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A NONOSCILLATION THEOREM FOR SECOND ORDER NONLINEAR DIFFERENTIAL EQUATIONS WITH DECAYING COEFFICIENTS

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ABSTRACT. The purpose of this paper is to give sufficient conditions for all nontrivial solutions of the nonlinear differential equation $x''+a(t)g(x)=0$ to be nonoscillatory. Here $g(x)$ satisfies the sign condition $x\,g(x)>0$ if $x\neq 0$, but is not assumed to be monotone increasing. This differential equation includes the generalized Emden-Fowler equation as a special case. Our main result extends some nonoscillation theorem for the generalized Emden-Fowler equation. Proof is given by means of some Liapunov functions and phase plane analysis.

1. INTRODUCTION

We consider the second order nonlinear differential equation

$$x''+a(t)g(x)=0 \tag{1.1}$$

in which $a(t)$ is positive, continuous and locally of bounded variation on some half line $[t_0,\infty)$, and $g(x)$ is continuous on $\mathbf{R}$ and satisfies

$$x\,g(x)>0 \quad \text{if} \quad x\neq 0. \tag{1.2}$$

But we do not necessarily require that $g(x)$ be monotone increasing. Since $a(t)$ is continuous and locally of bounded variation, $a(t)$ has the Jordan representation $a(t)=a_+(t)-a_-(t)$, where a_+ and a_- are continuous nondecreasing functions of t. Throughout this paper we assume that the uniqueness is guaranteed for the solutions of (1.1) to the initial value problem.

The generalized Emden-Fowler differential equation

$$x''+a(t)|x|^{\gamma}\mathrm{sgn}\,x=0 \tag{1.3}$$

is a special case of (1.1), where γ is a positive constant. Under the assumptions on $a(t)$, it is known that equation (1.3) has a unique solution satisfying given initial conditions and every solution of (1.3) is continuable in the future. For details, we refer to [3, 4, 9]. The oscillation problem for equation (1.3) has been widely researched in many papers (for example, see [1, 2, 4, 5, 7, 10, 12, 13, 17, 18] and the references cited therein).

A solution of (1.1) is said to be *nonoscillatory* if it is eventually of one sign. Our purpose here is to give conditions under which all nontrivial solutions of (1.1) are nonoscillatory in the case when the coefficient $a(t)$ goes to decay as t increases.

It is helpful to describe some nonoscillation criteria for equation (1.3) before stating our main result. For the linear case, $\gamma = 1$, Hille [11] showed that all nontrivial solutions of (1.3) are nonoscillatory if

$$\limsup_{t\to\infty} t^2 a(t) < \frac{1}{4}.$$

In the case $\gamma \neq 1$, equation (1.3) is customarily divided into two cases as follows. Equation (1.3) is of superlinear when $\gamma > 1$, of sublinear when $0 < \gamma < 1$. For the superlinear case, Atkinson [1] first proved the following result. Under the assumption that $a(t)$ is continuously differentiable and $a'(t) \leq 0$ for $t \geq t_0$, if

$$\int_{t_0}^{\infty} t^{\gamma} a(t)dt < \infty,$$

then all nontrivial solutions of (1.3) are nonoscillatory. For the sublinear case, under the same assumption, Heidel [10] gave the result corresponding to Atkinson's theorem. If

$$\int_{t_0}^{\infty} t a(t)dt < \infty, \tag{1.4}$$

then all nontrivial solutions of (1.3) are nonoscillatory.

Gollwitzer [7] investigated this problem under the assumption that $a(t)$ is locally of bounded variation and

$$\int_{t_0}^{\infty} \frac{da_+(t)}{a(t)} < \infty. \tag{1.5}$$

He showed that each of

$$\lim_{t\to\infty} t\int_{t}^{\infty} s^{\gamma-1} a(s)ds = 0,$$

$$\lim_{t\to\infty} t\int_{t}^{\infty} a(s)^{2/(\gamma+1)}ds = 0$$

and

$$\int_{t_0}^{\infty} a(t)^{1/(\gamma+1)} < \infty$$

is a nonoscillation criterion for equation (1.3) with $\gamma > 1$ and each of

$$\lim_{t\to\infty} a(t)^{(\gamma-1)/2}\int_{t}^{\infty} s^{\gamma} a(s)ds = 0, \tag{1.6}$$

$$\lim_{t\to\infty} a(t)^{(\gamma-1)/2(\gamma+1)}\int_{t}^{\infty} a(s)^{1/(\gamma+1)}ds = 0 \tag{1.7}$$

and (1.4) is a nonoscillation criterion for equation (1.3) with $0 < \gamma < 1$.

