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Existence of oscillatory solutions of neutral differential equations

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1. Introduction

In this paper we consider the neutral differential equation

$$\frac{d^n}{dt^n}[x(t) + \lambda x(t-\tau)] + f(t, x(g(t))) = 0. \tag{1.1}$$

Throughout this paper, the following conditions (H1)–(H3) are assumed:

(H1) $n \in \mathbf{N}$, $\lambda > 0$ and $\tau > 0$;

(H2) $g \in C[t_0, \infty)$ and $\lim_{t\to\infty} g(t) = \infty$;

(H3) $f \in C([t_0, \infty) \times \mathbf{R})$ and there exists a function $F \in C([t_0, \infty) \times [0, \infty))$ such that $F(t, u)$ is nondecreasing in $u \in [0, \infty)$ for each fixed $t \geq t_0$ and satisfies

$$|f(t,u)| \leq F(t, |u|), \quad (t,u) \in [t_0, \infty) \times \mathbf{R}.$$

By a solution of (1.1), we mean a function $x(t)$ which is continuous and satisfies (1.1) on $[t_x, \infty)$ for some $t_x \geq t_0$. Therefore, if $x(t)$ is a solution of (1.1), then $x(t) + \lambda x(t-\tau)$ is n-times continuously differentiable on $[t_x, \infty)$. Note that, in general, $x(t)$ itself is not continuously differentiable.

A solution of (1.1) is said to be oscillatory if it has arbitrarily large zeros; otherwise it is said to be nonoscillatory. This means that a solution $x(t)$ is oscillatory if and only if there is a sequence $\{t_i\}_{i=1}^{\infty}$ such that $t_i \to \infty$ as $i \to \infty$ and $x(t_i) = 0$ $(i = 1, 2, \ldots)$, and a solution $x(t)$ is nonoscillatory if and only if $x(t)$ is either eventually positive or eventually negative.

There has been much current interest in the existence of oscillatory solutions and nonoscillatory solutions of neutral differential equations, and many results have been obtained. For typical results, we refer to the paper [1, 5–15] and the monographs [2, 3].

Neutral differential equations find numerous applications in natural science and technology. For instance, they are frequently used for the study of distributed networks containing lossless transmission lines. See, for example, Hale [4].

Now consider the equation

$$\frac{d^n}{dt^n}[x(t) - \lambda x(t-\tau)] + f(t, x(g(t))) = 0. \tag{1.2}$$

Let ω and $\omega_- \in C(\mathbf{R})$ satisfy $\omega(t+\tau) = -\omega(t)$ and $\omega_-(t+\tau) = \omega_-(t)$, respectively, for $t \in \mathbf{R}$. For example, $\omega(t) = \sin(\pi t/\tau)$ and $\omega_-(t) = \cos(2\pi t/\tau)$ are such functions. We easily see that $\lambda^{t/\tau}\omega(t)$ and $\lambda^{t/\tau}\omega_-(t)$ are solutions of the unperturbed equations

$$\frac{d^n}{dt^n}[x(t) + \lambda x(t-\tau)] = 0 \quad \text{and} \quad \frac{d^n}{dt^n}[x(t) - \lambda x(t-\tau)] = 0,$$

respectively. Thus it is natural to expect that, if f is small enough in some sense, equation (1.1) [resp. (1.2)] has a solution $x(t)$ which behaves like the function $\lambda^{t/\tau}\omega(t)$ [resp. $\lambda^{t/\tau}\omega_-(t)$] as $t \to \infty$. In fact, the following results have been established by Jaroš and Kusano [7].

Theorem A. *Suppose that* $0 < \lambda \leq 1$ *and that there exist constants* $\mu \in (0, \lambda)$ *and* $a > 0$ *such that*

$$\int_{t_0}^{\infty} t^{n-1}\mu^{-t/\tau} F(t, a\lambda^{g(t)/\tau})dt < \infty.$$

