v_1, 0), \ldots, (v_k, 0), (v_j, 1), \ldots, (v_k, 1) \mid (v_1, \ldots, v_k) \in K, \ v_1 < \ldots < v_k \in K^{(0)}, \ 1 \leq i \leq j \leq k \}. \]

Then $K \times I$ is an ordered simplicial complex with the following order on $(K \times I)^{(0)} = K^{(0)} \times \{0, 1\}$:

\[ (v, i) \leq (v', i') \iff v \leq v' \quad \text{and} \quad i \leq i'. \]

Let $K$ and $L$ be ordered simplicial complexes and $f: K \to L$ a simplicial map. The simplicial mapping cylinder $M(f)$ is defined as follows:

\[ M(f) = K \cup L \cup \{(f(v_1), \ldots, f(v_k)) \mid (v_1, \ldots, v_k) \in K, \ v_1 < \ldots < v_k, \ 1 \leq i \leq j \leq k \}. \]

When $L$ is degenerate (i.e., a singleton), $M(f)$ is the simplicial cone $C(K)$ over $K$. We have the natural simplicial map $q_f: K \times I \to M(f)$ which is naturally defined by $q_f(v, 0) = f(v)$ and $q_f(v, 1) = v$ for $v \in K^{(0)}$. The simplicial collapsing map $c_f: M(f) \to L$ is defined by $c_f(v) = f(v)$ for $v \in K^{(0)}$ and $c_f(u) = u$ for $u \in L^{(0)}$. Then $c_f q_f = f \mathrm{pr}_X$ and $c_f \simeq \mathrm{id} \text{ rel. } |L|$ in $|M(f)|$. Extending the orders on $K^{(0)}$ and $L^{(0)}$ to $M(f)^{(0)} = K^{(0)} \cup L^{(0)}$ so that $u < v$ for each $u \in L^{(0)}$ and $v \in K^{(0)}$, $M(f)$ is an ordered simplicial complex. Let $f^{(n)} = f|K^{(n)}$: $K^{(n)} \to L^{(n)}$ be the restriction of $f$. Observe that

\[ M(f)^{(n)} \subset M(f^{(n)}) \subset M(f)^{(n+1)} \subset M(f^{(n)}) \cup K \cup L \]

and $c_f|M(f^{(n)}) = c_{f^{(n)}} \simeq \mathrm{id} \text{ rel. } |L^{(n)}|$ in $|M(f^{(n)})|$. 
Fact 2. For a simplicial map \( f: K \rightarrow L \), \( c_{f}|M(f)^{(n+1)} \cup K \cup L| \simeq^{n} \text{id rel. } |L| \) in \( |M(f)^{(n+1)} \cup K \cup L| \), hence \( f = c_{f}|K \simeq^{n} \text{id}_{K} \) in \( |M(f)^{(n+1)} \cup K \cup L| \).

Since \( K \times I \) can be regarded as \( M(\text{id}_{K}) \), we have the following:

Fact 3. Let \( p: |(K^{(n)} \times I) \cup (K \times \{0,1\})| \rightarrow |K \times \{0\}| \) be the retraction defined by \( p(x,t) = (x,0) \). Then, \( p \simeq^{n} \text{id rel. } |K \times \{0\}| \) in \( |(K^{(n)} \times I) \cup (K \times \{0,1\})| \), where we identify \( K = K \times \{0\} \).

Let \( K = (|K_{i}|, q_{i,i+1})_{i \in \mathbb{N}} \) be an inverse sequence of ordered simplicial complexes such that each \( q_{i,i+1}: K_{i+1} \rightarrow \delta K_{i} \) is simplicial, where \( \delta K_{i} \) is some subdivision of \( K_{i} \). Let \( q_{i}: \lim \rightarrow |K_{i}| \) be the projection of the inverse limit of \( K \) to \( |K_{i}| \) and denote

\[
q_{i,j} = q_{i,i+1} \circ \cdots \circ q_{j-1,j}: |K_{j}| \rightarrow |K_{i}|, \ i < j.
\]

