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Non-existence of periodic solutions in delayed Lotka-Volterra systems

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1 Introduction

In this paper we derive sufficient conditions for the non-existence of nonconstant periodic solutions of Volterra differential equations with distributed delays where the delay kernels are chosen among γ-functions or their suitable convex normalized combinations. The reason of this choice for the kernels is that the Volterra delay differential equations can thus be transformed in an expanded system of ordinary differential equations by the standard "linear chain trick" method [1]. To this expanded o.d.e. Volterra system we can apply the conditions, encoded by the logarithmic norm of some Jacobian related matrix, that Li and Muldowney [2] have obtained for the nonexistence of (nontrivial) periodic solutions for autonomous ordinary differential equations in $\mathbf{R}^N$, conditions that generalize to the case $N > 2$ the Bendixon and Dulac critera.

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2 General results

The Volterra delay differential systems with distributed delays can be written as

$$\begin{cases} \dot{x}_i = x_i(e_i + \sum_{j=1}^{n} a_{ij}x_j + \sum_{j=1}^{n} \gamma_{ij} \int_{-\infty}^{t} f_{ij}(t-u)x_j(u)du), \\ i \in \mathrm{N} \triangleq \{1, 2, \ldots, n\} \end{cases} \tag{2.1}$$

where for each $\gamma_{ij} \neq 0$, $f_{ij} : [0, +\infty) \rightarrow \mathbf{R}$ are continuous nonnegative functions obtained by convex combination

$$f_{ij}(u) = \sum_{k=1}^{p_{ij}} c_{ij}^{(k)} f_{ij}^{(k)}(u), \quad c_{ij}^{(k)} \geq 0, \quad \sum_{k=1}^{p_{ij}} c_{ij}^{(k)} = 1 \tag{2.2}$$

of functions which are solutions of linear differential equations with constant coefficients:

$$f_{ij}^{(k)}(u) = \frac{\alpha_{ij}^{k}}{(k-1)!}u^{k-1}\exp(-\alpha_{ij}u), \quad \alpha_{ij} \in \mathbf{R}_{+}, \quad k \in \{1, 2, \ldots, p_{ij}\} \tag{2.3}$$

and satisfy the normalized condition

$$\int_{0}^{+\infty} f_{ij}(u)du = 1.$$

We remind that the average time delay of (2.3) is $T = k/\alpha_{ij}$. We refer to (2.3) as to a γ-distribution (or γ-function) of order k. According to linear chain trick ([1]) we put

$$\begin{cases} x_{ij}^{(k)}(t) := \int_{-\infty}^{t} f_{ij}^{(k)}(t-u)x_j(u)du, \quad k = 1, \ldots, p_{ij}, \\ x_{ij}^{(0)}(t) := x_j(t), \quad i, j \in \mathrm{N}, \quad \gamma_{ij} \neq 0. \end{cases} \tag{2.4}$$

Let "p" the number of distinct functions $x_{ij}^{(k)}$ and $P = \{n+1, \ldots, n+p\}$ the set of all their indices. According to (2.4), system (2.1) is transformed in an expanded system of "$n+p$" ordinary differential equations

$$\begin{cases} \dot{x}_i = x_i(e_i + \sum_{j=1}^{n} a_{ij}x_j + \sum_{j=1}^{n} \gamma_{ij} \sum_{k=1}^{p_{ij}} c_{ij}^{(k)} x_{ij}^{(k)}), \quad i \in \mathrm{N} \\ \dot{x}_{ij}^{(k)} = \alpha_{ij}x_{ij}^{(k-1)} - \alpha_{ij}x_{ij}^{(k)}, \quad k = 1, \ldots, p_{ij}, \quad i, j \in \mathrm{N} : \gamma_{ij} \neq 0 \end{cases} \tag{2.5}$$

where the last "p" are linear differential equations with real constant coefficients. The initial conditions for (2.1) require the knowledge in the past of the nonnegative, continuous and bounded functions

$$x_i(u) = \varphi_i(u), \quad u \in (-\infty, 0] \quad \text{for all} \quad i \in \mathrm{N}. \tag{2.6}$$