Wong [18] and Kwong and Wong [13] clarifed that the equivalent among (1.6), (1.7),

$$\lim_{t\to\infty} t\int_{t}^{\infty} a(s)ds = 0$$

and

$$\lim_{t\to\infty} t^2 a(t) = 0. \tag{1.8}$$

By their works, we have the following result which are more easy to use than previous results.

THEOREM A (Wong [18]). *Let* $0 < \gamma < 1$ *and let* $a(t)$ *satisfy* (1.5). *Then* (1.8) *implies that all nontrivial solutions of* (1.3) *are nonoscillatory.*

THEOREM B (Wong [18]). *Let* $\gamma > 1$ *and let* $a(t)$ *satisfy* (1.5). *Then*

$$\lim_{t\to\infty} t^{\gamma+1}a(t) = 0$$

implies that all nontrivial solutions of (1.3) *are nonoscillatory.*

Later, Erbe [6] removed the restriction (1.5) and showed that $\int_{t_0}^{\infty}(da_+(t)/a(t)) = \infty$ is compatible with nonoscillation for equation (1.3). He improved the results in [1, 7, 18]. Unfortunately, his results are somewhat complicate and his conditions have no relations like the equivalent among (1.6)–(1.8). We intend to discuss the nonoscillation problem for equation (1.1) under the assumption (1.5) and relax restrictions on $g(x)$ rather than $a(t)$.

Our main result is as follows:

THEOREM 1.1. *Assume* (1.2) *and* (1.5). *Suppose that there exists* $\alpha \geq 1$ *satisfying*

$$\lim_{t\to\infty} t^{\alpha+1}a(t) = 0 \tag{1.9}$$

and

$$\limsup_{x\to\infty} \frac{g(x)}{|x|^{\alpha}\mathrm{sgn}\, x} < \infty \quad or \quad \limsup_{x\to-\infty} \frac{g(x)}{|x|^{\alpha}\mathrm{sgn}\, x} < \infty. \tag{1.10}$$

Then all nontrivial solutions of (1.1) *are nonoscillatory.*

It is safe to say that all nontrivial solutions of (1.1) have a tendency to be nonoscillatory as $a(t)g(x)$ grows less in some sense. Hence, in our problem, it is important to examine the relation between the decay of $a(t)$ and the growth of $g(x)$. Judging from previous results on nonoscillation, conditions (1.9) and (1.10) seem to be reasonable. The result above extends Theorem A when $\alpha = 1$ and extends Theorem B when $\alpha > 1$.

In the next section, using Liapunov's second method, we will prove that all solutions of (1.1) can be continued for all future time. In Section 3, we will discuss unbondedness of solutions of (1.1) by means of phase plane analysis for a system which is equivalent to (1.1). We call here the projection of a positive semitrajectory of the system onto the phase plane a *positive orbit*. In Section 4, we will give the proof of the main theorem. We will also give a simple example to illustrate our result in Section 5.

2. CONTINUATION OF SOLUTIONS

In this section, we will show that every solution of (1.1) exists in the future. Hara, Yoneyama and the author [8] discussed the continuation problem by means of two Liapunov functions for the system

$$\begin{aligned} \dot{x} &= F_1(t,x,y), \\ \dot{y} &= F_2(t,x,y), \end{aligned} \tag{2.1}$$

where $F_1 : [0,\infty) \times \mathbf{R}^m \times \mathbf{R}^n \to \mathbf{R}^m$ and $F_2 : [0,\infty) \times \mathbf{R}^m \times \mathbf{R}^n \to \mathbf{R}^n$ are continuous. Following Yoshizawa [19, 20], if $V\colon [0,\infty)\times\mathbf{R}^m\times\mathbf{R}^n \to \mathbf{R}$ is continuous and locally Lipschitz in (x, y), then we call $V(t,x,y)$ a *Liapunov function* for system (2.1) and define

$$\dot{V}_{(2.1)}(t,x,y) = \limsup_{h\to 0^+} \frac{1}{h}\{V(t+h, x+hF_1(t,x,y), y+hF_2(t,x,y)) - V(t,x,y)\}.$$