We define

\[
\text{Tel}_{[j,\infty)}(K) = \bigcup_{i=j}^{\infty} |M(q_{i,i+1})| \quad \text{and} \quad \text{Tel}_{[j,k]}(K) = \bigcup_{i=j}^{k-1} |M(q_{i,i+1})|, \ j < k,
\]

where \( |M(q_{i,i+1})| \cap |M(q_{i+1,i+2})| = |K_{i+1}| \) and \( |M(q_{i,i+1})| \cap |M(q_{j,j+1})| = \emptyset \) for \( |i - j| > 1 \). The polyhedron \( \text{Tel}_{[1,\infty)}(K) \) is called the \textit{polyhedral telescope} for \( K \). One should note that \( \bigcup_{i=1}^{\infty} M(q_{i}) \) is not a simplicial complex unless \( \delta K_{i} = K_{i} \) for every \( i \in \mathbb{N} \). Let

\[
\text{Tel}_{[0,\infty)}(K) = |C(K_{1})| \cup \text{Tel}_{[1,\infty)}(K) \quad \text{and} \quad \text{Tel}_{[0,k]}(K) = |C(K_{1})| \cup \text{Tel}_{[1,k]}(K),
\]

where \( |C(K_{1})| \cap \text{Tel}_{[1,\infty)}(K) = |K_{1}| \). We call \( \text{Tel}_{[0,\infty)}(K) \) the \textit{polyhedral conic telescope}.

The simplicial collapsing map \( c_{q_{i,i+1}}: M(q_{i,i+1}) \rightarrow \delta K_{i} \) extends to the deformation retraction

\[
c_{i,i+1}^{K}: \text{Tel}_{[0,i+1]}(K) = \text{Tel}_{[0,i]}(K) \cup |M(q_{i,i+1})| \rightarrow T_{[0,i]}(K).
\]

The following diagram is commutative:

\[
\begin{array}{ccc}
\text{Tel}_{[0,1]}(K) & \xleftarrow{c_{1,2}^{K}} & \text{Tel}_{[0,2]}(K) & \xleftarrow{c_{2,3}^{K}} & \text{Tel}_{[0,3]}(K) & \xleftarrow{c_{3,4}^{K}} & \cdots \\
|K_{1}| & \underset{q_{1,2}}{\leftarrow} & |K_{2}| & \underset{q_{2,3}}{\leftarrow} & |K_{3}| & \underset{q_{3,4}}{\leftarrow} & \cdots \\
\end{array}
\]

The inverse limit of the upper sequence is denoted by \( \text{Tel}_{[0,\infty)}(K) \) with the projection \( c_{i}^{K}: \text{Tel}_{[0,\infty)}(K) \rightarrow \text{Tel}_{[0,i]}(K) \). We denote

\[
c_{i,j}^{K} = c_{i,i+1}^{K} \circ \cdots \circ c_{j-1,j}^{K}: \text{Tel}_{[0,j]}(K) \rightarrow \text{Tel}_{[0,i]}(K), \ i < j.
\]

Regarding \( \text{Tel}_{[0,\infty)}(K) \) as an open subspace of \( \text{Tel}_{[0,\infty)}(K) \), we have

\[
\text{Tel}_{[0,\infty)}(K) \setminus \text{Tel}_{[0,\infty)}(K) = \lim \rightarrow K \quad \text{and} \quad c_{i}^{K}|\lim K = q_{i}, \ i \in \mathbb{N}.
\]
It is easy to see that each \( c^K_i \) is a strong deformation retraction. Hence, it follows that \( \text{Tel}_{[0,\infty]}(K) \) is homotopy dense in \( \text{Tel}_{[0,\infty]}(K) \), that is, there is a homotopy \( h: \text{Tel}_{[0,\infty]}(K) \times I \to \text{Tel}_{[0,\infty]}(K) \) such that \( h_0 = \text{id} \) and \( h_t(\text{Tel}_{[0,\infty]}(K)) \subset \text{Tel}_{[0,\infty]}(K) \) for \( t > 0 \). Since \( \text{Tel}_{[0,\infty]}(K) \) is a polyhedron, \( \text{Tel}_{[0,\infty]}(K) \) is an ANR by Hanner's characterization of ANR's (cf. [Hu]). Since \( \text{Tel}_{[0,\infty]}(K) \) is contractible, it is an AR. The above construction was founded in [Ko, Theorem 1 and Corollary 1]. For each \( j \in \mathbb{N} \), we can similarly define \( \text{Tel}_{[j,\infty]}(K) \), which is an ANR and a closed subspace of \( \text{Tel}_{[0,\infty]}(K) \). Clearly,

\[
\text{Tel}_{[j,\infty]}(K) = \text{Tel}_{[0,\infty]}(K) \setminus \text{Tel}_{[0,\infty]}(K) = \lim_{\longrightarrow} K.
\]