Then

(i) *for each* $\omega \in C(\mathbf{R})$ *such that* $\omega(t+\tau) = -\omega(t)$ *for* $t \in \mathbf{R}$ *and* $\max_{t\in\mathbf{R}} |\omega(t)| < a$, *equation* (1.1) *has a solution* $x(t)$ *satisfying*

$$x(t) = \lambda^{t/\tau}[\omega(t) + o(1)] \quad (t \to \infty), \tag{1.3}$$

(ii) *for each* $\omega_- \in C(\mathbf{R})$ *such that* $\omega_-(t+\tau) = \omega_-(t)$ *for* $t \in \mathbf{R}$ *and* $\max_{t\in\mathbf{R}} |\omega_-(t)| < a$, *equation* (1.2) *has a solution* $x(t)$ *satisfying*

$$x(t) = \lambda^{t/\tau}[\omega_-(t) + o(1)] \quad (t \to \infty). \tag{1.4}$$

Theorem B. *Suppose that* $\lambda > 1$ *and that there exist constants* $\mu \in (1, \lambda)$ *and* $a > 0$ *such that*

$$\int_{t_0}^{\infty} \mu^{-t/\tau} F(t, a\lambda^{g^*(t)/\tau})dt < \infty,$$

where $g^*(t) = \max\{g(t), t\}$. *Then* (i) *and* (ii) *of Theorem A follow.*

We note that a solution $x(t)$ satisfying (1.3) is oscillatory if $\omega(t) \not\equiv 0$, and that a solution $x(t)$ satisfying (1.4) is oscillatory or nonoscillatory according to whether the function $\omega_-(t)$ is oscillatory or nonoscillatory. In particular, Theorems A and B are first results concerning the existence of oscillatory solutions of nonlinear neutral differential equations.

For equation (1.2), Theorems A and B have been extended to the following results by Kitamura and Kusano [9]. (See also [5, 8, 10, 14].)

Theorem C. *Let $\lambda = 1$. Suppose that*

$$\int_{t_0}^{\infty} t^n F(t,a)dt < \infty \quad \textit{for some } a > 0.$$

Then, for each $\omega_- \in C(\mathbf{R})$ such that $\omega_-(t+\tau) = \omega_-(t)$ for $t \in \mathbf{R}$ and $\max_{t\in\mathbf{R}} |\omega_-(t)| < a$, equation (1.2) *has a solution $x(t)$ satisfying*

$$x(t) = \omega_-(t) + o(1) \quad (t \to \infty).$$

Theorem D. *Let $\lambda \neq 1$. Suppose that*

$$\int_{t_0}^{\infty} \lambda^{-t/\tau} F(t, a\lambda^{g(t)/\tau})dt < \infty \quad \textit{for some } a > 0. \tag{1.5}$$

Then (ii) *of Theorem A follows.*

However, very little is known about extensions of Theorems A and B for equation (1.1) such as Theorems C and D. In this paper we have the next results which are improvements of Theorems A and B for equation (1.1).

Theorem 1.1. *Let $\lambda = 1$. Suppose that*

$$\int_{t_0}^{\infty} t^{n-1} F(t,a)dt < \infty \quad \textit{for some } a > 0. \tag{1.6}$$

Then, for each $c \in \mathbf{R}$ and $\omega \in C(\mathbf{R})$ such that $\omega(t+\tau) = -\omega(t)$ for $t \in \mathbf{R}$ and $\max_{t\in\mathbf{R}} |\omega(t)| + |c| < a$, equation (1.1) *has a solution $x(t)$ satisfying*

$$x(t) = \omega(t) + c + o(1) \quad \textit{as } t \to \infty. \tag{1.7}$$

Theorem 1.2. *Let $\lambda \neq 1$. Suppose that* (1.5) *holds. Then* (i) *of Theorem A follows.*

Remark 1.1. The solution obtained in Theorem 1.1 is oscillatory or nonoscillatory according to whether the function $\omega(t) + c$ is oscillatory or nonoscillatory. Since condition (1.6) is independent of the choice of the function $\omega(t) + c$, equation (1.1) possesses both oscillatory solutions and nonoscillatory solutions if (1.6) holds. For the case $\omega(t) \not\equiv 0$, the solution of (1.1) obtained in Theorem 1.2 is oscillatory.

The proof of Theorem 1.1 is given in Section 2. The proof of Theorem 1.2 will be omitted. (See [16].) To prove the existence of solutions, we will use Schauder-Tychonoff fixed point theorem.