The (2.6) provide the i.c. for (2.5). In fact:

$$\begin{cases} \dot{x}_i(0) = \varphi_i(0), \quad i \in \mathbf{N}, \\ x_{ij}^{(k)}(0) = \int_{-\infty}^{0} f_{ij}^{(k)}(-u)\varphi_j(u)du, \quad k = 1, \dots, p_{ij}, \quad i, j \in \mathbf{N} \end{cases} \tag{2.7}$$

Consider the general system of differential equations

$$\frac{dx}{dt} = F(x) \tag{2.8}$$

where $F(x) \in \mathbf{R}^N$, $x \mapsto F(x)$ is C^1 in an open subset D_0 of $\mathbf{R}^N$. Denote by $J = (\partial F/\partial x)$ the Jacobian of (2.8) and by $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N$ the eigenvalues of $(1/2)[(\partial F/\partial x) + (\partial F/\partial x)^T]$. Denote by $J^{[2]}$ the $\binom{N}{2} \times \binom{N}{2}$ matrix which is the second additive compound matrix associated to the Jacobian matrix J ([2]) and remind that if $x \in \mathbf{R}^N$ then the corresponding logarithmic norms of $J^{[2]}$ (that we denote by $\mu(J^{[2]})$) endowed by the vector norms (i) $|x|_1 = \sum_i |x_i|$, (ii) $|x|_\infty = \sup_i |x_i|$ and (iii) $|x|_2 = (x^T x)^{1/2}$ respectively are:

$$\begin{aligned} &\text{(i)} \quad \mu_1(J^{[2]}) &=& \sup\left\{\frac{\partial F_r}{\partial x_r} + \frac{\partial F_s}{\partial x_s} + \sum_{j\neq r,s}\left(|\frac{\partial F_j}{\partial x_r}| + |\frac{\partial F_j}{\partial x_s}|\right) : 1 \leq r < s \leq N\right\}; \\ &\text{(ii)} \quad \mu_\infty(J^{[2]}) &=& \sup\left\{\frac{\partial F_r}{\partial x_r} + \frac{\partial F_s}{\partial x_s} + \sum_{j\neq r,s}\left(|\frac{\partial F_r}{\partial x_j}| + |\frac{\partial F_s}{\partial x_j}|\right) : 1 \leq r < s \leq N\right\}; \\ &\text{(iii)} \quad \mu_2(J^{[2]}) &=& \lambda_1 + \lambda_2; \end{aligned}$$

where $\mu_\infty(J^{[2]}) < 0$ implies the diagonal dominance by row of the matrix $J^{[2]}$ and $\mu_1(J^{[2]}) < 0$ means its diagonal dominance by column. Then the following holds[2]:

Theorem 2.1 *If $\Omega \subset \mathbf{R}^N$ is a compact global attractor of (2.8) on which $\mu(J^{[2]}) < 0$ for some logarithmic norm then in Ω there is no simple closed rectifiable curve which is invariant with respect to (2.8).*

3 2-dimensional Volterra systems with 2 delays

Now let us consider n-dimensional Volterra delay differential systems with distributed delays expressed by (2.1) with delay kernels (2.2) and (2.3). The systems can be expressed as (2.5) by using p new variables (2.4) and become $(n+p)$-demensional o.d.e.. Their Jacobian has a size $(n+p) \times (n+p)$ and its second additive compound, is $\binom{n+p}{2} \times \binom{n+p}{2}$. Hence, in the following we restrict our systems with $n = 2$ and $p \leq 2$, that is, we consider

2-dimensional Volterra systems with at most 2 delays, whose kernels are given by the first or second order γ-distributions ($k = 1$ or 2 in(2.3)). Hereafter, for the simplicity of notation, we denote $x_{ij}^{(k)}$ as $x_j^{(k)}$.

Because of the symmetry of the systems, they are described as follows:

- a system with one first order delay:

$$\begin{cases} \dot{x}_1 &= x_1(e_1 + a_{11}x_1 + a_{12}x_2 + \gamma x_j^{(1)}) \\ \dot{x}_2 &= x_2(e_2 + a_{21}x_1 + a_{22}x_2) \\ \dot{x}_j^{(1)} &= \alpha x_j - \alpha x_j^{(1)} \quad j = 1 \text{ or } 2. \end{cases} \tag{3.1}$$