Each \( d^K_j = c^K_j \mid \text{Tel}_{[j,\infty]}(K) : \text{Tel}_{[j,\infty]}(K) \to |K_j| \) is a strong deformation retraction and \( q_{i,j}d^K_j = d^K_i \mid \text{Tel}_{[j,\infty]}(K) \).

Now, we define

\[
\begin{align*}
\text{Tel}_{[j,\infty]}^{n+1}(K) &= \bigcup_{i=j}^{\infty} |K_i| \cup \bigcup_{i=j}^{\infty} |M(q_{i,i+1})^{(n+1)}| \text{ and} \\
\text{Tel}_{[j,k]}^{n+1}(K) &= \bigcup_{i=j}^{k} |K_i| \cup \bigcup_{i=j}^{k-1} |M(q_{i,i+1})^{(n+1)}|, \quad j < k.
\end{align*}
\]

These are subpolyhedra of \( \text{Tel}_{[1,\infty]}(K) \). Recall that \( \bigcup_{i=1}^{\infty} M(q_i) \) is not a simplicial complex in general. We call \( \text{Tel}_{[1,\infty]}^{n+1}(K) \) the \((n+1)\)-skeletal telescope for \( K \). Let

\[
\begin{align*}
\text{Tel}_{[0,\infty]}^{n+1}(K) &= |C(K_1)^{(n+1)}| \cup \text{Tel}_{[1,\infty]}^{n+1}(K) \text{ and} \\
\text{Tel}_{[0,k]}^{n+1}(K) &= |C(K_1)^{(n+1)}| \cup \text{Tel}_{[1,k]}^{n+1}(K).
\end{align*}
\]

These are \( n \)-connected. The polyhedron \( \text{Tel}_{[0,\infty]}^{n+1}(K) \) is called the \((n+1)\)-skeletal conic telescope for \( K \).

Observe that \( c^K(K_1) = \text{Tel}_{[0,i+1]}^{n+1}(K) = \text{Tel}_{[0,i]}^{n+1}(K) \). The following diagram is commutative:

\[
\begin{CD}
\text{Tel}_{[0,1]}^{n+1}(K) @>c^K_{1,2}>> \text{Tel}_{[0,2]}^{n+1}(K) @>c^K_{2,3}>> \text{Tel}_{[0,3]}^{n+1}(K) @>c^K_{3,4}>> \cdots \\
@. \cup @. \cup @. \cup @. \cdots \\
|K_1| @<q_{1,2}<< |K_2| @<q_{2,3}<< |K_3| @<q_{3,4}<< \cdots
\end{CD}
\]

Then the inverse limit of the upper sequence is the closed subspace

\[
\text{Tel}_{[0,\infty]}^{n+1}(K) = \text{Tel}_{[0,\infty]}^{n+1}(K) \cup \lim_{\longrightarrow} K \subset \text{Tel}_{[0,\infty]}(K).
\]

For each \( j \in \mathbb{N} \), let \( \text{Tel}_{[j,\infty]}^{n+1}(K) = \text{Tel}_{[j,\infty]}^{n+1}(K) \cup \lim_{\longrightarrow} K. \)
**Fact 4.** For each $j \in \mathbb{N} \cup \{0\}$, $\text{Tel}^{n+1}[j, \infty](K) \setminus \text{Tel}^{n+1}[j, \infty](K) = \lim_{\rightarrow} K$ is a $Z$-set in $\text{Tel}^{n+1}[j, \infty](K)$.