2. Proof of Theorem 1.1

In this section we give the proof of Theorem 1.1. Consider the neutral differential equation

$$\frac{d^n}{dt^n}[x(t) + x(t-\tau)] + f(t, x(g(t))) = 0. \tag{2.1}$$

Let T and T_* be constants with $T - \tau \geq T_* \geq t_0$. We denote by $U[T_*, \infty)$ the set of all functions $u \in C[T_*, \infty)$ such that the series

$$\sum_{i=1}^{\infty}(-1)^{i+1}u(t+i\tau)$$

converges for each fixed $t \in [T-\tau, \infty)$. For each $u \in U[T_*, \infty)$, we assign the function Φu on $[T_*, \infty)$ by

$$(\Phi u)(t) = \begin{cases} \displaystyle\sum_{i=1}^{\infty}(-1)^{i+1}u(t+i\tau), & t \geq T-\tau, \\ (\Phi u)(T-\tau), & t \in [T_*, T-\tau]. \end{cases}$$

Then we see that

$$(\Phi u)(t) + (\Phi u)(t-\tau) = u(t), \quad t \geq T, \quad u \in U[T_*, \infty). \tag{2.2}$$

In fact,

$$\begin{aligned}(\Phi u)(t) + (\Phi u)(t-\tau) &= \sum_{i=1}^{\infty}(-1)^{i+1}u(t+i\tau) + \sum_{i=1}^{\infty}(-1)^{i+1}u(t+(i-1)\tau) \\ &= \sum_{i=1}^{\infty}(-1)^{i+1}u(t+i\tau) - \sum_{i=0}^{\infty}(-1)^{i+1}u(t+i\tau) \\ &= u(t), \quad t \geq T, \quad u \in U[T_*, \infty).\end{aligned}$$

Hereafter, $C[T_*, \infty)$ is regarded as the Fréchet space of all continuous functions on $[T_*, \infty)$ with the topology of uniform convergence on every compact subinterval of $[T_*, \infty)$.

The following lemma will be used in the proof of Theorem 1.1.

Lemma 2.1. *Let T and T_* be constants with $T - \tau \geq T_* \geq t_0$. Suppose that $\eta \in C[T-\tau, \infty)$ such that $\eta(t) \geq 0$ for $t \geq T-\tau$ and $\lim_{t\to\infty}\eta(t) = 0$ and define*

$$V = \{v \in U[T_*, \infty) \; : \; |(\Phi v)(t)| \leq \eta(t), \quad t \geq T-\tau\}.$$

Then Φ maps V into $C[T_*, \infty)$ and is continuous on V in the $C[T_*, \infty)$-topology.

Proof. If $v \in V$, then

$$\begin{aligned}
(2.3)\qquad \sup_{t\in[T-\tau,\infty)}\left|\sum_{i=p+1}^{\infty}(-1)^{i+1}v(t+i\tau)\right| &= \sup_{t\in[T-\tau,\infty)}\left|\sum_{i=1}^{\infty}(-1)^{i+1}v(t+p\tau+i\tau)\right| \\
&\le \sup_{t\in[T-\tau,\infty)} \eta(t+p\tau) \\
&= \sup_{t\in[T+(p-1)\tau,\infty)} \eta(t), \quad p=0,1,2,\dots,
\end{aligned}$$

which means that the series $\sum_{i=1}^{\infty}(-1)^{i+1}v(t+i\tau)$ converges uniformly on $[T-\tau,\infty)$. Consequently, Φv is continuous on $[T_*,\infty)$ for each $v \in V$ and Φ maps V into $C[T_*,\infty)$.

Now we prove that Φ is continuous on V. It suffices to show that if $\{v_j\}_{j=1}^{\infty}$ is a sequence in $C[T_*,\infty)$ converging to $v \in C[T_*,\infty)$ uniformly on every compact subinterval of $[T_*,\infty)$, then Φv_j converges to Φv uniformly on every compact subinterval of $[T_*,\infty)$.

For any $\varepsilon > 0$, there is an integer $p \ge 1$ such that

$$(2.4)\qquad \sup_{t\in[T+(p-1)\tau,\infty)} \eta(t) < \frac{\varepsilon}{3}.$$

Take an arbitrary compact subinterval I of $[T-\tau,\infty)$. There exists an integer $j_0 \ge 1$ such that

$$\sum_{i=1}^{p}|v_j(t+i\tau)-v(t+i\tau)| < \frac{\varepsilon}{3}, \quad t\in I, \quad j \ge j_0.$$

It follows from (2.3) and (2.4) that

$$\begin{aligned}
|(\Phi v_j)(t)-(\Phi v)(t)| &\le \sum_{i=1}^{p}|v_j(t+i\tau)-v(t+i\tau)| \\
&\quad + \left|\sum_{i=p+1}^{\infty}(-1)^{i+1}v_j(t+i\tau)\right| \\
&\quad + \left|\sum_{i=p+1}^{\infty}(-1)^{i+1}v(t+i\tau)\right| \\
&< \varepsilon, \quad t\in I, \quad j\ge j_0,
\end{aligned}$$

which implies that Φv_j converges Φv uniformly on I. In view of the fact that $(\Phi v)(t) = (\Phi v)(T-\tau)$ for $t \in [T_*, T-\tau]$ and $v \in V$, we conclude that Φ is continuous on V. The proof is complete.

