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ANTI-ABEL INTEGRAL EQUATION AND INVERSE PROBLEMS

東京水産大 上村 豊 (Yutaka Kamimura)

§1. This article is concerned with the nonlinear integral equation

$$\frac{1}{\Gamma(\alpha)}\int_a^x \frac{g(t)}{\left(\int_t^x \varphi(r)dr\right)^{1-\alpha}}dt = f(x), \quad a < x < b \tag{1}$$

where φ is the unknown function and f, g are given functions. Here $-\infty < a < b \leq \infty$, $0 < \alpha < 1$, and Γ is Euler's gamma function. The objective is to establish global existence and uniqueness results for (1). We adopt the set $L^1_{loc}[a,b) \cap C_+(a,b)$ as a function space for solutions φ of (1), where $L^1_{loc}(I)$ denotes the set of functions which are measurable on an interval I and are integrable on all compact subintervals of I; while $C_+(I)$ denotes the set of continuous and positive functions on I.

Let $AC_{loc}(I)$ stand for the set of functions which are absolutely continuous on all compact subintervals of an interval I. Moreover we shall denote by $C^1_+(I)$ the set of continuously differentiable and positive functions on I. Then the main theorem in this article is stated as follows:

Theorem 1. *Assume that*

(i) $f \in AC_{loc}[a,b) \cap C^1_+(a,b)$,

(ii) $g \in L^1_{loc}[a,b) \cap C_+(a,b)$.

If, for some $c \in (a,b)$, there exists $\varphi \in L^1_{loc}[a,c) \cup C_+(a,c)$ satisfying (1) for $a < x < c$, then φ can be uniquely continued to the whole interval (a,b) as a solution of (1): that is, there exists a unique solution $\tilde{\varphi} \in L^1_{loc}[a,b) \cup C_+(a,b)$ of (1) such that $\tilde{\varphi}(x) = \varphi(x)$ for $a < x < c$.

Theorem 1, a global continuation result, guarantees that, under assumptions (i) and (ii) on given functions f and g, if once we find a *local* solution φ of (1) near a, then we can extend this solution uniquely to the whole interval as a *global* solution of it. Therefore, *local* existence and uniqueness results combined with Theorem 1 yield *global* existence and uniqueness results. For instance we can obtain the following results:

Theorem 2. *Assume that*

(i) $f \in C[a,b) \cap C^1_+(a,b)$, $f(a) = 0$, $\lim_{x\to a} \frac{f'(x)}{(x-a)^{\mu-1}} > 0$;

(ii) $g \in C_+(a,b)$, $\lim_{x\to a} \frac{g(x)}{(x-a)^{\lambda-1}} > 0$

for some μ, λ satisfying $0 < \mu < \lambda$. Then there exists a unique solution $\varphi \in C_+(a,b)$ of (1) such that

$$\lim_{x\to a} \frac{\varphi(x)}{(x-a)^{\frac{\lambda-\mu}{1-\alpha}-1}} > 0.$$

Theorem 3. *Suppose that*

(i) $f \in C_+[a,b) \cap C^1(a,b)$, $\lim_{x\to a} \dfrac{f'(x)}{(x-a)^{\mu-1}} = 0$;

(ii) $g \in C_+(a,b)$, $\lim_{x\to a} \dfrac{g(x)}{(x-a)^{\lambda-1}} > 0$

for some $\mu, \lambda > 0$. *Then there exists a unique solution* $\varphi \in C_+(a,b)$ *of* (1) *such that*

$$\lim_{x\to a} \frac{\varphi(x)}{(x-a)^{\frac{\lambda}{1-\alpha}-1}} > 0.$$

Theorem 2 asserts that if $f(x) \sim (x-a)^{\mu}$, $g(x) \sim (x-a)^{\lambda-1}$ as $x \to a$, then equation (1) has a unique solution φ such that $\varphi(x) \sim (x-a)^{\frac{\lambda-\mu}{1-\alpha}-1}$ as $x \to a$; while Theorem 3 implies that in the case $f(x)$ =positive constant+$o((x-a)^{\mu})$, $g(x) \sim (x-a)^{\lambda-1}$ as $x \to a$, then we have a solution φ such that $\varphi(x) \sim (x-a)^{\frac{\lambda}{1-\alpha}-1}$ as $x \to a$. In both cases equation (1) is satisfied even for $x = a$ by the solution φ.

