数理解析研究所講究録
1132 巻 2000 年 143-149

On the Nash equilibrium of partial cooperative games

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1 Introduction

In [4] a class of partial cooperative games with perfect information (PCGPI) is defined. PCGPI proceeds on a tree $K(x_0)$ of a finite non–cooperative game in extensive form with perfect information and without chance moves $\Gamma = \langle K(x_0), P, h\rangle$. Here, x_0 is the origin of $K(x_0)$; P denotes the player partition $P_1, \ldots, P_i, \ldots, P_n$, P_{n+1}, where P_i, $i \in N$, is the set of decision points of player i, and P_{n+1} is the set of the endpoints; $h : P_{n+1} \to R^n_+$ is the terminal payoff function. Denote the player set by $N = \{1, \ldots, n\}$. In PCGPI for each player i a set of points called the cooperative region is given. (In general case the cooperative region may be empty.) During the game, in a decision point $x \in P_i$ player i is purposed to use an individually rational behavior if x is not in his cooperative region. But, if x lies in the cooperative region of player i, then in x he forms a coalition involving all players whose cooperative regions contain x also.

Formalization of the concept of the players' cooperative region may be realized by various approaches. In [4] a timing interpretation of the cooperative region is considered. It is supposed that $K(x_0)$ has the following information structure:

1. For any evolution of the game players make decisions in accordance with their index order, i.e., in the point x_0 the decision is made by player 1, in the immediate successors of x_0 the decision is made by player 2 and so on until player n. After player n the decision is again made by player 1 and etc.

2. Each path has the same length.

For the given game tree, we shall say that a *stage* is the n sequential moves, where the first move is made by player 1. Let the length of $K(x_0)$ be $T + 1$ stages.

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In PCGPI a vector $s = (s_1, \ldots, s_i, \ldots, s_n)$, $s_i \in \overline{L} = \{0, 1, \ldots, T, T+1\}$, is given. The component s_i denotes the length of the player i's cooperative activity. If $s_i = 0$, then during the game player i plays non–cooperatively. If $s_i > 0$, then starting form the initial stage 0 until the stage $T - s_i$ player i plays non–cooperatively, and since the stage $t_i = T - s_i + 1$ until the end of the game player i is ready to cooperate with anybody. The given PCGPI is denoted by $\Gamma_s(x_0)$.

Suppose that $\{x_0, \ldots, \bar{x}\}$ is the path realized in $\Gamma_s(x_0)$. Let $S_s = \{i \in N | s_i > 0\}$ be a coalition formed to the end of $\Gamma_s(x_0)$. If $i \notin S$, then the payoff of player i is defined by the terminal payoff function h and equals $h_i(\bar{x})$. If $i \in S_s$, then the payoff of player i is defined by the Shapley value $\alpha(s)$ of the payoff of the coalition S_s, i.e.,

$$\sum_{j \in S_s} \alpha_j(s) = \sum_{j \in S_s} h_j(\bar{x})$$

It is considered that the purpose of a player in $\Gamma_s(x_0)$ is maximizing his payoff within the restrictions given by s.

Let $L = \prod_{i \in N} \overline{L}$ be the set of all vectors s that can be defined for $K(x_0)$. In [4] an approach to find the players' optimal behavior in $\Gamma_s(x_0)$, $s \in L$, is proposed. The scheme of construction of a path $\Phi_s(x_0) = \{x_0, \ldots, \phi_s(x_0)\}$, $\phi_s(x_0) \in P_{n+1}$, which is realized in $\Gamma_s(x_0)$ when players keep on their optimal behavior, is defined. The payoff–vector $r(s) = (r_1(s), \ldots, r_n(s))$,

$$r_i(s) = \begin{cases} h_i(\phi_s(x_0)), & \text{if } s_i = 0 \\ \alpha_i(s), & \text{if } s_i > 0, \end{cases} \quad i \in N$$

related to $\Phi_s(x_0)$ is called the *value* of $\Gamma_s(x_0)$.

In $\Gamma_s(x_0)$ the vector s is not regulated by players. In this paper we consider a generalization of $\Gamma_s(x_0)$, where players form a vector $s \in L$ themselves.

