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A mathematical method of fuzzy reasoning

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Abstract
In this paper, we introduce in a new natural method of fuzzy reasoning by defining some fuzzy mathematical concepts. It is well-known that Mamdani's method never holds the property of Monotony. Our proposal method holds the monotony property.

1. Introduction
The monotony is the important point for applying fuzzy reasoning to "educational evaluation", "clinical decision making", etc. Then, by defining some fuzzy mathematical concepts, we can mathematically introduce a new fuzzy reasoning method that could make it clear the above point.

2. Fuzzy Mathematical Concepts
[Definition 2.1] Normal
A fuzzy set A is normal if its membership function attains 1, that is,
$$\sup_{x \in \mathbb{R}} A(x) = 1.$$  

[Definition 2.2] Convex
A fuzzy set A is convex if its membership function is such that
$$A(\lambda x + (1-\lambda)y) \geq A(x) \land A(y)$$
for any \(x, y \in \mathbb{R}\) and \(\lambda \in [0, 1]\).

Lemme 2.1 Let A be a normal convex fuzzy set. If \(A(x)\) is a membership function of A, we have
$$A(x) = \begin{cases} 1(x), & x \in I_1 \\ 1, & x \in I_2 \\ r(x), & x \in I_3 \end{cases}$$
where \(1(x)\) and \(r(x)\) are a monotonous increasing function and a monotonous decreasing function, respectively, \(I_1, I_2, I_3\) are int-
ervals which satisfy $I_1 \cup I_2 \cup I_3 = \mathbb{R}$.

**Lemma 2.2** Let $A(x)$ be a membership function of normal convex fuzzy set $A$ defined on a bounded closed interval $I \subset \mathbb{R}$. Then $A(x)$ is integrable over $I$.

**[Definition 2.3] Fuzzy Partitioned Space**

Let $I$ be a bounded closed interval on $\mathbb{R}$ and let

$$FP(I)=\{A_i : i \in A\}$$

be a family of normal fuzzy sets on $I$. If $FP(I)$ satisfies the equation

$$\sum_{i \in A} A_i(x) = 1, \forall x \in I,$$

then we call $FP(I)$ a fuzzy partition of $I$, $I$ a fuzzy partitioned space and each $A_i$, which is a element of $FP(I)$, a fuzzy partition set.

**[Definition 2.4] $\alpha$-Cut**

The $\alpha$-cut of $A$, denoted by $[A]^\alpha$, is a set consisting of those elements of the universe $\mathbb{R}$ whose membership values exceed the threshold level $\alpha$,

$$[A]^\alpha = \{x | A(x) \geq \alpha\}$$

for any $\alpha \in [0,1]$.

**Lemma 2.3** Let $A$ be a normal convex fuzzy set in a closed interval $I$ on $\mathbb{R}$. Then the closure of $[A]^\alpha$ is a closed interval.

**[Definition 2.5] Ordered Fuzzy Partition Sets**

For two normal convex fuzzy sets $A, B$ in a closed interval $I$ on $\mathbb{R}$ with

$$[A]^\alpha = [a_\alpha, b_\alpha], \quad [B]^\alpha = [c_\alpha, d_\alpha],$$

we define an order relation "<" as follows:

$$A < B \iff a_\alpha \leq c_\alpha, \quad b_\alpha \leq d_\alpha, \forall \alpha \in [0,1].$$

**Lemma 2.4** Let $(A_1, A_2, \ldots, A_n)$ be a fuzzy partition of a closed interval $I$ on $\mathbb{R}$. After renumbering if necessary, this set can be well ordered such that

$$A_1 < A_2 < \cdots < A_n.$$

3. **Product-Sum-Gravity Method**

Let $FP(F_1)=(L_1, L_2, \ldots, L_m | L_1 < L_2 < \cdots < L_m)$, $FP(F_2)=(M_1, M_2, \ldots, M_n | M_1 < M_2 < \cdots < M_n)$ and $FP(F)=(A_1, A_2, \ldots, A, | A_1 < A_2 < \cdots < A_r)$ be ordered.
fuzzy partition sets of $F_1, F_2$ and $E$ respectively. And let the fuzzy rule be given by an onto-mapping

$$h: F_1 \times F_2 \to E$$

which is called rule mapping. Then we call numerical reasoning on fuzzy partitioned space to calculate the conclusion $E = \hat{\phi}(x_1, x_2)$ concerning with the fact $(x_1, x_2)$, where we call the function a numerical reasoning function. Especially the numerical reasoning function of product-sum-gravity method is as follows:

$$\phi(x_1, x_2) = \{\text{abscissa of the center of gravity of } \{L_i(x_1)M_j(x_2)A_k + L_i(x_1)M_{j+1}(x_2)A_k - L_{i+1}(x_1)M_j(x_2)A_k - L_{i+1}(x_1)M_{j+1}(x_2)A_k - \} \}

$$

where $L_s(x_i) = 0 (s \neq i, i+1)$, $M_t(x_2) = 0 (t \neq j, j+1)$ and the following rules satisfy:

$$L_i, M_j \to A_k, \quad L_i, M_{j+1} \to A_k, \quad L_{i+1}, M_j \to A_k, \quad L_{i+1}, M_{j+1} \to A_k.$$

