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Robustness and Global Bifurcation of Three-species Ecological Model

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§0. INTRODUCTION

We shall consider three-dimensional Lotka-Volterra equations defined by the following vector fields:

$$\text{(LV)} \qquad \dot{x_i} = \gamma_i x_i(1 + \sum_{j\in \mathbf{Z}_3} a_{ij}x_j), \qquad i \in \mathbf{Z}_3,$$

on the closed positive cone $\mathbf{R}^3_+ := \{x \in \mathbf{R}^3 : x \geq 0\}$, where $x = (x_1, x_2, x_3)$. We denote by $\dot{}$ a differential d/dt and $\mathbf{Z}_3 := \{1,2,3\}$ is considered cyclic.

In the case of the competitive systems (see Hirsch [H]), the global behaviors of the three-dimensional Lotka-Volterra systems (with $a_{ij} < 0$) have been studied by Driessche and Zeeman [DZ], Chi, Hsu and Wu [CHW].

The analysis of the global behaviors of orbits in this paper is featured that the system is not always competitive. For example in the case when $\gamma_1 = \gamma_2 = \gamma_3\ (> 0)$, we have Theorem 1.1. below.

Consider the following systems:

$$(\mathrm{LV}_1) \qquad \dot{x_i} = x_i(1 + \sum_{j\in \mathbf{Z}_3} a_{ij}x_j), \qquad i \in \mathbf{Z}_3.$$

THEOREM 1.1.[UO] *In the vector field of* (LV_1) *with* $\gamma_i > 0$, $a_{ii} < 0$ *and* $a_{i,i-1} < a_{ii} < a_{i,i+1}$ $(i \in \mathbf{Z}_3)$, *almost every orbit* ψ *in* $(\mathbf{R}^3_+)^\circ$ *satisfies one of the following*:
(1) *If* $\prod_{i\in\mathbf{Z}_3} (a_{i,i+1} - a_{ii}) > \prod_{i\in\mathbf{Z}_3} (a_{ii} - a_{i,i-1})$ *and* $\det A < 0$, *then every orbit* ψ *in* $(\mathbf{R}^3_+)^\circ$ *tends toward the equilibrium point* x^*.
(2) *If* $\prod_{i\in\mathbf{Z}_3} (a_{i,i+1} - a_{ii}) < \prod_{i\in\mathbf{Z}_3} (a_{ii} - a_{i,i-1})$, *then every orbit* ψ *in* $(\mathbf{R}^3_+)^\circ \setminus \Gamma_1$ *winds toward the heteroclinic cycle* T.

Here we denote by $(\cdot)^\circ$ the interior of $\cdot$, by $A = (a_{ij})$ called the *interactive matrix* of the system, by x^* the equilibrium point which is in $(\mathbf{R}^3_+)^\circ$ and by Γ_1 the half line from 0 passing through the equilibrium x^*. Three singularities $e_1 := (\frac{-1}{a_{11}}, 0, 0)$, $e_2 := (0, \frac{-1}{a_{22}}, 0)$, $e_3 := (0, 0, \frac{-1}{a_{33}})$ are connected one another by three orbits called the *heteroclinic orbits.* The union of these singularities and orbits forms a curved triangle T, called the *heteroclinic cycle.*

On the other hand, in the case when γ_i's are not necessaly the same, there is an impressive result Theorem 1.3. below.

DEFINITION 1.2. The system (LV) is said to be *permanent*, if there exists a compact set $K \subset (\mathbf{R}^3_+)^\circ$ such that for any $\psi(0) \in (\mathbf{R}^3_+)^\circ$, we have $\psi(t) \in K$ for t sufficiently large.

Namely the system (LV) is permanent, if $\partial\mathbf{R}^3_+$ is a repellor on $\mathbf{R}^3_+$, where $\partial\mathbf{R}^3_+$ denotes the boundary of $\mathbf{R}^3_+$ including points at infinity.

THEOREM 1.3.[HS] *Consider the vector field of* (LV) *with* $\gamma_i > 0$, $a_{ii} < 0$ *and* $a_{i,i-1} < a_{ii} < a_{i,i+1}$ $(i \in \mathbf{Z}_3)$.

(1) *If* $\prod_{i\in\mathbf{Z}_3}(a_{i,i+1}-a_{ii}) > \prod_{i\in\mathbf{Z}_3}(a_{ii}-a_{i,i-1})$ *and* $\det A < 0$, *then the system* (LV) *is permanent.*
(2) *If* $\prod_{i\in\mathbf{Z}_3}(a_{i,i+1}-a_{ii}) < \prod_{i\in\mathbf{Z}_3}(a_{ii}-a_{i,i-1})$, *then the heteroclinic cycle* T *is an attractor.*

The above results determine the behaviors of the orbits in the neighborhood of $\partial\mathbf{R}^3_+$ in $\mathbf{R}^3_+$.