Let $\psi : Z \to \text{Tel}^{n+1}[j, \infty](K)$ be a map of a space $Z$ with $\dim Z \leq n$. Then it is easy to construct a homotopy $h : Z \times I \to \text{Tel}^{n+1}[j, \infty](K)$ such that $h_0 = \psi$ and $h_t(Z) \subset \text{Tel}^{n+1}[j, \infty](K)$ for $t > 0$. In general, $\text{Tel}^{n+1}[0, \infty](K)$ is not an ANR, but we have the following:

**Fact 5.** Each $\text{Tel}^{n+1}[j, \infty](K)$ is $L^n$, hence it is an ANE$(n+1)$. Moreover, the space $\text{Tel}^{n+1}[0, \infty](K)$ is $n$-connected, so it is an $AE(n+1)$.\(^1\)

The following follows from Fact 2:

**Fact 6.** For $i < j \in \mathbb{N} \cup \{0\}$, $\partial^K_{i,j} \mid \text{Tel}^{n+1}[i,j](K) \cong^n \text{id}$ in $\text{Tel}^{n+1}[i,j](K)$, hence $q_{i,j} \cong^n \text{id}_{K_j}$ in $\text{Tel}^{n+1}[i,j](K)$. Moreover, $\partial^K_{i,j} \mid \text{Tel}^{n+1}[i,j](K) \cong^n \text{id}$ in $\text{Tel}^{n+1}[i,j](K)$, so $q_i \cong^n \text{id}_{K_j}$ in $\text{Tel}^{n+1}[i,j](K)$.

### 2. The strong $n$-shape category $\text{SH}_{n}^{S}$

Let $\mathcal{H}^n$ be the $n$-homotopy category of compacta and $\text{Sh}^n$ the $n$-shape category of compacta. In this section, we define the strong $n$-shape category $\text{Sh}_{n}^{S}$ of compacta and show that the $n$-shape functor from $\mathcal{H}^n$ to $\text{Sh}^n$ is factorized into two functors through the category $\text{Sh}_{n}^{S}$.

Every compactum $X$ is the limit of an inverse sequence $K = (K_i, q_i)_{i \in \mathbb{N}}$ of finite simplicial complexes such that each $q_{i,i+1} : K_{i+1} \to \text{Sd} K_i$ is simplicial for the barycentric subdivision $\text{Sd} K_i$ of $K_i$ and $\dim K_i \leq \dim X$ for all $i \in \mathbb{N}$ [Isb, Lemma 33] (cf. Proof of [Ko2, Theorem 1]). We call $K$ a barycentric sequence associated with $X$. It should be noted that $q_{i,i+1} : K_{i+1} \to K_i$ is not simplicial in general. In fact, there exists a 1-dimensional compact AR which is not the limit of any inverse sequence of simplicial complexes and simplicial maps [Ko1, Theorem 1(2)] (cf. [Ko2, p.536]). It should be also noted that a barycentric sequence associated with $X$ is an $L^n(n + 1)$-sequence associated with $X$ (cf. [Ch2]).

**Theorem 1.** Let $X$ and $Y$ be compacta and $K$, $L$ be barycentric sequences associated with $X$ and $Y$, respectively.

1. Every map $f : X \to Y$ extends to a map $\tilde{f} : \text{Tel}^{n+1}[0, \infty](K) \to \text{Tel}^{n+1}[0, \infty](L)$ such that $\tilde{f}(\text{Tel}^{k}[0, \infty](K)) \subset \text{Tel}^{k}[0, \infty](L)$ for each $k \in \mathbb{N}$.
2. For two maps $f, g : \text{Tel}^{n+1}[0, \infty](K) \to \text{Tel}^{n+1}[0, \infty](L)$ with $f^{-1}(Y) = g^{-1}(Y) = X$, if $f \mid X \cong^n g \mid X$ in $Y$ then $f \mid \text{Tel}^{n+1}[0, \infty](K) \cong^n g \mid \text{Tel}^{n+1}[0, \infty](K)$ in $\text{Tel}^{n+1}[0, \infty](L)$.

In Theorem 1(1) above, a proper map $\tilde{f} \mid \text{Tel}^{n+1}[0, \infty](K) : \text{Tel}^{n+1}[0, \infty](K) \to \text{Tel}^{n+1}[0, \infty](L)$ is said to be induced by $f$. By Theorem 1(2), the proper homotopy class of such a map is unique. The following is a direct consequence of Theorem 1.

\(^1\)A space $Y$ is an $AE(n + 1)$ (or an $\text{ANE}(n + 1)$) if every map of any closed set $A$ in an arbitrary metrizable space $X$ with $\dim X \leq n + 1$ extends over $X$ (or a neighborhood of $A$). A space $Y$ is an $AE(n + 1)$ if and only if $Y$ is an $n$-connected $\text{ANE}(n)$, and $Y$ is an $\text{ANE}(n + 1)$ if and only if $Y$ is $L^n$. 