Now let us show Theorem 1.1.

Proof of Theorem 1.1. Put $\delta = a - |c| - \max_{t\in\mathbf{R}} |\omega(t)| > 0$. We can take a number $T \geq t_0$ so large that

$$T_* \equiv \min\{T-\tau,\ \inf\{g(t)\ :\ t \geq T\}\} \geq t_0$$

and

$$\int_T^\infty s^{n-1}F(s,a)ds < \delta. \tag{2.5}$$

Let

$$G(t) = \begin{cases} \displaystyle\int_t^\infty \frac{(s-t)^{n-2}}{(n-2)!}F(s,a)ds, & n \geq 2, \\ F(t,a), & n = 1, \end{cases}$$

for $t \geq T$. Notice that

$$\int_t^\infty G(s)ds = \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!}F(s,a)ds, \quad t \geq T. \tag{2.6}$$

We consider the set Y of all functions $y \in C[T_*,\infty)$ such that

$$y(t) = y(T) \quad \text{for } t \in [T_*,T], \quad |y(t)| \leq \int_t^\infty G(s)ds \quad \text{for } t \geq T$$

and

$$|y(t) - y(t+\tau)| \leq \int_t^{t+\tau} G(s)ds \quad \text{for } t \geq T.$$

Obviously, Y is a closed convex subset of $C[T_*,\infty)$.

Now we claim that if $y \in Y$, then

$$\left|\sum_{i=1}^m (-1)^{i+1}y(t+i\tau)\right| \leq \int_{t+\tau}^\infty G(s)ds, \quad t \geq T-\tau \tag{2.7}$$

for $m = 1, 2, \ldots$. We see that if m is odd, then

$$\begin{aligned}\left|\sum_{i=1}^m (-1)^{i+1}y(t+i\tau)\right| &= \left|\sum_{j=1}^{(m-1)/2} [y(t+(2j-1)\tau) - y(t+2j\tau)] \right. \\ &\qquad \left. + y(t+m\tau)\right| \\ &\leq \sum_{j=1}^{(m-1)/2} \int_{t+(2j-1)\tau}^{t+2j\tau} G(s)ds + \int_{t+m\tau}^\infty G(s)ds \\ &\leq \int_{t+\tau}^\infty G(s)ds, \quad t \geq T-\tau, \quad y \in Y.\end{aligned}$$

For the case where m is even, using the equality

$$\sum_{i=1}^{m}(-1)^{i+1}y(t+i\tau)=\sum_{j=1}^{m/2}[y(t+(2j-1)\tau)-y(t+2j\tau)],\quad t\geq T-\tau,$$

we can conclude (2.7).

According to (2.7), if $m\geq p\geq 1$ and $t\in[T-\tau,\infty)$, then

$$\begin{aligned}\left|\sum_{i=p}^{m}(-1)^{i+1}y(t+i\tau)\right|&=\left|\sum_{i=1}^{m-p+1}(-1)^{i+p}y(t+(i+p-1)\tau)\right|\\&=\left|\sum_{i=1}^{m-p+1}(-1)^{i+1}y(t+(p-1)\tau+i\tau)\right|\\&\leq\int_{t+p\tau}^{\infty}G(s)ds\to 0\quad\text{as } p\to\infty.\end{aligned}$$

for each $y\in Y$. Hence, $Y\subset U[T_*,\infty)$. Letting $m\to\infty$ in (2.7), we obtain

$$|(\Phi y)(t)|\leq\int_{t+\tau}^{\infty}G(s)ds,\quad t\geq T-\tau,\quad y\in Y. \tag{2.8}$$

Lemma 2.1 implies that Φ maps Y into $C[T_*,\infty)$ and is continuous on Y. From (2.5), (2.6) and (2.8), it follows that

$$\lim_{t\to\infty}(\Phi y)(t)=0\quad\text{and}\quad|(\Phi y)(t)|\leq\delta,\quad t\geq T_*,\quad y\in Y.$$

Set

$$(\Omega y)(t)=\omega(t)+c+(-1)^{n-1}(\Phi y)(t),\quad t\geq T_*,\quad y\in Y.$$

Then we find that

$$(\Omega y)(t)=\omega(t)+c+o(1)\quad(t\to\infty) \tag{2.9}$$

and

$$|(\Omega y)(t)|\leq|\omega(t)|+|c|+\delta\leq a,\quad t\geq T_* \tag{2.10}$$

for each $y\in Y$.