We also call that a scalar function $\phi\colon [0,\infty)\times\mathbf{R}\to\mathbf{R}$ is of class $\mathcal{G}$ if, for any t_0 and $u_0 \in \mathbf{R}$, the maximal solution $u(t,t_0,u_0)$ of the equation

$$u' = \phi(t,u)$$

exists in the future. Then we have:

THEOREM C (Hara et al. [8]). *Let $V\colon [0,\infty)\times\mathbf{R}^m\times\mathbf{R}^n\to\mathbf{R}$ be a Liapunov function such that*

$$\begin{aligned} &V(t,x,y)\to\infty \quad \text{as } \|y\|\to\infty \text{ uniformly in } x\in\mathbf{R}^m \\ &\text{for each fixed } t \end{aligned} \tag{2.2}$$

and

$$\dot{V}_{(2.1)}(t,x,y) \le \phi(t, V(t,x,y)) \quad \text{for some } \phi\in\mathcal{G}. \tag{2.3}$$

Moreover, suppose that for each $K>0$ and $L>0$ there exists a Liapunov function W : $[0,L]\times\mathbf{R}^m\times S_K^n\to\mathbf{R}$, $S_K^n = \{y\in\mathbf{R}^n\colon \|y\|\le K\}$ which satisfies

$$\begin{aligned} &W(t,x,y)\to\infty \quad \text{as } \|x\|\to\infty \text{ uniformly in } y\in S_K^n \\ &\text{for each fixed } t \end{aligned} \tag{2.4}$$

and

$$\dot{W}_{(2.1)}(t,x,y) \le \psi(t, W(t,x,y)) \quad \text{for some } \psi\in\mathcal{G}. \tag{2.5}$$

Then every solution of (2.1) exists in the future.

Using Theorem C, we can prove the following continuation result.

THEOREM 2.1. *Assume (1.2). Then every solution of (1.1) and its derivative exist in the future.*

Proof. We consider the system

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -a(t)g(x) \end{aligned} \tag{2.6}$$

which is equivalent to (1.1). Define two Liapunov functions

$$V(t,x,y) = \frac{1}{2}y^2 + a(t)G(x),$$

where $G(x) = \int_0^x g(\xi)d\xi$, and

$$W(t,x,y) = |x|.$$

By assumption (1.2), we have $G(x) > 0$ if $x \neq 0$, and therefore, condition (2.2) is satisfied with $m = 1$. Also, condition (2.4) is satisfied with $n = 1$. Since $a(t)$ is continuous and locally of bounded variation, we have the Jordan decomposition

$$a(t) = a_+(t) - a_-(t),$$

where a_+ and a_- are continuous and nondecreasing. Hence, the upper right Dini derivatives $D^+a_+(t)$ and $D^+a_-(t)$ are nonnegative (see, for example, [14, pp. 347–348]). We obtain

$$\begin{aligned}\dot{V}_{(2.6)}(t,x,y) &= (D^+a(t))G(x) = (D^+a_+(t))G(x) - (D^+a_-(t))G(x)\\ &\leq (D^+a_+(t))G(x) \leq \frac{D^+a_+(t)}{a(t)}V(t,x,y)\end{aligned}$$

and

$$\dot{W}_{(2.6)}(t,x,y) \leq |y| \leq K \quad \text{on } S_K^1.$$

Since scalar functions $\phi(t,u) = (D^+a_+(t)/a(t))u$ and $\psi(t,u) = K$ belong to $\mathcal{G}$, conditions (2.3) and (2.5) are also satisfied. Thus, by Theorem C all solutions of (2.6) are continuable in the future. This means that every solution of (1.1) and its derivative exist in the future and completes the proof.

3. Unboundedness of solutions

A solution of (1.1) is said to be *oscillatory* if it has arbitrarily large zeros. In this section, we will show that all nontrivial oscillatory solutions of (1.1) are unbounded. To this end, we transform equation (1.1) into the equivalent planar system

$$\begin{aligned}\dot{u} &= v + u,\\ \dot{v} &= -e^{2s}a(e^s)g(u),\end{aligned} \tag{3.1}$$

where $\dot{} = d/ds$ and $u(s) = x(e^s) = x(t)$. From (1.2) and the vector field of (3.1), we see that each positive orbit of (3.1) corresponding to a nontrivial oscillatory solution of (1.1) keeps on rotating around the origin $(u,v) = (0,0)$.