Now we shall define the structural stable-like idea on a stability of the system (LV).

Consider the system (LV_ϵ) which is obtained by perturbing the system (LV) as follows:

$$(\mathrm{LV}_\epsilon) \qquad \dot{x_i} = \gamma_i x_i(1+\sum_{j\in\mathbf{Z}_3} a_{ij}x_j + \epsilon_i\phi_i(x)), \qquad i\in\mathbf{Z}_3,$$

where the ϕ_i are affine linear, the $|\epsilon_i|$ are sufficiently small and $\epsilon := (\epsilon_1,\epsilon_2,\epsilon_3)$.

DEFINITION 1.4. We say a property of the system (LV) is *robust*, if it holds in the system (LV_ϵ).

The following theorem is our first result.

THEOREM 1.5. *Consider a vector field of* (LV_1) *as in Theorem 1.1.*
(1) *The global property that* x^* *is a global attractor on* $(\mathbf{R}^3_+)^\circ$ *are robust.*
(2) *The global property that* T *is a global attractor on* $(\mathbf{R}^3_+)^\circ\setminus\Gamma_\epsilon$ *are robust.*

Here Γ_ϵ is the one-dimensional stable manifold of the x^*. This result has the following corollary.

COROLLARY 1.6. *Consider the system* (LV) *as in* Theorem 1.3.[HS]. *Then there exists sufficient small* $\epsilon > 0$, *such that for* $|\gamma_i-\gamma_j| < \epsilon \quad (i,j\in\mathbf{Z}_3)$, *every orbit* ψ *in* $(\mathbf{R}^3_+)^\circ\setminus\Gamma_\epsilon$ *satisfies one of the following*:
(1) *If* $\prod_{i\in\mathbf{Z}_3}(a_{i,i+1}-a_{ii}) > \prod_{i\in\mathbf{Z}_3}(a_{ii}-a_{i,i-1})$ *and* $\det A < 0$, *then* ψ *tends toward the equilibrium point* x^*.
(2) *If* $\prod_{i\in\mathbf{Z}_3}(a_{i,i+1}-a_{ii}) < \prod_{i\in\mathbf{Z}_3}(a_{ii}-a_{i,i-1})$ *then* ψ *winds toward the heteroclinic cycle* T.

In the case (1) of Theorem 1.3.[HS], the system is permanent but the x^* need not be locally stable. Infact it may be locally unstable with a certain condition. And with some additional conditions, the system may have a limit cycle which is robust in competitive systems. We shall show the above case in Proposition 2.2. and Corollary 2.3. as our second result.

PROPOSITION 2.2. *Consider the system* (LV) *with* $a_{ii}=-1$, $\det A<0$. *Suppose that* $\gamma_i x_i^*$ *are constant* k, $a_{i,i-1} \quad (i\in\mathbf{Z}_3)$ *have negative values (and not all the same) and satisfy the following conditions* (C1) *and* (C2):

$$(\mathrm{C1}) \qquad 1+\prod_{i\in\mathbf{Z}_3}(1+a_{i,i-1}) > 0.$$

$$(\mathrm{C2}) \qquad 8+\prod_{i\in\mathbf{Z}_3} a_{i,i-1} < 0.$$

If $a_{i,i+1}$ $(i\in\mathbf{Z}_3)$ *close enough to* 0, *then there exists a non-trivial* ω*-limit set in* $(\mathbf{R}^3_+)^\circ$. *In particular, when* $a_{i,i+1}$ $(i\in\mathbf{Z}_3)$ *close enough to* 0 *from below, the* ω*-limit set is a limit cycle.*

For our interests where the limit sets exist, we conclude by showing the existence of a positively invariant set which includes them in Theorem 3.1. and Corollary 3.2.

THEOREM 3.1. *Given system* (LV) *with* $a_{ij} + a_{ji} < 0$ $(i, j \in \mathbf{Z}_3)$. *If the set* I *satisfies* (G1), (G2) *and* (G3), *then the set* I *is positively invariant and the every orbit from* $(\mathbf{R}^3_+)^\circ$ *has an* ω*-limit in the set* I.

For details see §3.

§1. PROOF OF THEOREM 1.5.

