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On the number of zeros of principal solutions to second-order half-linear ordinary differential equations

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1. Introduction

First consider the half-linear differential equation

$$\text{(H)} \qquad (|x'|^{\alpha}\operatorname{sgn} x')' + q(t)|x|^{\alpha}\operatorname{sgn} x = 0, \quad t \geq a,$$

where $\alpha > 0$ is a constant and $q(t)$ is a continuous function on $[a, \infty), a > 0$, with the property that $q(t) > 0$ $(t \geq a)$. If $\alpha = 1$, then equation (H) becomes the linear equation

$$\text{(L)} \qquad x'' + q(t)x = 0, \quad t \geq a.$$

Although (H) is nonlinear for $\alpha \neq 1$, its qualitative behavior is essentially the same as that of the linear equation (L). For any initial condition $x(b) = x_0 \in \boldsymbol{R}$, $x'(b) = x_1 \in \boldsymbol{R}$ $(b \geq a)$, equation (H) has a unique solution $x(t)$ on the interval $[a, \infty)$. Therefore a nontrivial solution $x(t)$ of (H) has either a finite number of zeros on $[a, \infty)$, in which case $x(t)$ is called nonoscillatory, or an infinite number of zeros clustering at $t = \infty$, in which case $x(t)$ is called oscillatory. Furthermore Sturmian separation and comparison theorems can be established ([1, 5, 6]) for the half-linear equation (H) as a natural extension of (L). Thus nontrivial solutions of (H) are either all nonoscillatory or else all oscillatory. As usual, if the former occurs, then (H) is called nonoscillatory, and if the latter occurs, then (H) is called oscillatory.

Now let us consider the half-linear equation

$$(\mathrm{H}_\lambda) \qquad (|x'|^{\alpha}\operatorname{sgn} x')' + \lambda q(t)|x|^{\alpha}\operatorname{sgn} x = 0, \quad t \geq a,$$

containing a positive parameter $\lambda > 0$. As in the linear case, we say that (H_λ) is strongly nonoscillatory [resp. strongly oscillatory] if (H_λ) is nonoscillatory [resp. oscillatory] for every $\lambda > 0$.

A complete characterization of the strong nonoscillation and the strong oscillation is obtained in the following theorem, which is a direct generalization of a result of Nehari [7].

THEOREM A (Kusano, Y. Naito and Ogata [4]). (i) (H_λ) *is strongly nonoscillatory if and only if*

$$\lim_{t\to\infty} t^\alpha \int_t^\infty q(s)ds = 0. \tag{1}$$

(ii) (H_λ) *is strongly oscillatory if and only if*

$$\limsup_{t\to\infty} t^\alpha \int_t^\infty q(s)ds = \infty. \tag{2}$$

In this paper we are interested in the situation where (H_λ) is strongly nonoscillatory and are concerned with the problem of counting the number of zeros of (nonoscillatory) solutions of (H_λ). The main purpose of this paper is to show that, in the case $\alpha \geq 1$, precise information about the number of zeros can be drawn for some special type of solutions $x_\lambda(t)$ of (H_λ) such that

$$\lim_{t\to\infty} \frac{x_\lambda(t)}{\sqrt{t}} = 0. \tag{3}$$

It can be proved that if (H_λ) is strongly nonoscillatory, then, for each $\lambda > 0$, there is a nonoscillatory solution $x_\lambda(t)$ of (H_λ) satisfying (3) and $x_\lambda(t)$ is uniquely determined up to a nonzero constant multiple. Then we have the next theorem.

THEOREM 1. *Let $\alpha \geq 1$ and suppose that (H_λ) is strongly nonoscillatory. Then there exists a sequence $\{\lambda_n\}_{n=1}^\infty$ of positive parameters with the properties that*

(i) $0 = \lambda_0 < \lambda_1 < \cdots < \lambda_n < \cdots, \quad \lim_{n\to\infty} \lambda_n = \infty;$

(ii) *if $\lambda \in (\lambda_{n-1}, \lambda_n)$, $n = 1, 2, \cdots$, then $x_\lambda(t)$ has exactly $n-1$ zeros in the open interval (a, ∞) and $x_\lambda(a) \neq 0$;*

(iii) *if $\lambda = \lambda_n$, $n = 1, 2, \cdots$, then $x_\lambda(t)$ has exactly $n-1$ zeros in the open interval (a, ∞) and $x_\lambda(a) = 0$.*