§**2.** Let us introduce an operator $I^{\alpha}_{a,w}$ defined by

$$I^{\alpha}_{a,w}u(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{w(t)}{(\int_t^x w(r)dr)^{1-\alpha}} u(t)dt, \quad a < x < b, \tag{2}$$

where w is a positive function in $L^1_{loc}[a,b)$. This operator is referred to as a transformed Riemann-Liouville integral operator of the order α. In terms of this operator Equation (1) can be written as

$$I^{\alpha}_{a,\varphi} \frac{g}{\varphi} = f, \tag{3}$$

which is called a transformed Abel integral equation when it is regarded as an equation for the *unknown* function g with a *known* positive function $\varphi \in L^1_{loc}[a,b)$. Therefore one can consider (1) as an equation deduced by exchanging the roles of the unknown and known functions in a transformed Abel integral equation.

Let $L^1_{loc,w}[a,b)$ be the set of functions such that $wu \in L^1_{loc}[a,b)$. Then $I^{\alpha}_{a,w}$ is an operator from $L^1_{loc,w}[a,b)$ into itself or, from $L^1_{loc,w}[a,b) \cap C(a,b)$ into itself; and has the semigroup property

$$I^{\alpha}_{a,w} I^{\beta}_{a,w} = I^{\alpha+\beta}_{a,w} \tag{4}$$

for $\alpha, \beta > 0$. Moreover we define an operator $D^{\alpha}_{a,w}$ by

$$D^{\alpha}_{a,w} := D_w I^{1-\alpha}_{a,w}, \tag{5}$$

where $D_w := \frac{1}{w(x)} \frac{d}{dx}$. This operator is referred to as a transformed Riemann-Liouville differential operator of the order α. By definition and the semigroup property we have $D^{\alpha}_{a,w} I^{\alpha}_{a,w} u = u$ for any $u \in L^1_{loc,w}[a,b)$. Moreover $D^{\alpha}_{a,w}$ is an operator from $AC_{loc}[a,b)$ into $L^1_{loc,w}[a,b)$. If $u \in AC_{loc}[a,b)$ then $D^{\alpha}_{a,w}u$ is written as

$$D^{\alpha}_{a,w}u(x) = \frac{1}{\Gamma(1-\alpha)} \frac{u(a)}{(\int_a^x w(r)dr)^{\alpha}} + \frac{1}{\Gamma(1-\alpha)} \int_a^x \frac{u'(t)}{(\int_t^x w(r)dr)^{\alpha}} dt \tag{6}$$

and the equality $I^{\alpha}_{a,w} D^{\alpha}_{a,w} u = u$ holds. For these facts we refer to [1, 8].

By applying $D^{\alpha}_{a,\varphi}$ to both sides of (3) we have $\frac{g}{\varphi} = D^{\alpha}_{a,\varphi} f$. If we assume $f \in AC_{loc}[a,b)$, then, by (6), this equation is written as follows:

$$\varphi(x)\left\{\frac{1}{\Gamma(1-\alpha)}\frac{f(a)}{\left(\int_a^x \varphi(r)dr\right)^{\alpha}} + \frac{1}{\Gamma(1-\alpha)}\int_a^x \frac{f'(t)}{\left(\int_t^x \varphi(r)dr\right)^{\alpha}}dt\right\} = g(x), \quad a < x < b. \tag{7}$$

Conversely, if φ is a solution of (7) in $L^1_{loc}[a,b) \cap C_+(a,b)$ then $D^{\alpha}_{a,\varphi} f = \frac{g}{\varphi}$. Hence, by applying $I^{\alpha}_{a,\varphi}$ to both sides, we arrive at (1). Thus we arrive at:

Lemma 4. *Let $f \in AC_{loc}[a,b) \cap C^1(a,b)$, $g \in L^1_{loc}[a,b) \cap C(a,b)$. Then $\varphi \in L^1_{loc}[a,b) \cap C_+(a,b)$ is a solution of (1) if and only if the function φ is a solution of (7).*

We wish to note that (7) can be regarded as the α-th differentiation of (1).