Corollary 1. Let $K$ and $L$ be barycentric sequences associated with the same compactum $X$. Then a proper map $h: \text{Tel}_{[0,\infty)}^{n+1}(K) \rightarrow \text{Tel}_{[0,\infty)}^{n+1}(L)$ induced by $id_X$ is a proper $n$-homotopy equivalence.

Definition of $\text{Sh}_S^n$. Let $X$ and $Y$ be compacta. Let $K, K'$ be barycentric sequences associated with $X$ and $L, L'$ barycentric sequences associated with $Y$. Two proper maps $F: \text{Tel}_{[0,\infty)}^{n+1}(K) \rightarrow \text{Tel}_{[0,\infty)}^{n+1}(L)$ and $F': \text{Tel}_{[0,\infty)}^{n+1}(K') \rightarrow \text{Tel}_{[0,\infty)}^{n+1}(L')$ are $n$-fundamentally equivalent (written by $F \simeq_F^n F'$) if $hF \simeq_p^n F'h$ for some proper $n$-homotopy equivalences $h: \text{Tel}_{[0,\infty)}^{n+1}(K) \rightarrow \text{Tel}_{[0,\infty)}^{n+1}(K')$ and $h': \text{Tel}_{[0,\infty)}^{n+1}(L') \rightarrow \text{Tel}_{[0,\infty)}^{n+1}(L)$ induced by $id_X$ and $id_Y$, respectively. A strong $n$-shape morphism from $X$ to $Y$ is the $n$-fundamentally equivalence class of a proper map $F: \text{Tel}_{[0,\infty)}^{n+1}(K) \rightarrow \text{Tel}_{[0,\infty)}^{n+1}(L)$, where $K$ and $L$ are barycentric sequences associated with $X$ and $Y$ respectively. Thus, the strong $n$-shape category $\text{Sh}_S^n$ of compacta can be defined.

The following follows immediately from Theorem 1 and the definition above.

Corollary 2. There exists a functor $\Xi: \mathcal{H}^n \rightarrow \text{Sh}_S^n$ which maps objects identically.

For simplicity, let us assign each compactum $X$ to a barycentric sequence $K^X = (K_{i}^X, q_{i,i+1}^X)_{i\in \mathbb{N}}$ associated with $X$ and denote as follows:

$$
\text{Tel}_{[0,\infty)}^{n+1}(X) = \text{Tel}_{[0,\infty)}^{n+1}(K^X), \quad \text{Tel}_{[j,k]}^{n+1}(X) = \text{Tel}_{[j,k]}^{n+1}(K^X),
$$

$$
c_{i,i+1}^X = c_{i,i+1}^X | \text{Tel}_{[0,i+1]}^{n+1}(K^X), \quad c_i^X = c_i^X | \text{Tel}_{[0,\infty)}^{n+1}(K^X),
$$

$$
d_i^X = d_i^X | \text{Tel}_{[0,\infty)}^{n+1}(K^X), \quad \text{etc.}
$$

Thus, $X$ is assigned to the following commutative diagram of inverse sequences:

$$
\text{Tel}_{[0,1]}^{n+1}(X) \xleftarrow{c_{1,2}^X C} \text{Tel}_{[0,2]}^{n+1}(X) \xleftarrow{c_{2,3}^X C} \text{Tel}_{[0,3]}^{n+1}(X) \xleftarrow{c_{3,4}^X C} \cdots \\
\cup \\
\cup \\
\cup \\
\cdots \\
|K_1^X| \xleftarrow{q_{1,2}^X} |K_2^X| \xleftarrow{q_{2,3}^X} |K_3^X| \xleftarrow{q_{3,4}^X} \cdots \\
$$

Now, we prove the following:

Theorem 2. There exists a full\footnote{The functor is full if the induced maps of the sets of morphisms are surjective.} functor $\Theta: \text{Sh}_S^n \rightarrow \text{Sh}^n$ such that $\Theta \circ \Xi: \mathcal{H}^n \rightarrow \text{Sh}^n$ is the $n$-shape functor.

Remarks. The following proposition can be proved similarly to Theorem 1(1).

