SDEのEuler-丸山型近似解に対するSample Path Large Deviations (確率数値解析に於ける諸問題，IV)

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SDEのEuler-丸山型近似解に対するSample Path Large Deviations

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Introduction: Itô's SDE
Let \( \{B(t), 0 \leq t \leq 1\} \) be an \( r \)-dimensional standard Brownian motion. Consider Itô's stochastic differential equation (SDE) for a \( d \)-dimensional continuous process \( \{X(t), 0 \leq t \leq 1\} \) (\( d \geq 1 \)):

\[
\begin{cases}
  dX(t) = \sigma(t, X(t))dB(t) + b(t, X(t))dt, & 0 \leq t \leq 1 \\
  X(0) = X_0.
\end{cases}
\]

Suppose that \( \sigma(t, x) \) and \( b(t, x) \) satisfy the Lipschitz condition. Then their exists a unique solution of (1). (See e.g., Ikeda-Watanabe (1981).)

1. The Euler-Maruyama Algorithm
Maruyama (1955) showed the existence of the unique solution of (1) using an Euler type approximation solution \( Z_n := \{Z_n(t), 0 \leq t \leq 1\} \) defined by

\[
Z_n(t) := X_0 + \int_0^t \sigma_n(u)dB(u) + \int_0^t b_n(u)du, \quad 0 \leq t \leq 1,
\]

where

\[
\begin{align*}
\sigma_n(t) &:= \sigma\left(\frac{k-1}{n}, x_{k-1}\right), \quad k/n \leq t \leq (k+1)/n, \quad k = 0, \ldots, n-1, \\
b_n(t) &:= b\left(\frac{k-1}{n}, x_{k-1}\right), \quad k/n \leq t \leq (k+1)/n, \quad k = 0, \ldots, n-1, \\
x_k &:= X_0 + \sum_{j=1}^k \sigma\left(\frac{j-1}{n}, x_{j-1}\right)\eta_j + \sum_{j=1}^k b\left(\frac{j-1}{n}, x_{j-1}\right), \quad k = 0, 1, \ldots, n, \\
\eta_k &:= B\left(\frac{k}{n}\right) - B\left(\frac{k-1}{n}\right), \quad k = 1, \ldots, n.
\end{align*}
\]
Theorem A. (Maruyama (1955)) Suppose that $\sigma(t,x)$ and $b(t,x)$ satisfy the Lipschitz condition, i.e.

$$|\sigma(t,x) - \sigma(s,y)|^2 + |b(t,x) - b(s,y)|^2 \leq K_1(|x-y|^2 + |t-s|^2),$$

where $K_1$ is a positive constant independent of $x, y, t$ and $s$. Then for any $t \geq 0$

$$\lim_{n \to \infty} E\left( |X(t) - Z_n(t)|^2 \right) = 0.$$

The above scheme for the construction of $Z_n(t)$ has no practical value since it needs the complete knowledge about whole trajectory over the interval $[0,1]$ of the Brownian motion, which is also to be simulated. Therefore we find it necessary to introduce a stochastic process $X_n := \{X_n(t), 0 \leq t \leq 1\}$ in $D(0,1)$ by a slight modification of $Z_n$

$$\begin{cases} X_n(t) := x_k, & k/n \leq t < (k+1)/n, \ k = 0, \ldots, n-1 \\ X_n(1) := x_n, \end{cases}$$

here $\{\eta_k\}$ are i.i.d. random variables constructed from pseudo-random numbers with the $r$-dimensional normal distribution $N(0,1/n)$.

As for the error estimation for $Z_n$, Gihman-Skorohod (1979), Shimizu (1984) showed the rate of convergence of them to the real solution $X$ of (1) in $L^p$-mean for some $p \geq 2$. On the other hand Kanagawa (1988) considered the error of $X_n$ as follows.

Theorem 1. Kanagawa (1988) Suppose that for any $0 \leq s, t \leq 1$ and $x,y \in \mathbb{R}^d$

$$\begin{align*}
(4) & \quad |\sigma(t,x) - \sigma(s,y)|^2 + |b(t,x) - b(s,y)|^2 \leq K_1(|x-y|^2 + |t-s|^2), \\
(5) & \quad |\sigma(t,x)|^2 + |b(s,y)|^2 \leq K_2,
\end{align*}$$

where $K_1$ and $K_2$ are some positive constants independent of $s, t, x$ and $y$. Then for any $p \geq 2$ and for some $\varepsilon > p/2$