We define the mapping $\mathcal{F}:Y\longrightarrow C[T_*,\infty)$ as follows:

$$(\mathcal{F}y)(t)=\begin{cases}\displaystyle\int_t^{\infty}\frac{(s-t)^{n-1}}{(n-1)!}f(s,(\Omega y)(g(s)))ds, & t\geq T,\\ (\mathcal{F}y)(T), & t\in[T_*,T].\end{cases}$$

In view of (H3) and (2.10), we see that the mapping $\mathcal{F}$ is well defined. It can be shown that $\mathcal{F}(Y)\subset Y$. In fact, if $t\geq T$ and $y\in Y$, then

$$|(\mathcal{F}y)(t)|\leq\int_t^{\infty}\frac{(s-t)^{n-1}}{(n-1)}F(s,a)ds=\int_t^{\infty}G(s)ds,$$

by (2.6), and

$$\begin{aligned}|(\mathcal{F}y)(t)-(\mathcal{F}y)(t+\tau)| &= \left|\int_t^{t+\tau} f(s,(\Omega y)(g(s)))ds\right| \\ &\leq \int_t^{t+\tau} F(s,a)ds = \int_t^{t+\tau} G(s)ds\end{aligned}$$

for the case $n=1$, and

$$\begin{aligned}|(\mathcal{F}y)(t)-(\mathcal{F}y)(t+\tau)| &= \left|\int_t^{t+\tau}\int_s^{\infty}\frac{(r-s)^{n-2}}{(n-2)!}f(r,(\Omega y)(g(r)))drds\right| \\ &\leq \int_t^{t+\tau}\int_s^{\infty}\frac{(r-s)^{n-2}}{(n-2)!}F(r,a)drds \\ &= \int_t^{t+\tau} G(s)ds\end{aligned}$$

for the case $n \neq 1$.

Since Ω is continuous on Y, the Lebesgue dominated convergence theorem shows that $\mathcal{F}$ is continuous on Y.

Now we claim that $\mathcal{F}(Y)$ is relatively compact. We note that $\mathcal{F}(Y)$ is uniformly bounded on every compact subinterval of $[T_*,\infty)$, because of $\mathcal{F}(Y)\subset Y$. Ascoli-Arzelà theorem, it suffices to verify that $\mathcal{F}(Y)$ is equicontinuous on every compact subinterval of $[T_*,\infty)$. Observe that

$$|(\mathcal{F}y)'(t)| \leq \begin{cases} F(t,a), & n=1, \\ \displaystyle\int_T^{\infty} s^{n-2}F(s,a)ds, & n\neq 1, \end{cases} \qquad t\geq T, \quad y\in Y.$$

Let I be an arbitrary compact subinterval of $[T,\infty)$. Then we see that $\{(\mathcal{F}y)'(t) : y\in Y\}$ is uniformly bounded on I. The mean value theorem implies that $\mathcal{F}(Y)$ is equicontinuous on I. Since $|(\mathcal{F}y)(t_1)-(\mathcal{F}y)(t_2)|=0$ for t_1, $t_2\in[T_*,T]$, we conclude that $\mathcal{F}(Y)$ is equicontinuous on every compact subinterval of $[T_*,\infty)$. Thus $\mathcal{F}(Y)$ is relatively compact as claimed.

Consequently, we are able to apply the Schauder-Tychonoff fixed point theorem to the operator $\mathcal{F}$ and find that there exists a $\widetilde{y}\in Y$ such that $\widetilde{y}=\mathcal{F}\widetilde{y}$. Set $x(t)=(\Omega\widetilde{y})(t)$. From (2.9) it follows that $x(t)$ satisfies (1.7). In view of (2.2), we obtain

$$\begin{aligned}x(t)+x(t-\tau) &= \omega(t)+\omega(t-\tau)+2c+(-1)^{n-1}[(\Phi\widetilde{y})(t)+(\Phi\widetilde{y})(t-\tau)] \\ &= 2c+(-1)^{n-1}\widetilde{y}(t), \\ &= 2c+(-1)^{n-1}(\mathcal{F}\widetilde{y})(t), \quad t\geq T,\end{aligned}$$

Therefore we see that

$$\frac{d^n}{dt^n}[x(t) + x(t-\tau)] = (-1)^{n-1}(\mathcal{F}\widetilde{y})^{(n)}(t) = -f(t, x(g(t))), \quad t \geq T,$$

so that $x(t)$ is a solution of (2.1). The proof is complete.

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