We denote by $(\mathrm{LV}_{1\epsilon})$ the system (LV_ϵ) with $\gamma_1 = \gamma_2 = \gamma_3$ (> 0). The assertion of Theorem 1.5. means that, in the system $(\mathrm{LV}_{1\epsilon})$, every orbit ψ from $(\mathbf{R}^3_+)^\circ$ tends toward x^* or every orbit ψ from $(\mathbf{R}^3_+)^\circ \setminus \Gamma_\epsilon$ winds toward T if $\det A < 0$ and $\prod_{i\in\mathbf{Z}_3} (a_{i,i+1} - a_{ii}) + \prod_{i\in\mathbf{Z}_3} (a_{i,i-1} - a_{ii}) \neq 0$.

LEMMA 1.7. *There exists a open set* $K_\epsilon \subset (\mathbf{R}^3_+)^\circ$, *a neighborhood* $n(\Gamma_1)$ *of* Γ_1 *and a smooth scalar function* $G(x)$ *on* $(\mathbf{R}^3_+)^\circ \setminus \Gamma_1$ *such that* $(\mathrm{LV}_{1\epsilon})$ *is transverse to* $C_\theta := \{x \in (\mathbf{R}^3_+)^\circ : G(x) = \theta\}$ *for any* θ *in* $(0, \pi)$, *and* $\dot{G}$ *has a constant sign on* $K_\epsilon \setminus n(\Gamma_1)$.

Proof of Lemma 1.7. We consider the projected vector field $(\hat{x})$ of (LV_1) on $S^2_+ := \{x \in \mathbf{R}^3_+ : |x| = 1\}$ as follows:

$$\dot{x} = F - |x|^{-2}(x \cdot F)x, \tag{$\hat{\mathrm{x}}$}$$

where $F = (F_1, F_2, F_3)$ is the vector field of the system (LV_1). We denote by $\gamma_x(t)$ the orbit of $(\hat{\mathrm{x}})$ through the point x at $t = 0$.

We define the map $\hat{\psi} : \mathbf{R} \times S^2_+ \to S^2_+$ by $\hat{\psi}(t, x) = \gamma_x(t)$.

Then we have

$$\hat{\psi}(0, x) = x \qquad x \in S^2_+$$

and

$$\hat{\psi}(t_1 + t_0, x) = \hat{\psi}(t_1, \hat{\psi}(t_0, x)) \qquad t_1, t_0 \in \mathbf{R}.$$

For each $t \in \mathbf{R}$ we have a map

$$\widehat{\psi_t} : S^2_+ \to S^2_+$$

defined by

$$\widehat{\psi_t}(x) = \hat{\psi}(t, x) \qquad (t, x) \in \mathbf{R} \times S^2_+.$$

We shall consider the system (LV_1) in the case (1). Because x^* is locally asymptotically stable, in the vector field $(\hat{\mathrm{x}})$ there exists a positive real number d such that $(\hat{\mathrm{x}})$ is transverse to the closed curve $\widehat{C_d}$ inward, where $\widehat{C_d} := \{x \in (S^2_+)^\circ : |x - x^*| = d\}$.

For each $t < 0$ the closed curve $\widehat{\psi_t}(\widehat{C_d}) = \{\widehat{\psi_t}(x) \in (S^2_+)^\circ : x \in \widehat{C_d}\}$ is smooth because $\widehat{\psi_t}$ is a diffeomorphism. For each $t < 0$, let

$$C_\theta := \bigcup_{s>0} s\widehat{\psi_t}(\widehat{C_d}) \subset (\mathbf{R}^3_+)^\circ,$$

where $\theta = \cot^{-1} t$.

Now we consider the system $(\mathrm{LV}_{1\epsilon})$ in the case (1). We define the function $G : (\mathbf{R}^3_+)^\circ \to \mathbf{R}$ as follows:

$$G(x) = \begin{cases} \theta & (x \in (\mathbf{R}^3_+)^\circ \setminus \Gamma_1), \\ 0 & (x \in \Gamma_1). \end{cases}$$

We consider

$$\dot{G} := \nabla G \cdot f = \sum_{i \in \mathbf{Z}_3} \frac{\partial G}{\partial x_i} \dot{x}_i,$$

where $\nabla := (\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3})$ and $f := (f_1, f_2, f_3)$ is the vector field of the system $(\mathrm{LV}_{1\epsilon})$.