- a system with one-second order delay:

$$\begin{cases} \dot{x}_1 &= x_1(e_1 + a_{11}x_1 + a_{12}x_2 + \gamma x_j^{(2)}) \\ \dot{x}_2 &= x_2(e_2 + a_{21}x_1 + a_{22}x_2) \\ \dot{x}_j^{(1)} &= \alpha x_j - \alpha x_j^{(1)} \\ \dot{x}_j^{(2)} &= \alpha x_j^{(1)} - \alpha x_j^{(2)}, \quad j = 1 \text{ or } 2. \end{cases} \tag{3.2}$$

- a system with two-first order delays:

$$\begin{cases} \dot{x}_1 &= x_1(e_1 + a_{11}x_1 + a_{12}x_2 + \gamma_1 x_1^{(1)} + \gamma_2 x_2^{(1)}) \\ \dot{x}_2 &= x_2(e_2 + a_{21}x_1 + a_{22}x_2) \\ \dot{x}_1^{(1)} &= \alpha x_1 - \alpha x_1^{(1)} \\ \dot{x}_2^{(1)} &= \beta x_2 - \beta x_2^{(1)} \end{cases} \tag{3.3}$$

$$\begin{cases} \dot{x}_1 &= x_1(e_1 + a_{11}x_1 + a_{12}x_2 + \gamma_1 x_1^{(1)}) \\ \dot{x}_2 &= x_2(e_2 + a_{21}x_1 + a_{22}x_2 + \gamma_2 x_2^{(1)}) \\ \dot{x}_1^{(1)} &= \alpha x_1 - \alpha x_1^{(1)} \\ \dot{x}_2^{(1)} &= \beta x_2 - \beta x_2^{(1)} \end{cases} \tag{3.4}$$

$$\begin{cases} \dot{x}_1 &= x_1(e_1 + a_{11}x_1 + a_{12}x_2 + \gamma_1 x_2^{(1)}) \\ \dot{x}_2 &= x_2(e_2 + a_{21}x_1 + a_{22}x_2 + \gamma_2 x_1^{(1)}) \\ \dot{x}_1^{(1)} &= \alpha x_1 - \alpha x_1^{(1)} \\ \dot{x}_2^{(1)} &= \beta x_2 - \beta x_2^{(1)} \end{cases} \tag{3.5}$$

$$\begin{cases} \dot{x}_1 &= x_1(e_1 + a_{11}x_1 + a_{12}x_2 + \gamma_1 x_1^{(1)}) \\ \dot{x}_2 &= x_2(e_2 + a_{21}x_1 + a_{22}x_2 + \gamma_2 \tilde{x}_1^{(1)}) \\ \dot{x}_1^{(1)} &= \alpha x_1 - \alpha x_1^{(1)} \\ \dot{\tilde{x}}_1^{(1)} &= \beta x_1 - \beta \tilde{x}_1^{(1)}. \end{cases} \tag{3.6}$$

We will distinguish between two systems in (3.1) as $(3.1)_j$ for $j = 1, 2$. Similarly we define system $(3.2)_j$ for $j = 1, 2$. For all systems, we always assume that $a_{ii} < 0, e_i \neq 0$ $(i = 1, 2)$

and $\alpha, \beta > 0$. The first assumptions imply self-crowding effects biologically and the last comes from (2.3).

First, we consider the boundedness and 'partial permanence' of the solutions to systems $(3.1)_j$-(3.6). Note that $\mathbf{R}^3_+$ or $\mathbf{R}^4_+$ is positive invariant for each system.

Theorem 3.1 *Suppose that*

(a) for $(3.1)_1$; one of the following is satisfied

(a-1) $a_{12}a_{21} < 0$ *and* $a_{11} + \gamma < 0$

(a-2) $a_{12} \leq 0$, $a_{21} \leq 0$ *and* $a_{11} + \gamma < 0$

(a-3) $a_{11}a_{22} > a_{12}a_{21}$ *and* $\gamma < 0$*:*

(b) for $(3.1)_2$; one of the folowing is satisfied

(b-1) $a_{12}a_{21} < 0$ *and* $a_{11}a_{22} > -\gamma^2 a_{21}/(4a_{12})$

(b-2) $a_{12} \leq 0$ *and* $a_{21} \leq 0$

(b-3) $a_{11}a_{22} > a_{21}a_{21}$ *and* $\gamma \leq 0$*:*

(c) for $(3.2)_1$; one of the following is satisfied

(c-1) $a_{12}a_{21} < 0$ *and* $a_{11} + |\gamma| < 0$

(c-2) $a_{12} \leq 0$, $a_{21} \leq 0$ *and* $a_{11} + |\gamma| < 0$

(c-3) $a_{11}a_{22} > |a_{12}||a_{21}|$, $a_{11} + |a_{12}| < 0$ *and* $\gamma \leq 0$*:*