§**3.** In the case $f(a) = 0$, $\alpha = \frac{1}{2}$, Equation (7) has been studied by Jones [3, 4], Suzuki [9], Kamimura [5] in connection with the inverse problem of determining the time-dependent thermal conductivity of the one-dimensional heat equation by means of measurements of temperature and heat flux at the boundary of the semi-infinite, homogeneous conductor. A mathematical formulation of the inverse problem is as follows:

Problem 5. *Let $0 < T \leq \infty$. Given functions $f(t), g(t)$, determine $a(t) \in C_+[0,T)$ so that the parabolic system*

$$\begin{cases} u_t = a(t)u_{xx}, & 0 < x < \infty,\ 0 < t < T; \\ u(x,0) = 0, & 0 \leq x < \infty; \\ u(0,t) = f(t), & 0 \leq t < T; \\ -a(t)u_x(0,t) = g(t), & 0 < t < T \end{cases}$$

admits a bounded, classical solution $u(x,t)$.

By a *bounded* solution we mean that $u(x,t)$ satisfies an appropriate growth condition (for example, $|u(x,t)| \leq C_1 e^{C_2 x^2}$, $0 \leq x < \infty$, $0 \leq t \leq T'$ with some constants C_1, C_2 for each $T' < T$) which guarantees the uniqueness of the solutions of the system

$$\begin{cases} u_t = a(t)u_{xx}, & 0 < x < \infty,\ 0 < t < T; \\ u(x,0) = 0, & 0 \leq x < \infty; \\ -a(t)u_x(0,t) = g(t), & 0 < t < T. \end{cases} \tag{8}$$

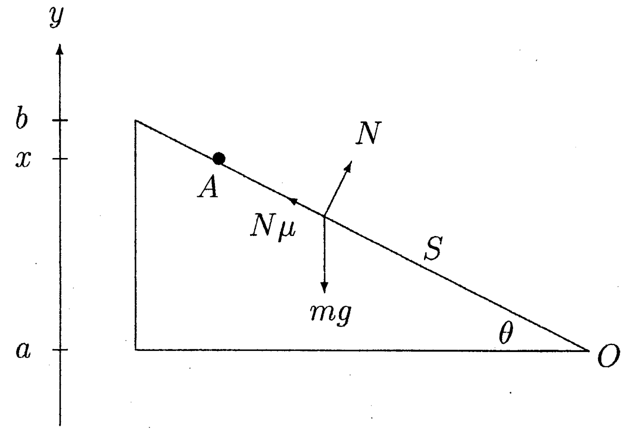
Although, in [3, 4, 5, 9], Problem 5 has been reduced to (7) with $f(a) = 0, \alpha = \frac{1}{2}$, it can be recast to (1) more directly. In fact, the solution $u(x,t)$ of (8) can be expressed as

$$u(x,t) = \frac{1}{\sqrt{\pi}}\int_0^t \frac{e^{-\frac{x^2}{4\int_s^t a(r)dr}}}{\left(\int_s^t a(r)dr\right)^{1/2}} g(s)\, ds,$$

provided that $g \in L^1_{loc}[0,T) \cap C(0,T)$, $a \in C_+[0,T)$, and hence, Problem 5 is equivalent to (1) with $\alpha = \frac{1}{2}, a = 0$.

Global existence and uniqueness results for Problem 5 were already established in [3, 4, 9] under the assumption that f is monotonically nondecreasing; and in [5] under the assumption that g is positive. We obtain the same result as in [5] as an immediate consequence of Theorem 2. Conversely speaking, the present article aims at providing a generalization of results in [5] to show that some other inverse problems mentioned below can be treated in a unified manner through (1).

§4. Equation (1) arises from several nonlinear inverse problems. As a example, we consider the following mechanical problem: determine a position-dependent coefficient of friction of a slope S such that a material point, starting with zero initial velocity at a given point A on S, slides down and reaches the lowest point O of S in an interval of time which is a given (or observed) function of the initial elevation.