Hence for the sufficiently small ϵ, there exists a open set $K_\epsilon \subset (\mathbf{R}^3_+)^\circ$ and the neighborhood $n(\Gamma_1)$ of Γ_1 such that

$$\dot{G} < 0 \qquad \text{on } K_\epsilon \setminus n(\Gamma_1).$$

Similarly in the case (2), we have

$$\dot{G} > 0 \qquad \text{on } K_\epsilon \setminus n(\Gamma_1).$$

□

Proof of Theorem 1.5. First we consider sail-like surfaces $D_r := \{x \in (\mathbf{R}^3_+)^\circ : D(x) = r\}$ and a family of cone-like annuluses C_θ , where $D(x) := |x|^2$ on $(\mathbf{R}^3_+)^\circ$ and $G(x)$ are smooth scalar functions defined in Lemma 1.7.

For $r_1, r_2 (0 < r_1 < r_2 < \infty)$ and $\theta_1, \theta_2 (0 < \theta_1 < \theta_2 < \pi)$ we suppose the domain:

$$D_{(r_1,r_2)} := \{x : r_1 < D(x) < r_2\}, \quad C_{(\theta_1,\theta_2)} := \{x : \theta_1 < G(x) < \theta_2\}.$$

Clearly we have

$$D_{(0,p)} \subseteq D_{(0,q)}, \quad C_{(0,p)} \subseteq C_{(0,q)},$$

for $p < q$. And we have

$$\lim_{\theta \to 0} C_\theta = \Gamma_1, \quad \lim_{\theta \to \pi} C_\theta = \partial \mathbf{R}^3_+ \setminus \{\infty\} \quad \text{and} \quad \lim_{r \to 0} D_r = \{0\}.$$

We shall consider the system (LV_1). If $\det A < 0$ and $\prod_{i \in \mathbf{Z}_3} (a_{i,i+1} - a_{ii}) + \prod_{i \in \mathbf{Z}_3} (a_{i,i-1} - a_{ii}) > 0$, then the system (LV_1) is permanent by Theorem 1.3. Therefore there is some bounded domain $I(q) \subset (\mathbf{R}^3_+)^\circ$ such that for any orbit ψ from any point in $I(q)$ and sufficiently large t, $\psi(t) \in (\mathbf{R}^3_+)^\circ \setminus I(q)$.

Hence there is some $\underline{\delta}, \overline{\delta}$ $(0 < \underline{\delta} < \overline{\delta} < \infty)$ such that for any orbit ψ from any point in $(\mathbf{R}^3_+)^\circ$, $\liminf_{t \to \infty} \psi^i(t) > \underline{\delta}$ and $\limsup_{t \to \infty} \psi^i(t) < \overline{\delta}$ $(i \in \mathbf{Z}_3)$, where the $\psi^i(t)$ are the components of $\psi(t)$.

When we consider the system (LV_ϵ), there exists some domain $I(q)_\epsilon$, the construction of which is similar to that of $I(q)$ in the (LV_1) , because the permanence of the system is robust in this case (see [HS]). In addition to existence of $I(q)_\epsilon$, from the Appendix 1 if ϵ is enough close to 0, then $I(q)_\epsilon$ is close to $I(q)$ enough.

Now we consider the system $(\mathrm{LV}_{1\epsilon})$ in the case (1). For sufficiently large l we have

$$\dot{D} \geq D(1 - l(x_1 + x_2 + x_3)),$$

where $\dot{D} := \nabla D \cdot f$.

Therefore for sufficiently small m we have

$$\dot{D}|_{D_{(0,m)}} > 0.$$

On the other hand for sufficiently large L and T we have

$$\psi(T) \in D_{(0,L)} \quad \text{if} \quad \psi(0) \in D_{(L,\infty)},$$

where $\psi(t)$ is the orbit of the system $(\mathrm{LV}_{1\epsilon})$.

In the system (LV_1), if necessary we shall exchange once, there exists a family $\{C_\theta\}$, such that $\dot{G} < 0$ on $\{x \in C_\theta : 0 < \theta < \infty\}$. Therefore in the system ($LV_{1\epsilon}$), for any $\underline{r}$, $\overline{r}$ $(0 < \underline{r} < \overline{r} < \infty)$ and any $\underline{\theta}$, $\overline{\theta}$ $(0 < \underline{\theta} < \overline{\theta} < \pi)$, if necessary by resetting ϵ_i enough small, then there is r_1, r_2 $(0 < r_1 < \underline{r} < \overline{r} < r_2 < \infty)$ and θ_1, θ_2 $(0 < \theta_1 < \underline{\theta} < \overline{\theta} < \theta_2 < \pi)$ such that

$$\dot{G} < 0 \quad \text{on} \quad D_{(r_1,r_2)} \cap C_{(\theta_1,\theta_2)}, \tag{1}$$