(d) for $(3.2)_2$; one of the following is satisfied

(d-1) $-a_{11} > |a_{12}| + |\gamma|$ *and* $-a_{22} > |a_{21}|$

(d-2) the same as (c-2)

(d-3) the same as (c-3):

(e) for (3.3); one of the following is satisfied

(e-1) $-a_{11} > |a_{12}| + |\gamma_1| + |\gamma_2|$ *and* $-a_{22} > |a_{21}|$

(e-2) $a_{12} \leq 0, a_{21} \leq 0$ *and* $-a_{11} > |\gamma_1| + |\gamma_2|$

(e-3) $a_{12}a_{22} > |a_{12}||a_{21}|$, $-a_{11} > |a_{12}|$, $\gamma_1 \leq 0$ *and* $\gamma_2 \leq 0$:

(f) for (3.4) or (3.5) or (3.6); one of the following is satisfied

(f-1) $a_{12} \leq 0$, $a_{21} \leq 0$, $-a_{11} > |\gamma_1|$ *and* $-a_{22} > |\gamma_2|$

(f-2) the same as (e-3).

Then the solutions of $(3.1)_j$*-(3.6) are bounded for any* $\alpha > 0$ *and* $\beta > 0$.

Theorem 3.2 *Suppose that the solutions of* $(3.1)_j$*-(3.6) are bounded and at least one of* e_i $(i = 1, 2)$ *is positive. Consider the solution* $x(t)$ *starting in* $\mathbf{R}^3_+$ *(system* $(3.1)_j$*) or in* $\mathbf{R}^4_+$ *(system* $(3.2)_j$*-(3.6)). Choose a sufficiently large number* $T > 0$ *and a sufficiently small number* $\varepsilon > 0$ *and define sets*

$$\begin{aligned}
\Omega^3_j &= \{x \in R^3_+ | x_1 + x_2 > \varepsilon, x^{(1)}_j > 0\}, \quad j = 1,2 \\
\Omega^4 &= \{x \in R^4_+ | x_1 + x_2 > \varepsilon, x^{(1)}_j > 0, j = 1,2\} \\
\bar{\Omega}^4 &= \{x \in R^4_+ | x_i > \varepsilon, x^{(i)}_1 > 0, i = 1,2\} \\
\tilde{\Omega}^4 &= \{x \in R^4_+ | x_1 + x_2 > \varepsilon, x^{(1)}_1 > 0, \tilde{x}^{(1)}_1 > 0\}.
\end{aligned}$$

(i) For $(3.1)_1$*, the solution stays in* Ω^3_1 *for* $t > T$*, if* $\gamma \leq 0$ *or* $-a_{11} > \gamma > 0$;

(ii) For $(3.1)_2$*, the solution stays in* Ω^3_2 *for* $t > T$;

(iii) Suppose that $-a_{11} > |\gamma|$. *Then for* $(3.2)_1$*, the solution stays in* $\bar{\Omega}^4$ *for* $t > T$*, if*

$$e_2 > a_{21}e_1/(a_{11} + \gamma) \quad \text{when} \quad e_1 > 0$$

$$\text{or} \quad e_1 > a_{12}e_2/a_{22} \quad \text{when} \quad e_2 > 0; \tag{3.7}$$

(iv) For $(3.2)_2$*, the solution stays in* $\bar{\Omega}^4$ *for* $t > T$*, if*

$$e_2 > a_{21}e_1/a_{11} \quad \text{when} \quad e_1 > 0$$

$$\text{or} \quad e_1 > e_2(a_{12} + \gamma)/a_{22} \quad \text{when} \quad e_2 > 0; \tag{3.8}$$

(v) For (3.3), the solution stays in Ω^4 *for* $t > T$*, if* $-a_{11} > |\gamma_1|$;

(vi) For (3.4), the solution stays in Ω^4 for $t > T$, if

$$-a_{ii} > |\gamma_i| \quad (i = 1, 2); \tag{3.9}$$

(vii) For (3.5), the solution stays in Ω^4 for $t > T$;

(viii) For (3.6), the solution stays in $\tilde{\Omega}^4$ for $t > T$, if $-a_{11} > |\gamma_1|$.