Proposition. Let $K$ and $L$ be barycentric sequences associated with compacta $X$ and $Y$, respectively. Every proper map $f: \text{Tel}_{[0,\infty)}^{n+1}(K) \rightarrow \text{Tel}_{[0,\infty)}^{n+1}(L)$ is properly homotopic to a proper map $\bar{f}: \text{Tel}_{[0,\infty)}^{n+1}(K) \rightarrow \text{Tel}_{[0,\infty)}^{n+1}(L)$ such that $\bar{f}(\text{Tel}_{[0,\infty)}^{n+1}(K)) \subset \text{Tel}_{[0,\infty)}^{n+1}(L)$ for each $k \in \mathbb{N}$.

By the same proof, Theorem 1(2) is valid even if $\text{Tel}_{[0,\infty)}^{n+1}$ is replaced with $\text{Tel}_{[0,\infty)}^{0}$. Then, in the definition of $\text{Sh}_S^n$, replacing $\text{Tel}_{[0,\infty)}^{n+1}$ by $\text{Tel}_{[0,\infty)}^{0}$, we can define the
category \( \overline{\text{Sh}}_S^n \) which factorizes the \( n \)-shape functor into two functors through \( \text{Sh}_S^n \).

In fact, the functor \( \Xi \) in Corollary 2 is factorized into two natural functors through \( \overline{\text{Sh}}_S^n \), where the natural functor from \( \text{Sh}_S^n \) to \( \text{Sh}_S^n \) can be obtained by the proposition above. As is easily observed, the functor from \( \text{Sh}_S^n \) to \( \overline{\text{Sh}}_S^n \) is injective, but it is a problem whether it is surjective or not.

\[
\begin{array}{c}
\mathcal{H}^n \\
\downarrow \\
\text{Sh}_S^n \\
\downarrow \\
\overline{\text{Sh}}_S^n
\end{array} \longrightarrow \begin{array}{c}
\text{Sh}_S^n \\
\downarrow \\
\overline{\text{Sh}}_S^n \\
\downarrow \\
\text{Sh}_S^n \\
\downarrow \\
\text{Sh}_S^n
\end{array} \longrightarrow \text{Sh}
\]

In the definition of \( \text{Sh}_S^n \), replacing \( \text{Tel}_{[0,\infty)}^{n+1} \) and \( \simeq^n_p \) by \( \text{Tel}_{[0,\infty)} \) and \( \simeq_p \), we can obtain the strong shape category \( \text{Sh}_S \) (cf. [DS]). Then, we can easily obtain the natural functor from \( \text{Sh}_S \) to \( \overline{\text{Sh}}_S^n \). Let \( \mathcal{H} \) be the homotopy category of compacta. We have the following diagram of categories and functors:

\[
\begin{array}{c}
\mathcal{H} \\
\downarrow \\
\text{Sh}_S \\
\downarrow \\
\overline{\text{Sh}}_S \\
\downarrow \\
\text{Sh}_S^n \\
\downarrow \\
\overline{\text{Sh}}_S^n \\
\downarrow \\
\text{Sh}_S^n \\
\downarrow \\
\text{Sh}
\end{array} \longrightarrow \begin{array}{c}
\text{Sh}_S \\
\downarrow \\
\overline{\text{Sh}}_S \\
\downarrow \\
\text{Sh}_S^n \\
\downarrow \\
\overline{\text{Sh}}_S^n \\
\downarrow \\
\text{Sh}_S^n \\
\downarrow \\
\text{Sh}
\end{array}
\]

Restricting the objects to compacta with \( \dim \leq k \), we have the subcategories \( \text{Sh}(k), \text{Sh}_S(k), \text{Sh}_S^n(k) \) and \( \overline{\text{Sh}}_S(n) \) of \( \text{Sh}, \text{Sh}_S, \text{Sh}_S^n \) and \( \overline{\text{Sh}}_S^n \), respectively. Then, \( \text{Sh}_S^n(n) = \overline{\text{Sh}}_S^n(n) \) because \( \text{Tel}_{[0,\infty)}^{n+1}(X) = \text{Tel}_{[0,\infty)}(X) \) if \( \dim X \leq n \).