$$\begin{align*}
(6) & \quad E\left( \max_{0 \leq s \leq 1} |X(t) - X_n(t)|^p \right) = o\left(n^{-\frac{p}{2}}(\log n)^\varepsilon\right) \quad \text{as } n \to \infty, \\
(7) & \quad E\left( \max_{0 \leq s \leq 1} |X(t) - Z_n(t)|^p \right) = O\left(n^{-\frac{p}{2}}\right) \quad \text{as } n \to \infty.
\end{align*}$$
Furthermore, in the case when the approximate solutions are constructed from $r$-dimensional i.i.d. random variables $\{\xi_k\}$ which do not always obey the normal distribution, Kangawa (1989) showed the rate of convergence of $E\left(\max_{0 \leq t \leq 1} |X(t) - Y_n(t)|^p\right)$ to zero, where $Y_n := \{Y_n(t), 0 \leq t \leq 1\}$ is a stochastic process in $D(0,1)$ defined by for $k=0,1,\ldots,n$

$$\begin{cases} Y_n(t) := y_k, & k/n \leq t < (k+1)/n, \quad k=0,\ldots,n-1 \\ Y_n(1) := y_n \end{cases}$$

where

$$y_k := X_0 + \sum_{j=1}^{k} \sigma\left(\frac{j-1}{n}, Y_{j-1}\right) \xi_j / \sqrt{n} + \sum_{j=1}^{k} b\left(\frac{j-1}{n}, Y_{j-1}\right) / n.$$ 

**Theorem 2.** Kanagawa (1989) Let $\{\xi_k, k \geq 1\}$ be $r$-dimensional i.i.d. random variables with

$$E(\xi_1) = 0, E\left(|\xi_1|^2\right) = 1, E\left(|\xi_1|^{2+\delta}\right) < \infty$$ for some $0 < \delta \leq 1$.

Assume $\sigma(t,x)$ and $b(t,x)$ satisfy (4) and (5). Then we can redefine $\{X(t), 0 \leq t \leq 1\}$ and $\{Y_n(t), 0 \leq t \leq 1\}$ on a common probability space such that for any $p \geq 2$ and $\epsilon > (2 + \delta)^2 / (2 + \delta)$,

$$E\left(\max_{0 \leq t \leq 1} |X(t) - Y_n(t)|^p\right) = o\left(n^{-\delta/2(2+\delta)}(\log n)^\epsilon\right)$$ as $n \to \infty$,

where the power of $n$ cannot be improved by better one.

Furthermore, under the Cramér's condition instead of $E\left(|\xi_1|^{2+\delta}\right) < \infty$, we have the next result.

**Theorem 3.** Kanagawa (1995) Let $\{\xi_k, k \geq 1\}$ be $r$-dimensional i.i.d. random variables with $E(\xi_1) = 0, E\left(|\xi_1|^2\right) = 1$. Suppose that $E\left(e^{4x_1}\right) \leq \infty$ in a neighborhood of $t=0$. Assume $\sigma(t,x)$ and $b(t,x)$ satisfy (4) and (5). Then we can redefine $\{X(t), 0 \leq t \leq 1\}$ and $\{Y_n(t), 0 \leq t \leq 1\}$ on a common probability space such that for any $p \geq 2, \epsilon > p/2$ and for sufficiently large $n$

$$E\left(\max_{0 \leq t \leq 1} |X(t) - Y_n(t)|^p\right) = o\left(n^{-4\epsilon}(\log n)^\epsilon\right)$$ as $n \to \infty$. 

2. Sample Path Large Deviations
We can apply Schiler's Brownian motion sample path large deviations to Euler-Maruyama approximate solutions $X_n := \{X_n(t), 0 \leq t \leq 1\}$ for SDE's.

**Theorem 4.** Consider the following SDE,
$$
\begin{cases}
    dX(t) = dB(t) + b(t, X(t)) dt, & 0 \leq t \leq 1 \\
    X(0) = X_0.
\end{cases}
$$

Suppose that $b(t, x)$ satisfies the Lipschitz condition (4). Let $\{\xi_k, k \geq 1\}$ be $r$-dimensional i.i.d. random variables with $E(\xi_1) = 0, E(|\xi_1|^2) = 1$. Suppose that $E(e^{\xi_1}) \leq \infty$ in a neighborhood of $t = 0$. Put
$$
\Lambda(\lambda) = \log E(e^{\lambda \xi_1}), \quad \Lambda^*(x) = \sup_{\lambda} E(\lambda x - \Lambda(\lambda)),
$$
$$
I(\phi) := \left\{ \begin{array}{ll}
\int_0^1 \Lambda^*(\phi(t) - \frac{1}{\sqrt{n}} b(\sqrt{n}\phi(t))) dt, & \text{if } \phi \in AC, \phi(0) = 0 \\
0, & \text{otherwise}
\end{array} \right.
$$

Then we have for any closed $F \in \mathcal{C}[0,1]$

$$
\limsup_{n \to \infty} \frac{1}{n} \log P\left[ \sqrt{n}X_n \in F \right] = O\left( -\inf_{x \in F} \Lambda^*(x) \right) \quad \text{as } n \to \infty.
$$

References