Theorem 3.1. *Assume* (1.2) *and* (1.5). *Then* (1.8) *implies that all nontrivial oscillatory solutions of* (1.1) *are neither bounded from above nor bounded from below.*

To prove Theorem 3.1, we need the following lemma given by Gollwitzer[7].

Lemma 3.1. *Let*

$$E(t) = \frac{1}{2}(x'(t))^2 + a(t)G(x(t))$$

and

$$B(t) = \frac{E(t)}{a(t)}.$$

Then, for $t \geq t_0$ *we have the estimations*

$$E(t) \leq E(t_0)\exp\int_{t_0}^{t}\frac{da_+(s)}{a(s)} \tag{3.2}$$

and

$$B(t_0) \le B(t) \exp \int_{t_0}^{t} \frac{da_+(s)}{a(s)}. \tag{3.3}$$

Proof of Theorem 3.1. Let $x(t)$ be any nontrivial oscillatory solution of (1.1). Then, by (1.2) we have $x(t)x''(t) = -a(t)x(t)g(x(t)) < 0$ if t is not a zero of $x(t)$, and therefore, the local maxima and minima of $x(t)$ alternate with each other. Let $\{t_n\}$ be a sequence such that $x'(t_n) = 0$. We may assume without loss of generality that $x(t_n) > 0$ if n is odd and $x(t_n) < 0$ if n is even. From (3.3) with $t = t_{2m}$ we have

$$B(t_0) \le B(t_{2m}) \exp \int_{t_0}^{t_{2m}} \frac{da_+(s)}{a(s)} = G(x(t_{2m})) \exp \int_{t_0}^{t_{2m}} \frac{da_+(s)}{a(s)}$$

and so from (1.5) we obtain

$$G(x(t_{2m})) \not\to 0 \quad \text{as } m \to \infty.$$

Hence, by (1.2) again, we get

$$x(t_{2m}) \not\to 0^- \quad \text{as } m \to \infty.$$

Thus, there exists a $\rho > 0$ such that

$$\liminf_{m\to\infty} x(t_{2m}) \le -\rho. \tag{3.4}$$

Suppose that $x(t)$ is bounded from above, that is, there exists an $M > 0$ such that

$$x(t) \le M \quad \text{for } t \ge t_0.$$

Let

$$L = \max\{g(x) \colon 0 \le x \le M\}.$$

Then, by (1.8) and (3.4) we can choose an integer l so large that

$$x(t_{2l}) \le -\rho \tag{3.5}$$

and

$$t^2 a(t) < \frac{\rho^2}{4LM} \quad \text{for } t \ge t_{2l}. \tag{3.6}$$

Let $(u(s), v(s))$ be the solution of (3.1) corresponding to $x(t)$ and let $s_n = \log t_n$. Since $u(s) = x(t)$ and $\dot{u}(s) = tx'(t)$, we have $u(s_{2l}) = x(t_{2l})$ and $v(s_{2l}) = -u(s_{2l})$. Hence, by (3.5) we obtain $v(s_{2l}) \ge \rho$. Let $A = (u(s_{2l}), v(s_{2l}))$ and consider the positive orbit $\gamma^+_{(3.1)}(A)$ of (3.1) starting at the point A. Since $x(t)$ is oscillatory, $\gamma^+_{(3.1)}(A)$ rotates around the origin clockwise. Let τ be the first time when $\gamma^+_{(3.1)}(A)$ crosses the positive v-axis. From the vector field of (3.1) we see that

$$u(\tau) = 0 \quad \text{and} \quad v(\tau) > v(s_{2l}) \ge \rho. \tag{3.7}$$

Hence, $\gamma^+_{(3.1)}(A)$ meets the line $v = \rho/2$. Let σ be the first intersecting time of $\gamma^+_{(3.1)}(A)$ with the line. Then $\sigma > \tau > s_{2l}$,

$$0 < u(\sigma) \le M \quad \text{and} \quad v(\sigma) = \frac{\rho}{2}. \tag{3.8}$$

Note that

$$v(s) + u(s) \ge v(s) \ge \frac{\rho}{2} \quad \text{for } \tau \le s \le \sigma.$$