COROLLARY 1. *Consider the singular eigenvalue problem*

$$\begin{cases} (|x'|^\alpha \mathrm{sgn}\, x')' + \lambda q(t)|x|^\alpha \mathrm{sgn}\, x = 0, & t \geq a, \\ x(a) = 0 \quad \text{and} \quad \lim_{t\to\infty} \dfrac{x(t)}{\sqrt{t}} = 0. \end{cases} \tag{4}$$

Let $\alpha \geq 1$ and suppose that (1) holds. Then the totality of eigenvalues of (4) is written as a sequence $\{\lambda_n\}_{n=1}^\infty$, where $0 < \lambda_1 < \cdots < \lambda_n < \cdots$, $\lim_{n\to\infty} \lambda_n = \infty$, and the eigenfunction $x_{\lambda_n}(t)$ of (4) associated with $\lambda = \lambda_n$ has exactly $n-1$ zeros on (a, ∞).

Theorem 1 is closely related to the results in [2], and Theorem 1 for the case $\alpha = 1$ is given in [3].

2. Proof of the Theorem

PROPOSITION 1. *Let $\alpha \geq 1$ and suppose that* (1) *holds. Then, for each $\lambda > 0$, there is an eventually positive solution $x_\lambda(t)$ of* (H_λ) *satisfying* (3). *Further, such a solution $x_\lambda(t)$ is uniquely determined up to a positive constant multiple.*

Note: The condition $\alpha \geq 1$ is used for showing that $x_\lambda(t)$ is uniquely determined up to a positive constant multiple. The existence of a solution $x_\lambda(t)$ is valid for the case $0 < \alpha < 1$.

If we require that a solution $x_\lambda(t)$ obtained in Proposition 1 satisfies the nomalized condition

$$[x_\lambda(a)]^2 + [x'_\lambda(a)]^2 = 1,$$

then $x_\lambda(t)$ is uniquely determined. We denote this normalized solution of (H_λ) by $x(t;\lambda)$. Thus $x(t;\lambda)$ is a unique solution of (H_λ) such that $x(t;\lambda)$ is eventually positive and satisfies

$$\lim_{t\to\infty} \frac{x(t;\lambda)}{\sqrt{t}} = 0 \tag{5}$$

and

$$[x(a;\lambda)]^2 + [x'(a;\lambda)]^2 = 1. \tag{6}$$

By the proof of Proposition 1 we see that

- $\dfrac{x(t;\lambda)}{\sqrt{t}} \to 0 \quad \text{as} \quad t \to \infty;$
- $\sqrt{t}x'(t;\lambda) \to 0 \quad \text{as} \quad t \to \infty;$
- $x(a;\lambda) \to 1 \quad \text{as} \quad \lambda \to +0;$
- $x'(a;\lambda) \to 0 \quad \text{as} \quad \lambda \to +0;$

and

- $x(t;\lambda)$ is a continuous function of $\lambda \in (0,\infty)$ for each fixed $t \in [a,\infty)$.

Moreover we find that

- there is $\lambda_* > 0$ such that if $0 < \lambda < \lambda_*$, then $x(t;\lambda) > 0$ for $t \geq a$;

and

- for any $N \in \boldsymbol{N}$, there is $\lambda^* > 0$ such that if $\lambda > \lambda^*$, then $x(t;\lambda)$ has at least N zeros in the interval $[a, a+1]$.

Now let us define the generalized trigonometric functions $S(\tau)$ and $C(\tau)$. The generalized sine function $S(\tau)$ is defined as the solution of the specific half-linear equation

$$(|\dot{S}|^{\alpha-1}\dot{S})^{\cdot} + \alpha|S|^{\alpha-1}S = 0 \qquad \left(\cdot = \frac{d}{d\tau} \right)$$

satisfying the initial condition

$$S(0) = 0, \qquad \dot{S}(0) = 1.$$

The generalized cosine function $C(\tau)$ is the derivative of $S(\tau)$: $C(\tau) = \dot{S}(\tau)$. The generalized trigonometric functions $S(\tau)$ and $C(\tau)$ have the same properties as the classical sine function $\sin\tau$ and the classical cosine function $\cos\tau$. (See [1] for the details.) They are defined on $\boldsymbol{R}$ and are periodic with period $2\pi_\alpha$, where

$$\pi_\alpha = \frac{2\pi}{\alpha+1} \Big/ \sin\frac{\pi}{\alpha+1}.$$

Further, the generalized Pythagorean theorem holds for $S(\tau)$ and $C(\tau)$:

$$|S(\tau)|^{\alpha+1} + |C(\tau)|^{\alpha+1} = 1 \quad \text{for all } \tau.$$

The generalized sine and cosine functions may be used for the generalized Prüfer transformation. For the solution $x(t;\lambda)$, we perform the next transformation, which consists in associating with $x(t;\lambda)$ the polar functions $\rho(t;\lambda)$ and $\theta(t;\lambda)$ defined by

$$x(t;\lambda) = \rho(t;\lambda)S(\theta(t;\lambda)), \qquad x'(t;\lambda) = \rho(t;\lambda)C(\theta(t;\lambda)). \tag{7}$$

It is easy to see that

$$\rho(t;\lambda) = \left(|x(t;\lambda)|^{\alpha+1} + |x'(t;\lambda)|^{\alpha+1} \right)^{1/(\alpha+1)}.$$

Moreover it can be shown that $\theta = \theta(t;\lambda)$ satisfies the first order differential equation

$$\theta' = |C(\theta)|^{\alpha+1} + \frac{\lambda}{\alpha} q(t)|S(\theta)|^{\alpha+1}. \tag{8}$$

By the properties of $x(t;\lambda)$ and $x'(t;\lambda)$, we have

$$\lim_{t\to\infty} \theta(t;\lambda) = \frac{\pi_\alpha}{2} + 2m\pi_\alpha \quad \text{for some } m \in \boldsymbol{Z}.$$

We suppose without loss of generality that

$$\lim_{t\to\infty} \theta(t;\lambda) = \frac{\pi_\alpha}{2}.$$

The basic properties of $\theta(t;\lambda)$ are as follows:

- $\theta(t;\lambda)$ is a continuous function of $\lambda \in (0,\infty)$ for each fixed $t \in [a,\infty)$;
- $\theta(t;\lambda)$ is strictly increasing in $t \in [a,\infty)$ for each fixed $\lambda \in (0,\infty)$;
- $\theta(t;\lambda)$ is strictly decreasing in $\lambda \in (0,\infty)$ for each fixed $t \in [a,\infty)$;
- $\lim_{\lambda\to+0} \theta(a;\lambda) = \frac{\pi_\alpha}{2}$;
- $\lim_{\lambda\to\infty} \theta(a;\lambda) = -\infty$.

For the proofs of the strict increasingness in $t \in [a,\infty)$ and the strict decreasingness in $\lambda \in (0,\infty)$ of $\theta(t;\lambda)$, the equation (8) is effectively used. The other properties of $\theta(t;\lambda)$ are easily proved by the above-mentioned properties of $x(t;\lambda)$ and $x'(t;\lambda)$.

From the above discussions we see that, for each $n = 1, 2, \cdots$, there exists $\lambda_n > 0$ such that

$$\theta(a;\lambda_n) = -(n-1)\pi_\alpha.$$

Then, in view of the generalized Prüfer transformation (7), we find that the sequence $\{\lambda_n\}_{n=1}^{\infty}$ satisfies the properties (i), (ii) and (iii) in Theorem 1.

References

[1] Á. Elbert, A half-linear second order differential equation, Colloq. Math. Soc. J. Bolyai **30**: Qualitative Theory of Differential Equations, (Szeged) (1979), 153–180.

[2] Á. Elbert, T. Kusano and M. Naito, On the number of zeros of nonoscillatory solutions to second-order half-linear differential equations, Ann. Univ. Sci. Budapest. Etotvos Sect. Math., to appear.

[3] T. Kusano and M. Naito, A singular eigenvalue problem for the Sturm-Liouville equation, Differentsial'nye Uravneniya, 34 (1998), 303–312.

[4] T. Kusano, Y. Naito and A. Ogata, Strong oscillation and nonoscillation of quasilinear differential equations of second order, Differential Equations and Dynamical Systems, **2** (1994), 1–10.

[5] H. J. Li and C. C. Yeh, Sturmian comparison theorem for half-linear second-order differential equations, Proc. Roy. Soc. Edinburgh, **125A** (1995), 1193–1204.

[6] J. D. Mirzov, On some analogs of Sturm's and Kneser's theorems for nonlinear systems, J. Math. Anal. Appl., **53** (1976), 418–425.

[7] Z. Nehari, Oscillation criteria for second-order linear differential equations, Trans. Amer. Math. Soc., **85** (1957), 428–445.