and

$$D_{(0,r_1)},\ D_{(r_2,\infty)},\ C_{(\theta_2,\pi)} \ \subset\ I(q). \tag{2}$$

On the other hand, there exists a tubular neighborhood $n(\Gamma_\epsilon)$ of one dimensional manifold Γ_ϵ which is tangent to the eigenspace spanned by an eigenvector with real eigen value of the Jacobian matrix $D_{x_\epsilon^*}f$ of f at x_ϵ^* such that for any orbit ψ from any point in $n(\Gamma_\epsilon)$ we have $\omega(\psi) = x_\epsilon^*$, where $\omega(\cdot)$ is ω-limit set of $\cdot$ and x_ϵ^* is an interior equilibrium point of the system (LV_ϵ).

Now we define a domain J_ϵ:

$$J_\epsilon := \max_{r_2',\theta'} \min_{r_1'} \{\ x \in (\mathbf{R}_+^3)^\circ : x \in D_{(r_1',r_2')} \cap C_{(0,\theta')} \ \subset\ n(\Gamma_\epsilon)\ \}.$$

If necessary by resetting ϵ_i more close to 0, r_1' can be close to 0 enough, r_2' can be large enough and θ' can be close to π enough, namely for any r_1, r_2 (> 0) and $\theta_1 \in (0, \pi)$, there are ϵ_i $(0 < |\epsilon_i| \ll 1)$, such that $0 < r_1' < r_1 < r_2 < r_2' < \infty$ and $0 < \theta_1 < \theta' < \pi$. Hence,

$$C_{(0,\theta_1)} \ \subset\ \{I(q)_\epsilon \cup J_\epsilon\} \ \subset\ \{I(q)_\epsilon \cup n(\Gamma_\epsilon)\}. \tag{3}$$

Therefore by (1), (2) and (3), for any orbit ψ from any point in $(\mathbf{R}_+^3)^\circ$, we have $\omega(\psi) = x_\epsilon^*$.

In the other case that $\prod_{i \in \mathbf{Z}_3} (a_{i,i+1} - a_{ii}) + \prod_{i \in \mathbf{Z}_3} (a_{i,i-1} - a_{ii}) < 0$, the proof is done similarly.
□

This theorem has the following corollary.

COROLLARY 1.6. *Consider the system* (LV) *as in* Theorem 1.3.[HS]. *Then there exists sufficient small* $\epsilon > 0$, *such that for* $|\gamma_i - \gamma_j| < \epsilon$ $(i, j \in \mathbf{Z}_3)$, *every orbit* ψ *in* $(\mathbf{R}_+^3)^\circ \setminus \Gamma_\epsilon$ *satisfies one of the following*:

(1) *If* $\prod_{i \in \mathbf{Z}_3} (a_{i,i+1} - a_{ii}) > \prod_{i \in \mathbf{Z}_3} (a_{ii} - a_{i,i-1})$ *and* $\det A < 0$, *then* ψ *tends toward the equilibrium point* x^*.

(2) *If* $\prod_{i \in \mathbf{Z}_3} (a_{i,i+1} - a_{ii}) < \prod_{i \in \mathbf{Z}_3} (a_{ii} - a_{i,i-1})$ *then* ψ *winds toward the heteroclinic cycle* T.

§2. EXISTENCE OF A LIMIT CYCLE

In another case that the system is not so near the system (LV_1), an ω-limit set which is another type one in Theorem 1.1. is happened. In some case, it is a limit cycle. We shall show these cases in Proposition 2.2.

PROPOSITION 2.2. *Consider the system* (LV) *with* $a_{ii} = -1$, $\det A < 0$. *Suppose that* $\gamma_i x_i^*$ *are constant* k, $a_{i,i-1}$ $(i \in \mathbf{Z}_3)$ *have negative values (and not all the same) and satisfy the following conditions* (C1) *and* (C2):

(C1)
$$1 + \prod_{i \in \mathbf{Z}_3} (1 + a_{i,i-1}) > 0.$$

$$\text{(C2)} \qquad 8 + \prod_{i \in \mathbf{Z}_3} a_{i,i-1} < 0.$$

If $a_{i,i+1}$ $(i \in \mathbf{Z}_3)$ close enough to 0, then there exists a non-trivial ω-limit set in $(\mathbf{R}^3_+)^\circ$. In particular, when $a_{i,i+1}$ $(i \in \mathbf{Z}_3)$ close enough to 0 from below, the ω-limit set is a limit cycle.

The above proposition has the following corollary.