Let the reaction force at a point on S be N and the friction coefficient be μ, which is a function of the elevation y. Then, by Newton's second law of motion, the elevation $y(t)$ of the material point at time t satisfies

$$my''(t) = -mg + N\cos\theta + N\mu(y)\sin\theta,$$

where θ is the angle of S to the horizontal, m is the mass of the material point, and g is the gravity acceleration. Noting $N = mg\cos\theta$ we obtain

$$y''(t) = -g\sin\theta(\sin\theta - \mu(y)\cos\theta).$$

To simplify notation, set

$$\varphi(y) = g\sin\theta(\sin\theta - \mu(y)\cos\theta)$$

Then the differential equation reads: $y'' + \varphi(y) = 0$.

Suppose, as indicated in the above figure, that the material point slides down from height x and that the friction coefficient is so small that $\phi(y)$ is positive for any $y \in (a,b)$. Recalling the initial velocity is zero we get the initial conditions: $y(0) = x$,

$y'(0) = 0$. Thus the inverse problem can be formulated as: to determine a positive function φ so that the first zero of the solution $y(t)$ to

$$\begin{cases} y'' + \varphi(y) = 0, & t > 0; \\ y(0) = x, \ y'(0) = 0. \end{cases} \tag{9}$$

coincides with a predetermined function $T(x)$ for each $x \in (a, b)$.

We suppose φ is an integrable function. Then, by a standard calculation, namely, multiplying the differential equation by $2y'$, integrating the resulting equation, and taking into account the initial conditions, we find the first integral

$$y'(t)^2 = 2 \int_{y(t)}^{x} \varphi(r)dr.$$

On the other hand, the positivity of φ, implies that $y'(t) = -\int_0^t \varphi(y(s))ds < 0$. Therefore the inverse function $t(y)$ of $y(t)$ exists and satisfies

$$\frac{dt}{dy} = -\frac{1}{\left(2\int_y^x \varphi(r)dr\right)^{1/2}}, \qquad a < y < x.$$

Integrating this from a to x, we find that the time T is given by

$$T(x) = \frac{1}{\sqrt{2}} \int_a^x \frac{dy}{\left(\int_y^x \varphi(r)dr\right)^{1/2}}, \qquad a < x < b. \tag{10}$$

In this way the inverse problem can be modeled in terms of the integral equation (1). It is interesting to point out that Abel's original integral equation was found in connection with a similar mechanical problem: find a curve along which a material point will fall, without friction, so that the time of fall is a given function of the distance fallen.

§5. An inverse problem of such type as in §4 occurs in nonlinear oscillations: find a nonlinear term φ of the autonomous differential equation

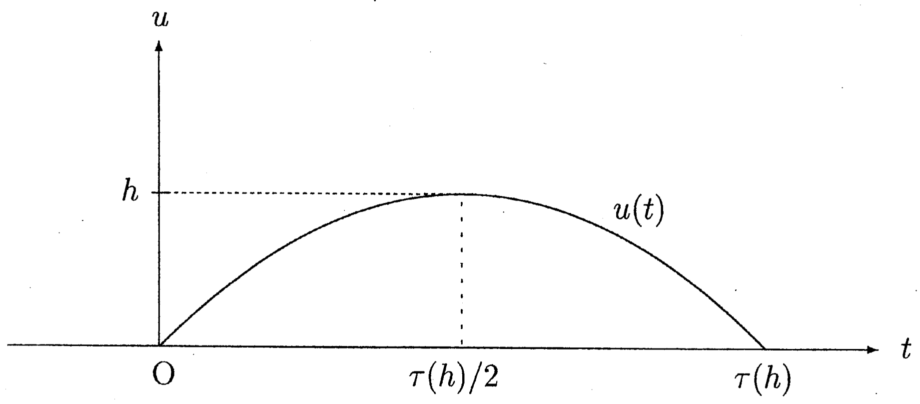
$$u'' + \varphi(u) = 0, \qquad ' = \tfrac{d}{dt} \tag{11}$$

from a prescribed relation between the half periods and the half amplitudes of the solutions to (11). Let us denote the half period by $\tau(h)$, which is a function of the half amplitude h. We assume $\varphi \in L^1_{loc}[0, H) \cap C_+(0, H)$. Under this assumption, if a solution u of (11) satisfies $u(0) = u(\tau) = 0$, $u(t) \neq 0$ for $0 < t < \tau$, then the derivative of u vanishes only at $t = \tau/2$, and so, u takes the maximum at $t = \tau/2$. Accordingly, the inverse problem is formulated as:

Problem 6. *Given positive function $\tau(h)$ defined on the interval $(0, H)$, find a function φ so that (11) admits a solution $u(t)$ satisfying the conditions*

$$u(0) = u(\tau(h)) = 0, \quad u(\tau(h)/2) = h, \quad u(t) \neq 0 \quad (0 < t < \tau(h)), \tag{12}$$

for each $h \in (0, H)$.



This problem is the simplest one among those of determining nonlinear terms from knowledge of period functions, which has been studied in [6, 7, 10]. The existence of φ realizing a given $\tau(h)$ was shown by Urabe [10] in the local sense, namely, in the case H is small. We shall establish the existence of the nonlinear term φ even in the case H is large or $H = \infty$.

By a quite similar way to that used for the deduction of (10), we have

Lemma 7. *Let $0 < H \leq \infty$. For a positive function $\tau(h)$ defined on $(0, H)$, a function $\varphi \in L^1_{loc}[0, H) \cap C_+(0, H)$ is a solution of Problem 6 if and only if φ satisfies*

$$\sqrt{2} \int_0^h \frac{du}{\left(\int_u^h \varphi(r)dr\right)^{1/2}} = \tau(h), \qquad 0 < h < H. \tag{13}$$

In this way, Problem 6 is reduced to (1) with $\alpha = \frac{1}{2}, a = 0, g \equiv 1$. Hence as immediate consequences of Theorems 2, 3, we obtain global existence results. By means of the notation $f(h) \sim h^\gamma$ $(h \to 0)$, which means $\lim_{h\to 0} h^{-\gamma} f(h) > 0$, the results are stated as follows:

Theorem 8. *Let $< H \leq \infty$.*
(a) If $\tau \in C[0, H) \cap C^1_+(0, H)$, $\tau(0) = 0$, $\tau'(h) \sim h^{\mu-1}$ $(h \to 0)$ with some $\mu \in (0, 1)$, then there exists a unique solution φ of Problem 6 such that

$$\varphi \in C_+(0, H), \quad \varphi(u) \sim u^{1-2\mu} \ (u \to 0).$$

(b) If $\tau \in C_+[0, H) \cap C^1(0, H)$, $\tau'(h) = o(h^{\mu-1})$ $(h \to 0)$ with some $\mu > 0$, then there exists a unique solution φ of Problem 6 such that

$$\varphi \in C_+[0, H), \quad \varphi(u) \sim u \ (u \to 0).$$

§6. It is clear from (1) that if g is positive in (a, b) then f must be positive in the interval for the existence of solutions $\varphi \in L^1_{loc}[a, b) \cap C_+(a, b)$ to (1); and from (7) that if $f(a) \geq 0$, $f'(x) \geq 0$, $f'(x) \not\equiv 0$ for $a < x < b$ then g must be positive in the interval for the existence of solutions. The core of the assumptions in Theorem 1 is the positivity of g in the interval (a, b). Unlike the case, if g is negative somewhere in the interval, solutions φ can not necessarily be continued uniquely to the whole interval. For such an example see [5].

Now we give an outline of the proof of Theorem 1. More detailed proof and the proofs of Theorems 2, 3 will be published elsewhere. A basic ingredient in the proof of Theorem 1 consists in proving that solutions φ of (1) do not blow up as long as g is positive. The basic idea in proving it is by the so-called fractional calculus based upon a manipulation of the operators $I^{\alpha}_{a,w}$ and $D^{\alpha}_{a,w}$ defined in (2) and (5), respectively. Some remarks on the properties of these operators may be helpful at this stage:

• The semigroup $\{I^{\alpha}_{a,w}\}_{\alpha>0}$ is continuous in the sense that if $u \in L_{loc,w}[a,b)$ then

$$\lim_{\alpha\to 0} I^{\alpha}_{a,w}u(x) = u(x) \tag{14}$$

for any point x $(> a)$ where u is continuous. For this fact, see for instance [1, §6.1], [8, §2.7 and §18.2].

