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SDEのEuler-丸山型近似解に対するSample Path Large Deviations

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Introduction* : Itô's *SDE

Let $\{B(t), 0\le t\le 1\}$ be an r-dimensional standard Brownian motion. Consider Itô's stochastic differential equation (SDE) for a d-dimensional continuous process $\{X(t), 0\le t\le 1\}$ $(d\ge 1)$:

$$(1)\qquad \begin{cases} dX(t)=\sigma(t,X(t))dB(t)+b(t,X(t))dt, 0\le t\le 1 \\ X(0)=X_0. \end{cases}$$

Suppose that $\sigma(t,x)$ and $b(t,x)$ satisfy the Lipschitz condition. Then their exists a unique solution of (1). (See e.g., Ikeda-Watanabe (1981).)

1. The Euler-Maruyama Algorithm

Maruyama (1955) showed the existence of the unique solution of (1) using an Euler type approximation solution $Z_n := \{Z_n(t), 0\le t\le 1\}$ defined by

$$(2)\qquad Z_n(t) := X_0 + \int_0^t \sigma_n(u)dB(u) + \int_0^t b_n(u)du, \quad 0\le t\le 1,$$

where

$$\sigma_n(t) := \sigma\left(\frac{k-1}{n}, x_{k-1}\right), \quad k/n\le t\le (k+1)/n, \quad k=0,\cdots,n-1,$$

$$b_n(t) := b\left(\frac{k-1}{n}, x_{k-1}\right), \quad k/n\le t\le (k+1)/n, \quad k=0,\cdots,n-1,$$

$$x_k := X_0 + \sum_{j=1}^{k}\sigma\left(\frac{j-1}{n}, x_{j-1}\right)\eta_j + \sum_{j=1}^{k} b\left(\frac{j-1}{n}, x_{j-1}\right)\Big/ n, k=0,1,\cdots,n,$$

$$\eta_k := B\left(\frac{k}{n}\right) - B\left(\frac{k-1}{n}\right), \quad k=1,\cdots,n.$$

Theorem A. *(Maruyama (1955)) Suppose that $\sigma(t,x)$ and $b(t,x)$ satisfy the Lipschitz condition, i.e.*

$$|\sigma(t,x)-\sigma(s,y)|^2+|b(t,x)-b(s,y)|^2 \le K_1\left(|x-y|^2+|t-s|^2\right),$$

where K_1 is a positive constant independent of x, y, t and s. Then for any $t \ge 0$

$$\lim_{n\to\infty} \boldsymbol{E}\left(|X(t)-Z_n(t)|^2\right)=0.$$

The above scheme for the construction of $Z_n(t)$ has no practical value since it needs the complete knowledge about whole trajectory over the interval [0,1] of the Brownian motion, which is also to be simulated. Therefore we find it necessary to introduce a stochastic process $X_n := \{X_n(t), 0\le t\le 1\}$ in $D(0,1)$ by a slight modification of Z_n

$$\begin{cases} X_n(t):=x_k, & k/n \le t < (k+1)/n, \quad k=0.\cdots,n-1 \\ X_n(1):=x_n, \end{cases}$$

here $\{\eta_k\}$ are i.i.d. random variables constructed from pseudo-random numbers with the r-dimensional normal distribution $N(0,1/n)$.

As for the error estimation for Z_n, Ghiman-Skorohod (1979), Shimizu (1984) showed the rate of convergence of them to the real solution X of (1) in L^p-mean for some $p\ge 2$. On the other hand Kanagawa (1988) considered the error of X_n as follows.

Theorem 1. *Kanagawa(1988) Suppose that for any $0\le s,t\le 1$ and $x,y\in \mathbf{R}^d$*

(4) $$|\sigma(t,x)-\sigma(s,y)|^2+|b(t,x)-b(s,y)|^2 \le K_1\left(|x-y|^2+|t-s|^2\right),$$

(5) $$|\sigma(t,x)|^2+|b(s,y)|^2 \le K_2,$$

where K_1 and K_2 are some positive constants independent of s, t, x and y. Then for any $p\ge 2$ and for some $\varepsilon>p/2$

(6) $$\boldsymbol{E}\left(\max_{0\le t\le 1}|X(t)-X_n(t)|^p\right)=o\left(n^{-p/2}(\log n)^{\varepsilon}\right) \quad \textit{as } n\to\infty,$$

(7) $$E\left(\max_{0\le t\le 1}|X(t)-Z_n(t)|^p\right)=O\left(n^{-p/2}\right) \quad \text{as } n\to\infty.$$

Furthermore, in the case when the approximate solutions are constructed from r-dimensional i.i.d. random variables $\{\xi_k\}$ which do not always obey the normal distribution, Kangawa (1989) showed the rate of convergence of $E\left(\max_{0\le t\le 1}|X(t)-Y_n(t)|^p\right)$ to zero, where $Y_n:=\{Y_n(t), 0\le t\le 1\}$ is a stochastic process in $D(0,1)$ defined by for $k=0,1,\cdots,n$

$$\begin{cases} Y_n(t):=y_k, & k/n\le t<(k+1)/n, \quad k=0,\cdots,n-1 \\ Y_n(1):=y_n, & \end{cases} \tag{8}$$

where

$$y_k:=X_0+\sum_{j=1}^{k}\sigma\left(\frac{j-1}{n},y_{j-1}\right)\xi_j/\sqrt{n}+\sum_{j=1}^{k}b\left(\frac{j-1}{n},y_{j-1}\right)\Big/n.$$

<u>**Theorem 2.**</u> *Kanagawa(1989) Let* $\{\xi_k, k\ge 1\}$ *be* r-dimensional *i.i.d. random variables with*

$$E(\xi_1)=0, E\left(|\xi_1|^2\right)=1, E\left(|\xi_1|^{2+\delta}\right)<\infty \text{ for some } 0<\delta\le 1. \tag{9}$$

Assume $\sigma(t,x)$ *and* $b(t,x)$ *satisfy (4) and (5). Then we can redefine* $\{X(t), 0\le t\le 1\}$ *and* $\{Y_n(t), 0\le t\le 1\}$ *on a common probability space such that for any* $p\ge 2$ *and* $\varepsilon>(2+\delta)^2/2(3+\delta)$,

$$E\left(\max_{0\le t\le 1}|X(t)-Y_n(t)|^p\right)=o\left(n^{-p\delta/2(2+\delta)}(\log n)^{\varepsilon}\right) \quad \text{as } n\to\infty, \tag{10}$$

where the power of n cannot be improved by better one.