COROLLARY 2.3. *Consider the system* (LV) *with* $a_{ii} = -1$, $\det A < 0$, $\gamma_i > 0$ *and* $x^* \in (\mathbf{R}^3_+)^\circ$ $(i \in \mathbf{Z}_3)$. *Suppose that* $a_{i,i-1}$ *are negative and not all the same, and the parameters of the system holds the following condition* (D1) *and* (D2)*:*

$$\text{(D1)} \qquad \prod_{i \in \mathbf{Z}_3} (1 + a_{i,i+1}) + \prod_i (1 + a_{i,i-1}) > 0.$$

$$\text{(D2)} \qquad \Big(\sum_{i \in \mathbf{Z}_3} \gamma_i x_i^*\Big)\Big\{\sum_{i \neq j \in \mathbf{Z}_3} \gamma_i \gamma_j x_i^* x_j^* (1 - a_{ij} a_{ji})\Big\} + \Big(\prod_{i \in \mathbf{Z}_3} \gamma_i x_i^*\Big) \det A \quad < \quad 0.$$

Then there exists a non-trivial ω-limit set in $(\mathbf{R}^3_+)^\circ$. In particular, if $a_{i,i+1} < 0$ $(i \in \mathbf{Z}_3)$, then the ω-limit set is a limit cycle.

§3. STATEMENT AND PROOF OF THEOREM 3.1.

For our interests where the limit set exists, we shall consider the following set I and conditions (G1), (G2) and (G3).

$$I := \{x \in \mathbf{R}^3_+ : C_{\min} \leq \frac{x_1}{\gamma_1} + \frac{x_2}{\gamma_2} + \frac{x_3}{\gamma_3} \leq C_{\max}\}$$

where

$$C_{\min} := \min(C'_{\min}, C_{\bar{x}}) \quad \text{and} \quad C_{\max} := \max(C'_{\max}, C_{\bar{x}}).$$

Here

$$C'_{\min} := \inf\{c > 0 : \text{(G1)} \cap \text{(G2)}\}, \quad C'_{\max} := \sup\{c > 0 : \text{(G1)} \cap \text{(G2)}\}$$

and $C_{\bar{x}}$ is defined in (G3).

(G1) : For some $i, j \in \mathbf{Z}_3$ $(i \neq j)$,

$$(1 - c\gamma_i)(1 - c\gamma_j) < 0. \tag{6}$$

(G2) : For some $i, j \in \mathbf{Z}_3$ $(i \neq j)$ if

$$\gamma_i^2 + (a_{ij} + a_{ji})\gamma_i\gamma_j + \gamma_j^2 > 0,$$

$$c(2\gamma_i^2 + (a_{ij} + a_{ji})\gamma_i\gamma_j) > \gamma_i - \gamma_j \quad \text{and} \quad c(2\gamma_j^2 + (a_{ij} + a_{ji})\gamma_i\gamma_j) > \gamma_j - \gamma_i$$

holds, then

$$h_c(1 - c\gamma_i) < 0 \quad \text{or} \quad h_c(1 - c\gamma_j) < 0, \tag{7}$$

where

$$h_c := \frac{\gamma_i^2\gamma_j^2(a_{ij} + a_{ji} + 2)(a_{ij} + a_{ji} - 2)c^2 + 2\gamma_i\gamma_j(a_{ij} + a_{ji} + 2)(\gamma_i\gamma_j)c + (\gamma_i - \gamma_j)^2}{\gamma_i^2 + (a_{ij} + a_{ji})\gamma_i\gamma_j + \gamma_j^2}.$$

(G3) : We put ${}_1a_1$, ${}_\gamma a_\gamma$, s and $\tilde{x}$ as follows.

$$ {}_1a_1 := (1,1,1)\ (A + {}^tA)^{-1}\ {}^t(1,1,1), $$

$$ {}_\gamma a_\gamma := (\frac{1}{\gamma_1}, \frac{1}{\gamma_2}, \frac{1}{\gamma_3})\ (A + {}^tA)^{-1}\ {}^t(\frac{1}{\gamma_1}, \frac{1}{\gamma_2}, \frac{1}{\gamma_3}). $$

Thus

$$ {}_1a_\gamma := (1,1,1)\ (A + {}^tA)^{-1}\ {}^t(\frac{1}{\gamma_1}, \frac{1}{\gamma_2}, \frac{1}{\gamma_3}) = {}_\gamma a_1. $$

And

$$ s := -({}_1a_1/{}_\gamma a_\gamma)^{\frac{1}{2}}, $$

$$ \tilde{x} := \{s(\frac{1}{\gamma_1}, \frac{1}{\gamma_2}, \frac{1}{\gamma_3}) - (1,1,1)\}\ (A + {}^tA)^{-1}. $$

If $\tilde{x} \in (\mathbf{R}^3_+)^\circ$, then

$$ C_{\tilde{x}} = -{}_1a_\gamma - ({}_1a_1 \cdot {}_\gamma a_\gamma)^{\frac{1}{2}}, \tag{8} $$

where $\cdot$ is the ordinary multiplication of numbers.