**Theorem 3.1** If the rule mapping $h: F_1 \times F_2 \to E$ satisfies the following conditions, the numerical reasoning function of product-sum-gravity method on fuzzy partition space $E = \phi(x_1, x_2)$ is monotonous increasing.

i) Boundary Conditions

$$L_1, M_1 \to A_1, \quad L_m, M_n \to A_r.$$

ii) Monotonicity Condition

If $L_{i_1}, M_{j_1} \to A_r, \quad L_{i_2}, M_{j_2} \to A_q (i_1 \leq i_2, j_1 \leq j_2)$, then $p \leq q$.

From i), ii) it follows that $(L_i, M_j)$ is a complete lattice.

iii) Continuity Condition

If $L_{i_1}, M_{j_1} \to A_r, \quad L_{i_2}, M_{j_2} \to A_q (i_1 \leq i_2, j_1 \leq j_2, i_1 + j_1 + 1 = i_2 + j_2)$, then $\mid p - q \mid \leq 1$.

iv) $|A_1| \geq |A_2| = \cdots = |A_{r-1}| \leq |A_r|$

where

$$|A_i| = \int A_i(x) dx, \quad i = 1, 2, \cdots, r.$$

Let $FP(F_1) = \{L_1, L_2, \cdots, L_m | L_1 < L_2 < \cdots < L_m \}$, $FP(F_2) = \{M_1, M_2, \cdots, M_n | M_1 < M_2 < \cdots < M_n \}$, $FP(F_3) = \{N_1, N_2, \cdots, N_n | N_1 < N_2 < \cdots < N_n \}$ and $FP(E) = \{A_1, A_2, \cdots, A_r | A_1 < A_2 < \cdots < A_r \}$ be respectively ordered fuzzy partition sets of $F_1, F_2, F_3$ and $E$.

**Theorem 3.2** If the rule mapping $h: F_1 \times F_2 \times F_3 \to E$ satisfies the following conditions, the numerical reasoning function of product-sum
-gravity method on fuzzy partition space $E=\emptyset(x_1,x_2,x_3)$ is monotonous increasing.

i) Boundary Conditions

$$L_1, M_1, N_1 \rightarrow A_1, \quad L_m, M_n, N_n \rightarrow A_r$$

ii) Monotonicity Condition

If $L_{i_1}, M_{j_1}, N_{k_1} \rightarrow A_p, \quad L_{i_2}, M_{j_2}, N_{k_2} \rightarrow A_q$ $(i_1 \leq i_2, j_1 \leq j_2, k_1 \leq k_2)$, then $p \leq q$.

iii) Continuity Condition

If $L_{i_1}, M_{j_1}, N_{k_1} \rightarrow A_p, \quad L_{i_2}, M_{j_2}, N_{k_2} \rightarrow A_q$ $(i_1 \leq i_2, j_1 \leq j_2, k_1 \leq k_2, i_1 + j_1 + k_1 + 1 = i_2 + j_2 + k_2)$, then $|p-q| \leq 1$.

iv) $|A_1| \geq |A_2| = \cdots = |A_{r-1}| \leq |A_r|$.

Remark The same result holds if the rule mapping $h:F_1XF_2XF_3\rightarrow E$ satisfies the Okuda's condition$^4$)

$$|P_k(\lambda)| \cdot |Q_k(\lambda+1)| \geq |P_k(\lambda+1)| \cdot |Q_k(\lambda)|$$

instead of the condition iv), where

$$P_k(\lambda) = \{(\tau_1, \tau_2, \tau_3): h(\cdots, H_{\tau k}^r, \cdots) = \lambda, \tau_i = \tau_i', \tau_i' + 1\}$$

$$Q_k(\lambda) = \{(\tau_1, \tau_2, \tau_3): h(\cdots, H_{\tau k}^r, \cdots) = \lambda, \tau_i = \tau_i', \tau_i' + 1\}.$$ 

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