Moreover, \( \text{Sh}_S^n(n-1) = \overline{\text{Sh}}_S^n(n-1) = \text{Sh}_S(n-1) \) because \( \dim \text{Tel}_{[0,\infty)}(X) \leq n \) if \( \dim X \leq n-1 \). Although \( \text{Sh}_S^n(n) = \text{Sh}(n) \), it is not known whether \( \text{Sh}_S^n(n) = \text{Sh}_S(n) \) or not.

3. AN ISOMORPHISM BETWEEN \( \text{Sh}_S^n(Z(\mu^{n+1})) \) AND \( \mathcal{H}_P^p(\mathcal{M}_{n+1}) \)

Let \( Z(\mu^{n+1}) \) be the class of \( Z \)-sets in \( \mu^{n+1} \) and \( \mathcal{M}_{n+1} \) the class of \( \mu^{n+1} \)-manifolds \( \mu^{n+1} \setminus X, X \in Z(\mu^{n+1}) \). In this section, we prove that the strong \( n \)-shape category \( \text{Sh}_S^n(Z(\mu^{n+1})) \) of \( Z(\mu^{n+1}) \) is categorically isomorphic to the proper \( n \)-homotopy category \( \mathcal{H}_P^p(\mathcal{M}_{n+1}) \) of \( \mathcal{M}_{n+1} \).

Lemma 1. Let \( f: X \to Y \) be a map from a locally compact separable metrizable space \( X \) with \( \dim X \leq n+1 \) to a completely metrizable ANE(\( n+1 \)) \( Y \). For any closed set \( A \subset X \) and a \( Z \)-set \( B \subset Y \), \( f \) is approximated by maps \( g: X \to Y \) such that \( g|A = f|A \) and \( g(X \setminus A) \subset Y \setminus B \).

As in §2, we assign each \( X \in Z(\mu^{n+1}) \) to the following diagram:

\[
\begin{array}{c}
\text{Tel}_{[0,1]}^{n+1}(X) \xleftarrow{c_{2,3}^{X}} \text{Tel}_{[0,2]}^{n+1}(X) \xleftarrow{c_{3,4}^{X}} \text{Tel}_{[0,3]}^{n+1}(X) \xleftarrow{c_{4,5}^{X}} \cdots \\
\text{U} \quad \text{U} \quad \text{U} \quad \text{U} \quad \cdots \\
|K_1^{X}| \xleftarrow{q_{1,2}^{X}} |K_2^{X}| \xleftarrow{q_{2,3}^{X}} |K_3^{X}| \xleftarrow{q_{3,4}^{X}} \cdots,
\end{array}
\]
where the lower sequence is a barycentric sequence associated with \( X \). To prove

Theorem 3, we apply the construction in [Sa] to this diagram.

Let \( M_1^X = C(K_1^X)^{(n+1)} \). Then \( |M_1^X| = \text{Tel}_{[0,1]}^{n+1}(X) \). We inductively define a

simplicial complex

\[
M_{i+1}^X = ( \text{Sd} M_i^X \times I )^{(n+1)} \cup M(q_{i,i+1}^X)^{(n+1)},
\]

where we identify \( \text{Sd} M_i^X = M_i^X \times \{0\} \). So we have

\[
M(q_{i,i+1}^X)^{(n+1)} \cap ( \text{Sd} M_i^X \times I ) = M(q_{i,i+1}^X)^{(n+1)} \cap \text{Sd} M_i^X = \text{Sd} K_i.
\]

Observe that \( \text{Tel}_{[0,i]}^{n+1}(X) = \text{Tel}_{[0,i]}^{n+1}(X) \cup |M(q_{i,i+1}^X)^{(n+1)}| \subset |M_i^X| \). The simplicial

collapsing map \( c_{q_{i,i+1}^X} : M(q_{i,i+1}^X) \rightarrow \text{Sd} K_i^X \) extends to the simplicial retraction

\[
c_{i,i+1} : M_{i+1}^X \rightarrow ( \text{Sd} M_i^X \times I )^{(n+1)} \cup M(q_{i,i+1}^X)^{(n+1)} \rightarrow ( \text{Sd} M_i^X \times I )^{(n+1)}.
\]