Furthermore, under the Cramér's condition instead of $E\left(|\xi_1|^{2+\delta}\right)<\infty$, we have the next result.

<u>**Theorem 3.**</u> *Kanagawa(1995) Let* $\{\xi_k, k\ge 1\}$ *be* r-dimensional *i.i.d. random variables with* $E(\xi_1)=0, E\left(|\xi_1|^2\right)=1$. *Suppose that* $E\left(e^{t|\xi_1|}\right)\le\infty$ *in a neighborhood of* $t=0$. *Assume* $\sigma(t,x)$ *and* $b(t,x)$ *satisfy (4) and (5). Then we can redefine* $\{X(t), 0\le t\le 1\}$ *and* $\{Y_n(t), 0\le t\le 1\}$ *on a common probability space such that* for any $p\ge 2, \varepsilon>p/2$ and for sufficiently large n

$$E\left(\max_{0\le t\le 1}|X(t)-Y_n(t)|^p\right)=o\left(n^{-p/4}(\log n)^{\varepsilon}\right) \quad \text{as } n\to\infty. \tag{11}$$

2. *Sample Path Large Deviations*

We can apply Schiler's Brownian motion sample path large deviations to Euler-Maruyama approximate solutions $X_n := \{X_n(t), 0 \le t \le 1\}$ for SDE's.

Theorem 4. *Consider the following SDE,*

$$\begin{cases} dX(t) = dB(t) + b(t, X(t))dt, \ 0 \le t \le 1 \\ X(0) = X_0. \end{cases}$$

Suppose that $b(t,x)$ satisfies the Lipschitz condition (4). Let $\{\xi_k, k \ge 1\}$ be r-dimensional i.i.d. random variables with $E(\xi_1) = 0, E\left(|\xi_1|^2\right) = 1$. Suppose that $E\left(e^{t|\xi_1|}\right) \le \infty$ in a neighborhood of $t = 0$. Put

$$\Lambda(\lambda) := \log E\left(e^{\lambda \xi_1}\right), \quad \Lambda^*(x) := \sup_{\lambda} E(\lambda x - \Lambda(\lambda)),$$

$$I(\phi) := \begin{cases} \int_0^1 \Lambda^*\left(\dot{\phi}(t) - \frac{1}{\sqrt{n}} b\left(\sqrt{n}\phi(t)\right)\right) dt, & \text{if } \phi \in AC, \phi(0) = 0 \\ 0, & \text{otherwise} \end{cases}.$$

Then we have for any closed $F \in \mathbf{C}[0,1]$

$$\limsup_{n \to \infty} \frac{1}{n} \log P\left\{\sqrt{n} X_n \in F\right\} = O\left(-\inf_{x \in F} I(x)\right) \quad \text{as } n \to \infty. \tag{12}$$

References

Ikeda, N. and Watanabe, S., Stochastic Differential Equations and Diffusion Processes, North-Holland Publ. Co., Amsterdam, Oxford, New York; Kodansha Ltd., Tokyo 1981.

Kanagawa, S. (1988), On the rate of convergence for Maruyama's approximate solutions of stochastic differential equations,Yokohama Math. J., 36, 79-85.

Kanagawa, S. (1989), The rate of convergence for approximate solutions of stochastic differential equations, Tokyo J. Math.,12, 33-48.

Kanagawa, S., (1995) Error estimation for the Euler-Maruyama approximate solutions of stochastic differential equations, Monte Carlo MethodsAppl., 1, 165 - 171.

Kanagawa, S., Convergence rates for the Euler-Maruyama type approximate solutions of stochastic differential equations, Probability Theory and Mathematical Statistics, Proceedings of the Seventh Japan-Russia Symposium, pp. 183-192, World Scientific Publ., Singapore, 1996.

Kanagawa, S., Confidence intervals of discretized Euler-Maruyama approximate solutions of SDE's, Theory, Methods & Applications (Proc. 2nd World Congress of Nonlinear Analysts, Greece, 1996), Vol. 30, No. 7 (1997) 4101-4104.

Kloeden, P. E. and Platen, E., Numerical Solutions of Stochastic Differential Equations}, Springer-Verlag, Berlin, 1992.

Kloeden, P. E., Platen, E. and Schurz, H., Numerical Solutions of SDE Through Computer Experiments, Springer-Verlag, Berlin, 1994.

Maruyama, G. (1955), Continuous Markov processes and stochastic equations, Rend Circ. Mat. Palermo, 4, 48-90.

Ogawa, S. (1992), Monte Carlo Simulation of Nonlinear Diffusion processes, Japan J. of Industrial and Appl. Math., 9, 22-33.

Schilder, M. (1966), Some asymptotic formulae for Wiener integrals, Trans. Amer. Math. Soc., 125, 63-85.

Shimizu, A. (1984), Approximation solutions for stochastic differential equations, Proc. Symp. on Stochastic Differential Equations for Population Genetics, 201-210 (in Japanese).