• Let $C^{\beta}[a,b]$ be the Hölder space with exponent $0 < \beta \le 1$, that is, the set of functions f on $[a,b]$ for which there exists a constant C such that $|f(x) - f(y)| \le C|x-y|^{\beta}$ for all $x, y \in [a,b]$. Then, under the assumption $w \in C_+[a,b]$, we have

$$u \in C[a,b] \Longrightarrow I^{\alpha}_{a,w}u \in C^{\alpha}[a,b]. \tag{15}$$

This is a basic smoothing property of the operator $I^{\alpha}_{a,w}$, which goes back to Hardy and Littlewood [2]. See also [1, §4.2], [8, §3.1].

• Let $0 < \beta < \beta + \alpha \le 1$. Then, for any $u \in C^{\beta+\alpha}[a,b]$,

$$D^{\alpha}_{a,w}u(x) = \frac{1}{\Gamma(1-\alpha)}\frac{u(x)}{(\int_a^x w(r)dr)^{\alpha}} + \frac{\alpha}{\Gamma(1-\alpha)}\int_a^x \frac{u(x)-u(t)}{(\int_t^x w(r)dr)^{\alpha+1}}w(t)dt, \quad a < x \le b, \tag{16}$$

provided that $w \in C_+[a,b]$. See [2, Theorem 19], [8, (18.30)].

In what follows, we suppose that assumptions (i), (ii) in Theorem 1 are satisfied; and let $\varphi \in L^1_{loc}[a,c) \cup C_+(a,c)$ satisfy (1) for $a < x < c$, where $a < c < b$. By Lemma 4, $\varphi(x)$ satisfies (7) for $a < x < c$. It follows from (7) that $\inf\limits_{a+\delta\le x<c} \varphi(x) > 0$ for each $\delta > 0$. This, together with (1) and the assumption $f(c) > 0$, shows that

$$\int_a^c \varphi(r)dr < \infty. \tag{17}$$

Noting that the function φ satisfies

$$\frac{g(x)}{\varphi(x)} = \frac{1}{\Gamma(1-\alpha)}\frac{f(a)}{(\int_a^x \varphi(r)dr)^{\alpha}} + \frac{1}{\Gamma(1-\alpha)}\int_a^x \frac{f'(t)}{(\int_t^x \varphi(r)dr)^{\alpha}}dt, \quad a < x < c, \tag{18}$$

and letting $x \to c$ one can show that there exists a finite limit of $\varphi(x)^{-1}$ as x tends to c. We shall prove that $\lim\limits_{x\to c}\varphi(x)^{-1} > 0$ by contradiction, namely we assume the contrary: $\lim\limits_{x\to c}\varphi(x)^{-1} = 0$. Let d be a number such that $a < d < c$. Then by (1) we have

$$\frac{1}{\Gamma(\alpha)}\int_a^d \frac{g(t)}{(\int_t^x \varphi(r)dr)^{1-\alpha}}dt + \frac{1}{\Gamma(\alpha)}\int_d^x \frac{g(t)}{(\int_t^x \varphi(r)dr)^{1-\alpha}} = f(x), \quad d \le x \le c.$$

Note that by (17) the above equality holds even for $x = c$. By setting

$$q(x) := \frac{1}{\Gamma(\alpha)} \int_a^d \frac{g(t)}{\left(\int_t^x \varphi(r)dr\right)^{1-\alpha}} dt, \quad d \le x \le c,$$

the above equality can be written as

$$I_{d,\varphi}^{\alpha} \frac{g(x)}{\varphi(x)} = f(x) - q(x), \qquad d \le x \le c,$$

in terms of the integral operator defined in (2). We let $0 < \epsilon < 1 - \alpha$ and apply the differential operator $D_{d,\varphi}^{1-\epsilon}$ defined in (5) to both sides of this equality. Then we have

$$D_{d,\varphi}^{1-\epsilon} I_{d,\varphi}^{\alpha} \frac{g(x)}{\varphi(x)} = D_{d,\varphi}^{1-\epsilon}[f - q](x), \qquad d \le x \le c. \tag{19}$$