Hence, together with (3.6), we have

$$\frac{\dot v(s)}{\dot u(s)} = -\frac{e^{2s}a(e^s)g(u(s))}{v(s)+u(s)} > -\frac{\rho}{2M}$$

for $\tau \le s \le \sigma$, and therefore, by (3.7) and (3.8) we conclude that

$$\begin{aligned} -\frac{\rho}{2} &> v(\sigma) - v(\tau) \\ &> -\frac{\rho}{2M}(u(\sigma) - u(\tau)) \ge -\frac{\rho}{2}. \end{aligned}$$

This is a contradiction. Thus, no nontrivial oscillatory solutions of (1.1) are bounded from above. Using the same argument, we can show that no nontrivial oscillatory solutions of (1.1) are bounded from below. The proof is complete.

4. Proof of the main theorem

We consider only the former case of (1.10) because we can use the same argument in the latter case of (1.10). Then there exist constants $B > 0$ and $C > 0$ such that

$$g(x) \le Bx^\alpha \quad \text{for } x \ge C. \tag{4.1}$$

The proof is by contradiction. Suppose that equation (1.1) has a nontrivial oscillatory solution $x(t)$. By the estimation (3.2) in Lemma 3.1 and (1.5), we see that $x'(t)$ is bounded for $t \ge t_0$, and therefore, there exists a $K > 0$ such that

$$|x(t)| \le Kt \quad \text{for } t \ge t_0. \tag{4.2}$$

From (1.9) we can select a T so large that

$$t^{\alpha+1}a(t) < \frac{1}{4BK^{\alpha-1}} \quad \text{for } t \ge T. \tag{4.3}$$

Recall that equation (1.1) is transformed into system (3.1) by putting $s = \log t$ and $u(s) = x(t)$ and that every nontrivial solution of (1.1) corresponds to a positive orbit of (3.1) which rotates around the origin in clockwise direction. Let $(u(s), v(s))$ be the solution of (3.1) corresponding to $x(t)$. Note that (1.9) with $\alpha \ge 1$ implies (1.8). By virtue of Theorem 3.1 we see that there exists an $s_1 \ge \log T$ such that

$$u(s_1) \ge C \quad \text{and} \quad v(s_1) = 0.$$

For simplicity, let

$$P_1 = (u_1, 0) = (u(s_1), v(s_1)).$$

Now, we consider the autonomous linear system

$$\begin{aligned} \dot u &= v + u, \\ \dot v &= -\frac{1}{4}u \end{aligned} \tag{4.4}$$

and compare the positive orbit of $\gamma^+_{(3.1)}(P_1)$ with the positive orbit of $\gamma^+_{(4.4)}(P_1)$. Then the slopes of $\gamma^+_{(3.1)}(P_1)$ and $\gamma^+_{(4.4)}(P_1)$ at the point P_1 are

$$-\frac{e^{2s_1}a(e^{s_1})g(u_1)}{u_1} \quad \text{and} \quad -\frac{1}{4},$$

respectively. It follows from (4.1)–(4.3) that

$$0 > -\frac{e^{2s_1}a(e^{s_1})g(u_1)}{u_1} = -\frac{e^{(\alpha+1)s_1}a(e^{s_1})}{u_1}\frac{u_1^{\alpha-1}}{e^{(\alpha-1)s_1}}\frac{g(u_1)}{u_1^{\alpha-1}} > -\frac{1}{4}. \tag{4.5}$$

It is well known that $\gamma^+_{(4.4)}(P_1)$ remains in the region

$$R = \{(u,v) : u > 0 \text{ and } -\frac{1}{2}u < v < 0\}$$

and runs to infinity. On the other hand, $\gamma^+_{(3.1)}(P_1)$ rotates around the origin. Hence, from (4.5) it turns out that $\gamma^+_{(3.1)}(P_1)$ has an intersecting point $P_2 \in R$ with $\gamma^+_{(4.4)}(P_1)$ and lies above $\gamma^+_{(4.4)}(P_1)$ as far as P_2. Let $P_2 = (u_3, v_3)$. Since the arc P_1P_2 of $\gamma^+_{(3.1)}(P_1)$ lies above the arc P_1P_2 of $\gamma^+_{(4.4)}(P_1)$, there exist two points $P_3(u_2, v_1) \in R$ and $P_4(u_2, v_2) \in R$ with

$$0 < u_1 < u_2 \le u_3 \quad \text{and} \quad v_3 \le v_2 \le v_1 < 0$$

satisfying the following conditions:

(i) $\gamma^+_{(3.1)}(P_1)$ passes through the point P_3 at $s = \tau$ and $\gamma^+_{(4.4)}(P_1)$ passes through the point P_4 at $s = \sigma$;

(ii) the slope of $\gamma^+_{(3.1)}(P_1)$ at the point P_3 is steep than that of $\gamma^+_{(4.4)}(P_1)$ at the point P_4.