We define \( r_{i,i+1}^X = \text{pr}_i c_{i,i+1} : M_{i+1}^X \rightarrow M_i^X \), where \( \text{pr}_i : ( \text{Sd} M_i^X \times I ) \rightarrow M_i^X \)

is the projection. Let \( \pi_i^X = \text{id} : |M_i^X| \rightarrow \text{Tel}_{[0,i]}^{n+1}(X) = |M_i^X| \) and inductively

define the retraction \( \pi_{i+1}^X : |M_{i+1}^X| \rightarrow \text{Tel}_{[0,i+1]}^{n+1}(X) \) by \( \pi_{i+1}^X| |M(q_{i,i+1}^X)^{(n+1)}| = \text{id} \)

and \( \pi_{i+1}^X| |(\text{Sd} M_i^X \times I )^{(n+1)}| = \pi_i^X | \text{pr}_i^X \). Thus, we obtain the following commutative

diagram of the inverse sequences:

\[
\begin{array}{ccccccc}
|M_1^X| & \xrightarrow{r_{1,2}^X} & |M_2^X| & \xrightarrow{r_{2,3}^X} & |M_3^X| & \xrightarrow{r_{3,4}^X} & \ldots \\
\left. \begin{array}{c}
|M_i^X| \\
\uparrow \\
\text{Tel}_{[0,1]}^{n+1}(X) \simeq \text{Tel}_{[0,2]}^{n+1}(X) \simeq \text{Tel}_{[0,3]}^{n+1}(X) \simeq \text{Tel}_{[0,4]}^{n+1}(X) \simeq \cdots
\end{array} \right| \quad \text{and} \\
\text{Tel}_{[0,i]}^{n+1}(X) \simeq \text{Tel}_{[0,i+1]}^{n+1}(X) \simeq \text{Tel}_{[0,i+2]}^{n+1}(X) \simeq \cdots
\end{array}
\]

Recall that \( \text{Tel}_{[0,\infty]}^{n+1}(X) = \bigcup_{i \in \mathbb{N}} \text{Tel}_{[0,i]}^{n+1}(X) \), \( \text{Tel}_{[0,\infty]}^{n+1}(X) \cup X \)

is the inverse limit of the middle sequence and \( X \) is the inverse limit of the bottom sequence. Let \( M^X \)

be the inverse limit of the upper sequence. Then \( X \subseteq \text{Tel}_{[0,\infty]}^{n+1}(X) \)

but \( M^X \neq X \cup \bigcup_{i \in \mathbb{N}} |M_i^X| \). Applying Bestvina’s characterization of \( \mu^{n+1} \) [Be], one can see that \( M^X \approx \mu^{n+1} \) (cf. [Sa] and [Iwa, Proposition 2.1]). It is easily seen that \( X \) is a \( Z \)-set in \( M^X \) (it is also a \( Z \)-set in \( \text{Tel}_{[0,\infty]}^{n+1}(X) \) [Sa]). Since \( (M^X, X) \approx (\mu^{n+1}, X) \) by the \( Z \)-set unknotting theorem [Be], we have a

homeomorphism \( h_X : M^X \setminus X \rightarrow \mu^{n+1} \setminus X \). On the other hand, we have the

retraction \( \pi_X : M^X \rightarrow \text{Tel}_{[0,\infty]}^{n+1}(X) \) induced by \( \pi_X \). Observe that \( \pi_X| X = \text{id} \)

and \( \pi_X(M^X \setminus X) = \text{Tel}_{[0,\infty]}^{n+1}(X) \).

**Lemma 2.** \( \pi_X| M^X \setminus X \approx_p \text{id} \) in \( M^X \setminus X \).

Now we have the following:

**Theorem 3.** There is a categorical isomorphism \( \Phi : \text{Sh}_S^n(Z(\mu^{n+1})) \rightarrow \mathcal{H}_p^n(M_{n+1}) \)

such that \( \Phi(X) = \mu^{n+1} \setminus X \) for \( X \in Z(\mu^{n+1}) \).
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Y. Iwamoto: YUGE NATIONAL COLLEGE OF MARITIME TECHNOLOGY, YUGE 794-2593, JAPAN

E-mail address: iwamoto@gen.yuge.ac.jp

K. Sakai: INSTITUTE OF MATHEMATICS, UNIVERSITY OF TSUKUBA, TSUKUBA 305-8571, JAPAN

E-mail address: sakaiktr@sakura.cc.tsukuba.ac.jp