Since $f(d) - q(d) = 0$ by the definition of q, it follows from (6) that

$$(D_{d,\varphi}^{1-\epsilon}[f-q])(x) = \frac{1}{\Gamma(\epsilon)} \int_d^x \frac{f'(t) - q'(t)}{\left(\int_t^x \varphi(r)dr\right)^{1-\epsilon}} dt = I_{d,\varphi}^{\epsilon} \left\{ \frac{f'(x)}{\varphi(x)} - \frac{q'(x)}{\varphi(x)} \right\}.$$

Hence, by (14),

$$\lim_{\epsilon \to 0} \left(D_{d,\varphi}^{1-\epsilon}[f-q]\right)(c) = \left\{ \frac{f'(x)}{\varphi(x)} - \frac{q'(x)}{\varphi(x)} \right\}\Bigg|_c = \frac{(\alpha - 1)}{\Gamma(\alpha)} \int_a^d \frac{g(t)}{\left(\int_t^c \varphi(r)dr\right)^{2-\alpha}} dt,$$

where we have used the assumption $\lim_{x \to c} \varphi(x)^{-1} = 0$. This, together with the assumption $g(x) > 0$ for $a < x < b$, leads to

$$\lim_{\epsilon \to 0} \left(D_{d,\varphi}^{1-\epsilon}[f-q]\right)(c) > 0. \tag{20}$$

On the other hand, it follows from (18) and (15) that $\frac{g(x)}{\varphi(x)} \in C^{1-\alpha}[d,c]$. Hence, by (5), (4), (16), we have for $d \le x \le c$

$$\begin{aligned}
D_{d,\varphi}^{1-\epsilon} I_{d,\varphi}^{\alpha} \frac{g(x)}{\varphi(x)} &= D_{\varphi} I_{d,\varphi}^{\epsilon} I_{d,\varphi}^{\alpha} \frac{g(x)}{\varphi(x)} = D_{d,\varphi}^{1-(\alpha+\epsilon)} \frac{g(x)}{\varphi(x)} \\
&= \frac{1}{\Gamma(\alpha+\epsilon)} \frac{\varphi(x)^{-1} g(x)}{\left(\int_d^x \varphi(r)dr\right)^{1-\alpha-\epsilon}} + \frac{1-\alpha-\epsilon}{\Gamma(\alpha+\epsilon)} \int_d^x \frac{\varphi(x)^{-1}g(x) - \varphi(t)^{-1}g(t)}{\left(\int_t^x \varphi(r)dr\right)^{2-\alpha-\epsilon}} \varphi(t) dt.
\end{aligned}$$

This, together with the assumption $\lim_{x \to c} \varphi(x)^{-1} = 0$, yields

$$\left(D_{d,\varphi}^{1-\epsilon} I_{d,\varphi}^{\alpha} \frac{g}{\varphi}\right)(c) = \frac{\alpha + \epsilon - 1}{\Gamma(\alpha+\epsilon)} \int_d^c \frac{g(t)}{\left(\int_t^c \varphi(r)dr\right)^{2-\alpha-\epsilon}} dt.$$

The integrand in the right side is monotonically increasing as $\epsilon \to 0$, and so, by Beppo Levi's theorem, we arrive at

$$\lim_{\epsilon \to 0} \frac{\alpha+\epsilon-1}{\Gamma(\alpha+\epsilon)} \int_d^c \frac{g(t)}{\left(\int_t^c \varphi(r)dr\right)^{2-\alpha-\epsilon}} dt = \frac{\alpha - 1}{\Gamma(\alpha)} \int_d^c \frac{g(t)}{\left(\int_t^c \varphi(r)dr\right)^{2-\alpha}} dt.$$

Hence, by the assumption $g(x) > 0$ for $a < x < b$, we have

$$-\infty \le \lim_{\epsilon \to 0} \left(D_{d,\varphi}^{1-\epsilon} I_{d,\varphi}^{\alpha} \frac{g}{\varphi}\right)(c) < 0.$$

That is, the value of the left side at $x = c$ has a negative (finite or infinite) limit as $\epsilon \to 0$. This contradicts (20). Thus we have proved that solutions φ of (1) do not blow up as long as g is positive.

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