4 Non-existence of periodic solutions

Let us apply Li-Muldowney's criteria (Theorem 2.1) for the non-existence of periodic solutions of systems $(3.1)_j$-(3.6) $(j = 1, 2)$. The Jacobian matrix of $(3.1)_1$ becomes

$$J = \begin{pmatrix} e_1 + 2a_{11}x_1 + a_{12}x_2 + \gamma x_1^{(1)} & a_{12}x_1 & \gamma x_1 \\ a_{21}x_2 & e_2 + a_{21}x_1 + 2a_{22}x_2 & 0 \\ \alpha & 0 & -\alpha \end{pmatrix}.$$

The logarithmic norm μ_1 endowed by the norm $|x|_1$ of the second additive compound matrix $J^{[2]}$ associated to J is negative in $\mathbf{R}^3_{+0}$ if and only if the supremums of the following functions satisfy

$$\begin{aligned} &(e_1 + 2a_{11}x_1 + a_{12}x_2 + \gamma x_1^{(1)}) + (e_2 + a_{21}x_1 + 2a_{22}x_2) + \alpha < 0 \\ &(e_1 + 2a_{11}x_1 + a_{12}x_2 + \gamma x_1^{(1)}) - \alpha + |a_{21}|x_2 < 0 \\ &(e_2 + a_{21}x_1 + 2a_{22}x_2) - \alpha + |a_{12}|x_1 + |\gamma|x_1 < 0, \end{aligned}$$

in $\mathbf{R}^3_{+0}$. From the second and third inequalities, we have $a_{12} + |a_{21}| \leq 0$ and $a_{21} + |a_{12}| + |\gamma| \leq 0$ as necessary conditions for $\mu_1 < 0$ in $\mathbf{R}^3_{+0}$. These two conditions hold true only for $\gamma = 0$, which gives us a Lotka-Volterra system without a delay term. This shows that the *direct* application of Li-Muldowney's method does not work for $(3.1)_1$.

Now let us transform $(3.1)_1$ by change of variables

$$\begin{aligned} \dot{y}_1 &= (e_1 + a_{11}e^{\lambda_1 y_1} + a_{12}e^{\lambda_2 y_2} + \gamma x_1^{(1)})/\lambda_1 \\ \dot{y}_2 &= (e_2 + a_{21}e^{\lambda_1 y_1} + a_{22}e^{\lambda_2 y_2})/\lambda_2 \\ \dot{x}_1^{(1)} &= \alpha e^{\lambda_1 y_1} - \alpha x_1^{(1)} \end{aligned} \tag{4.1}$$

where new variables y_i $(i = 1, 2)$ are defined by $y_i = (\log x_i)/\lambda_i$, for some positive constants λ_i chosen later. The Jacobian matrix of (4.1) is

$$J_1^1 = \begin{pmatrix} a_{11}e^{\lambda_1 y_1} & \lambda_2 a_{12}e^{\lambda_2 y_2}/\lambda_1 & \gamma/\lambda_1 \\ \lambda_1 a_{21}e^{\lambda_1 y_1}/\lambda_2 & a_{22}e^{\lambda_2 y_2} & 0 \\ \alpha\lambda_1 e^{\lambda_1 y_1} & 0 & -\alpha \end{pmatrix}.$$

The logarithmic norm $\mu_1(J_1^{1[2]})$ is negative in $\mathbf{R}^3$ (note that it must be negative in $\mathbf{R}^3$, not in $\mathbf{R}^3_{+0}$, because of change of variables) if and only if the folowing is satisfied in $\mathbf{R}^3$

$$\begin{aligned} &\sup\{a_{11}e^{\lambda_1 y_1} + a_{22}e^{\lambda_2 y_2} + \alpha\lambda_1 e^{\lambda_1 y_1}\} < 0 \\ &\sup\{a_{11}e^{\lambda_1 y_1} - \alpha + \lambda_1 |a_{21}| e^{\lambda_1 y_1}/\lambda_2\} < 0 \\ &\sup\{a_{22}e^{\lambda_2 y_2} - \alpha + \lambda_2 |a_{12}| e^{\lambda_2 y_2}/\lambda_1 + |\gamma|/\lambda_1\} < 0. \end{aligned} \tag{4.2}$$