THEOREM 3.1. *Given system* (LV) *with* $a_{ij} + a_{ji} < 0$ $(i, j \in \mathbf{Z}_3)$. *If the set* I *satisfies* (G1), (G2) *and* (G3), *then the set* I *is positively invariant and the every orbit from* $(\mathbf{R}^3_+)^\circ$ *has an* ω*-limit in the set* I.

Proof We consider the function S as follows:

$$ S(x) := \frac{x_1}{\gamma_1} + \frac{x_2}{\gamma_2} + \frac{x_3}{\gamma_3}, $$

on $\mathbf{R}^3_+$. Thus the differential of S is

$$ \dot{S} = x_1 + x_2 + x_3 + xA^tx. \tag{9} $$

Define a plane S_c for $c > 0$ and a quadratic surface Q as follows:

$$ S_c := \{x \in \mathbf{R}^3_+ : S(x) = c\}, $$

$$ Q := \{x \in \mathbf{R}^3_+ : \dot{S} = 0\}. $$

Remark that the surface Q includes all equilibrium points of the system (LV).
For enough large $l \in \mathbf{R}$ and the matrix E_1 whose all elements are equal to 1,

$$ \dot{S} \geq x_1 + x_2 + x_3 - l\ xE_1{}^tx = (x_1 + x_1 + x_2)\{1 - l(x_1 + x_2 + x_3)\}. $$

Thus for an enough small $c' > 0$,

$$ \dot{S}|_{x \in S_{c'}} > 0 \quad \text{on} \quad (\mathbf{R}^3_+)^\circ. $$

And for the identity matrix E_0,

$$ \dot{S} \leq x_1 + x_2 + x_3 - xE_0{}^tx = x_1(1 - x_1) + x_2(1 - x_2) + x_3(1 - x_3). $$

Thus for an enough large $C' > 0$,

$$\dot{S}|_{x\in S_{C'}} < 0 \quad \text{on} \quad (\mathbf{R}^3_+)^\circ.$$

If for some $c > 0$

$$Q \cap S_c \cap \mathbf{R}^3_+ = \emptyset,$$

then for an arbitrary initial value $\psi(0) \in \mathbf{R}^3_+ \cap S_c$ and arbitrary $t > 0$,

$$\psi(t) \in \{x : S(x) > c\} \quad \text{or} \quad \psi(t) \in \{x : S(x) < c\}.$$

Here we denote the minimum value and maxmum value of c such that $Q \cap S_c \cap \mathbf{R}^3_+ \neq \emptyset$ by $C_{\min}$ and $C_{\max}$ respectively.

Now γ_i and a_{ij} $(i,j \in \mathbf{Z}_3)$ are fixed. The quadratic curve $Q \cap S_c$ does not change the type of it (but may degenerate).

Hence we shall consider the value of c such that this quadratic curve $Q \cap S_c$ and the compact set $S_c \cap \mathbf{R}^3_+$ have a common point.

In the case of (G1) and (G2), we shall consider the common set of $S_c \cap \partial\mathbf{R}^3$ and $Q \cap S_c$, where $\partial\mathbf{R}^3 := \{x \in \mathbf{R}^3 : \prod_i x_i = 0\}$. We may assume $x_3 = 0$ without loss of generality. So if

$$\{x \in \mathbf{R}^3_+ : x \in Q \cap S_c \text{ and } x_3 = 0\} \subset \{x \in \mathbf{R}^3_+ : x \in S_c \cap \partial\mathbf{R}^3\},$$

then $Q \cap S_c \cap \mathbf{R}^3_+ \neq \emptyset$.

It becomes a simple problem where the solution of the quadratic equation exist.

In the case of (G3), we can not determine the value of $C_{\min}$ and $C_{\max}$ in this way. But just on $C_{\min}$ or $C_{\max}$, the following equality holds.

$$s\nabla S_c = \nabla Q|_{\{x:S=C_{\min} \text{ or } C_{\max}\}}, \tag{10}$$

where $s\ (\neq 0) \in \mathbf{R}$.