However, this is impossible. In fact, since

$$\tau \ge s_1 \quad \text{and} \quad v_1 + u_2 \ge v_2 + u_2 > 0,$$

it follows from (i) and (4.1)–(4.3) that

$$0 > -\frac{e^{2\tau}a(e^{\tau})g(u_2)}{v_1 + u_2} \ge -\frac{e^{2\tau}a(e^{\tau})g(u_2)}{v_2 + u_2} \ge -\frac{u_2/4}{v_2 + u_2}.$$

This is a contradiction to (ii). We have thus proved the theorem.

5. Discussion

Our main result, Theorem 1.1, shows that the monotonicity of $g(x)$ is not essential in the nonoscillation problem for equation (1.1). We illustrate our result by a simple example.

EXAMPLE 5.1. Consider equation (1.1) with

$$a(t) = \frac{1}{t^3} \quad \text{and} \quad g(x) = (2 + \sin x)x. \tag{5.1}$$

Then all nontrivial solutions are nonoscillatory.

Clearly, conditions (1.2) and (1.5) hold. We have

$$t^2 a(t) = \frac{1}{t} \to 0 \quad \text{as } t \to \infty$$

and

$$\frac{g(x)}{x} = 2 + \sin x < \infty \quad \text{for } x \in \mathbf{R},$$

and therefore, conditions (1.9) and (1.10) are satisfied with $\alpha = 1$. Hence, from Theorem 1.1 we see that equation (1.1) with (5.1) has no nontrivial oscillatory solutions.

For lack of Sturm's separation theorem, the nonlinear equation (1.1) may possess oscillatory and nonoscillatory solutions at the same time. Theorem 1.1 guarantees, however, that there is no oscillatory solutions except the trivial solution $x(t) \equiv 0$ when $a(t)$ decays rapidly.

It is clear that under the same assumptions in Theorem 1.1, all nontrivial solutions of

$$x'' + \lambda a(t)g(x) = 0$$

are nonoscillatory for all positive λ, that is, equation (1.1) is strongly nonoscillatory. If $a(t)$ decays slowly, then equation (1.1) is not always strongly nonoscillatory. For example, it is well known that all nontrivial solutions of the Euler equation

$$x'' + \frac{\lambda}{t^2}x = 0$$

are oscillatory if $\lambda > 1/4$ and nonoscillatory if $\lambda \le 1/4$. In this case, condition (1.10) is satisfied with $\alpha = 1$, but $a(t) = 1/t^2$ decays slowly, and so condition (1.9) does not hold.

Since the balance between the decay of $a(t)$ and the growth of $g(x)$ is significant, even in the case that $a(t) = 1/t^2$, all nontrivial solutions of (1.1) are nonoscillatory when $g(x)$ grows slowly. The author and Hara [16] considered this case and gave the following result.

THEOREM D (Sugie and Hara). *Assume* (1.2) *and suppose that there exists a* μ *with* $0 < \mu < 1/4$ *such that*

$$\frac{g(x)}{x} \le \frac{1}{4} + \left(\frac{\mu}{\log|x|}\right)^2 \tag{5.2}$$

for $x > 0$ *or* $x < 0$, $|x|$ *sufficiently large. Then all nontrivial solutions of*

$$t^2x'' + g(x) = 0 \tag{5.3}$$

are nonoscillatory.

In case $a(t) = 1/t^2$, condition (1.9) holds for an arbitrary $\alpha < 1$. If we can choose an α with $0 < \alpha < 1$ so that condition (1.10) is satisfied, then condition (5.2) is also satisfied, and therefore, by Theorem D we conclude that all nontrivial solutions of (5.3) are nonoscillatory. Thus, Theorem D indicates that the restriction $\alpha \ge 1$ in Theorem 1.1 is relaxed. At present, however, we cannot answer whether this assertion is true or not.

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