Suppose that for sufficiently small $\varepsilon > 0$ and large $T > 0$, the following is satisfied by the solution $y(t) = (y_1(t), y_2(t), x_1^{(1)}(t))$ of (4.1)

$$y(t) \in \Omega_{1y}^3 = \{y \in R^3 | e^{\lambda_1 y_1} + e^{\lambda_2 y_2} > \varepsilon, x_1^{(1)} > 0\} \quad \text{for} \quad t > T. \tag{4.3}$$

Under the assumption (4.3), the conditon given in Theorem 2.1 is ensured if

$$\begin{aligned} &a_{11} + \alpha\lambda_1 < 0, \quad a_{11} + \lambda_1 |a_{21}|/\lambda_2 \le 0, \\ &a_{22} + \lambda_2 |a_{12}|/\lambda_1 \le 0, \quad -\alpha + |\gamma|/\lambda_1 < 0. \end{aligned}$$

The above is equivalent to

$$-\frac{|a_{21}|}{a_{11}} \le \frac{\lambda_2}{\lambda_1} \le -\frac{a_{22}}{|a_{12}|}, \quad \frac{|\gamma|}{\lambda_1} < \alpha < -\frac{a_{11}}{\lambda_1}. \tag{4.4}$$

Suppose that $a_{11}a_{22} \ge |a_{12}||a_{21}|$ and $-a_{11} > |\gamma|$. Then it is easy to check that we can choose $\lambda_i > 0 (i = 1, 2)$ satisfying (4.4) for each $\alpha > 0$. Note that Ω_{1y}^3 corresponds to Ω_1^3 defined in Section 3 and (4.3) is equivalent that the solution of $(3.1)_1$ stays in Ω_1^3 for $t > T$. For the last property, a sufficient condition is given in Theorem 3.2 (i). This proves the following Theorem 4.1 (i):

Theorem 4.1 *Suppose that the solutions of $(3.1)_j$-(3.6) are bounded and at least one of e_i $(i = 1, 2)$ is positive. Then each system has no periodic solutions for any $\alpha > 0$ and $\beta > 0$ if the following conditions are satisfied:*

(i) For $(3.1)_1$,

$$a_{11}a_{22} \ge |a_{12}||a_{21}|, \quad -a_{11} > |\gamma|; \tag{4.5}$$

(ii) For $(3.1)_2$,

$$a_{11}a_{22} \ge |a_{12}||a_{21}|, \quad a_{11}a_{22} > |a_{21}||\gamma|; \tag{4.6}$$

(iii) For (3.2)$_1$, (3.11) and

$$a_{22}(|\gamma| + a_{11}) > |a_{12}||a_{21}|; \tag{4.7}$$

(iv) For (3.2)$_2$, (3.12) and

$$a_{11}a_{22} > |a_{21}|(|\gamma| + |a_{12}|); \tag{4.8}$$

(v) For (3.3),

$$a_{22}(|\gamma_1| + a_{11}) > |a_{21}|(|\gamma_2| + |a_{12}|); \tag{4.9}$$

(vi) For (3.4), (3.13) and

$$(a_{11} + |\gamma_1|)(a_{22} + |\gamma_2|) > |a_{12}||a_{21}|; \tag{4.10}$$

(vii) For (3.5),

$$a_{11}a_{22} > (|\gamma_1| + |a_{12}|)(|\gamma_2| + |a_{21}|); \tag{4.11}$$

(viii) For (3.6),

$$a_{22}(|\gamma_1| + a_{11}) > |a_{12}||a_{21}|, \quad a_{11}a_{22} > |a_{12}|(|a_{21}| + |\gamma_2|); \tag{4.12}$$

and

$$|a_{21}| > |\gamma_2|; \tag{4.13}$$

or (4.12) and

$$-a_{11}|a_{21}| > (|a_{21}| + |\gamma_2|)|\gamma_1|, \quad 2|\gamma_1| + a_{11} < 0. \tag{4.14}$$

References

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