Since $\nabla S_c = (\frac{1}{\gamma_1}, \frac{1}{\gamma_2}, \frac{1}{\gamma_3})$, and Q is quadratic, the point $\tilde{x} \in \mathbf{R}^3$ which holds (10) are less than two or equal two. For those points, the just value of C such that $\tilde{x} \in S_c$ is $C_{\min}$ or $C_{\max}$.

We shall calculate in the state as follows:

$$\nabla Q = g + xD_x g = (1,1,1) + (x_1, x_2, x_3)(A + {}^tA),$$

where we put the system (LV), $\dot{x}_i = \gamma_i x_i g_i$ $(i \in \mathbf{Z}_3)$ and $g := (g_1, g_2, g_3)$.

From the equation (10),

$$\tilde{x} = \{s(\frac{1}{\gamma_1}, \frac{1}{\gamma_2}, \frac{1}{\gamma_3}) - (1,1,1)\}(A + {}^tA)^{-1}. \tag{11}$$

Since $\tilde{x} \in Q$,

$$(1,1,1)\tilde{x} + \frac{1}{2}\tilde{x}(A + {}^tA){}^tx = 0. \tag{12}$$

By (11) and (12),

$$s^2 = {}_1a_1/{}_\gamma a_\gamma. \tag{13}$$

When $\tilde{x} \in \mathbf{R}^3_+$, we may put $\tilde{c} := \{c : \tilde{x} \in S_c\}$.

Thus

$$\tilde{c} = (\frac{1}{\gamma_1}, \frac{1}{\gamma_2}, \frac{1}{\gamma_3}){}^t\tilde{x} = -(\frac{1}{\gamma_1}, \frac{1}{\gamma_2}, \frac{1}{\gamma_3})(A + {}^tA)^{-1}\,{}^t(1,1,1) - ({}_1a_1 \cdot {}_\gamma a_\gamma)^{\frac{1}{2}},$$

where we put $s = -({}_1a_1/{}_\gamma a_\gamma)^{\frac{1}{2}}$ by (13), because if $s = ({}_1a_1/{}_\gamma a_\gamma)^{\frac{1}{2}}$ then $\tilde{c} \leq 0$. The value of $\tilde{c}$ is $C_{\min}$ or $C_{\max}$ in the case. □

For arbitrary $\gamma_i' > 0$ $(i \in \mathbf{Z}_3)$, we shall consider the set:

$$I_{S_\gamma} := \{x \in \mathbf{R}_+^3 : C_{\min} \leq \frac{x_1}{\gamma_1'} + \frac{x_2}{\gamma_2'} + \frac{x_3}{\gamma_3'} \leq C_{\max}\}.$$

Here the value of $C_{\min}$ and $C_{\max}$ are the minimum and the maxmum value respectively such that

$$Q_\gamma \quad \cap \quad S_\gamma c \quad \cap \quad \mathbf{R}_+^3 \quad \neq \emptyset,$$

where

$$S_\gamma c := \{x \in \mathbf{R}^3 : S_\gamma := \frac{x_1}{\gamma_1'} + \frac{x_2}{\gamma_2'} + \frac{x_3}{\gamma_3'} = c\},$$

$$\dot{S}_\gamma = \frac{\gamma_1}{\gamma_1'}x_1 + \frac{\gamma_2}{\gamma'2}x_2 + \frac{\gamma_3}{\gamma_3'}x_3 + x \ \mathrm{diag}\{\frac{\gamma_1}{\gamma_1'}, \frac{\gamma_2}{\gamma_2'}, \frac{\gamma_3}{\gamma_3'}\}A^t x,$$

$$Q_\gamma := \{x \in \mathbf{R}_+^3 : \dot{S}_\gamma = 0\}.$$

Here diag$\{\cdot\}$ is a diagonal matrix with components $\cdot$. Remark that all equilibrium points of the system (LV) are included in Q_γ, too.

COROLLARY 3.2. *Consider the system as in Theorem 3.1. For an arbitrary vector* $\gamma' := (\gamma_1', \gamma_2', \gamma_3')$ $(\gamma_i' > 0 \quad i \in \mathbf{Z}_3)$, *we put*

$$I := \bigcap_\gamma I_{S_\gamma}.$$

Then the set I is positively invariant, and every orbit from $(\mathbf{R}_+^3)^\circ$ has an ω-limit in the set I.

Remark that in the case of (G3), it is well known that if the Q_γ is elliptic, then the x^* is exactly a global attractor on $(\mathbf{R}_+^3)^\circ$.

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