2 Model.

On the tree $K(x_0)$ consider a new game $\Gamma_L(x_0)$. In pre–play communications of $\Gamma_L(x_0)$ players form a vector $s \in L$. Then, players play in accordance with the vector s. Hence, $\Gamma_L(x_0)$ evolves along the optimal path $\Phi_s(x_0)$ and players get payoffs defined by the value $r(s)$. It is supposed that in $\Gamma_L(x_0)$ each player tries to maximize his own payoff.

Definition. A vector $s^* \in L$ is called the *Nash equilibrium* of $\Gamma_L(x_0)$ if for all $s_i \in \overline{L}$ and $i \in N$ there is

$$r_i(s^*) \geq r_i(s^* | s_i), \tag{2.1}$$

where $s^*|s_i = (s_1^*, \ldots, s_{i-1}^*, s_i, s_{i+1}^*, \ldots, s_n^*)$.

Theorem. The Nash equilibrium in $\Gamma_L(x_0)$ always exists.

Proof. We prove the theorem if propose the Nash equilibrium construction method for $\Gamma_L(x_0)$.

Knowing the formed vector s we know the path $\Phi_s(x_0)$ of the game evolution and players' payoffs $r(s)$. Therefore, the set $\overline{L}$ may be considered as the set of the player's strategies in $\Gamma_L(x_0)$. For each player i and his decision point $x \in P_i$, if player i cooperates in x or not that is all we need to know.

Basing on $K(x_0)$, define an auxiliary binary tree $\overline{K}(x_0)$. The length of $\overline{K}(x_0)$ is $T+1$ stages (the definition of a stage is given in section 1). For each decision point x, we shall call the branches going out from x by *Left* and *Right* respectively. We shall consider that if player i does not cooperate in a stage t, then on $\overline{K}(x_0)$ player i has to go Left in his decision points in the stage t. Otherwise, if player i cooperates in the stage t, then on $\overline{K}(x_0)$ player i has to go Right in his decision points in the stage t. For the given relation between the rules of $\Gamma_L(x_0)$ and $\overline{K}(x_0)$ to be one–to–one, we suppose that $\overline{K}(x_0)$ satisfies the following condition.

Let x_r and x_ℓ be immediate successors of a decision point x. Assume that x_r related to the decision Right in x, and x_ℓ related to the decision Left in x. Then,

$$K(x_r) \cap P_i = \emptyset \tag{2.2}$$

for each player $i \in N$ and his decision point $x \in P_i$. Here, $K(x_r)$ denotes the subtree with the initial point x_r.

Let $\overline{P}_1, \ldots, \overline{P}_n, \overline{P}_{n+1}$ be the player partition on $\overline{K}(x_0)$, where $\overline{P}_{n+1}$ is the set of endpoints. By the condition (2.2) there is one–to–one correspondence between the sets L and $\overline{P}_{n+1}$. Define a payoff function $\bar{h} : \overline{P}_{n+1} \to R^n_+$ by

$$\bar{h}(\hat{x}) = r(s), \quad \hat{x} \in \overline{P}_{n+1} \tag{2.3}$$

where $\hat{x}$ related to s. Consider a non–cooperative game $\overline{\Gamma} = \langle \overline{K}(x_0), \overline{P}, \bar{h} \rangle$. Let $\pi = (\pi_1, \ldots, \pi_n)$ denote a situation in $\overline{\Gamma}$, where π_i, $i \in N$, is a player i's strategy. Denote the set of all situations in $\overline{\Gamma}$ by Π. Suppose that π^* is the Nash equilibrium in $\overline{\Gamma}$. From the construction of the game $\overline{\Gamma}$ it follows that there is one–to–one correspondence between Π and L. Hence, the vector s^* related to π^* satisfies the definition of the Nash equilibrium in $\Gamma_L(x_0)$. □

Remark. During the theorem proof a construction method of the Nash equilibrium in $\Gamma_L(x_0)$ was proposed.

Example. Consider a three person non-cooperative game Γ with the game tree $K(x_0)$ given in Figure 1. $N = \{1, 2, 3\}$. The player 1's decision points are denoted by single circle, player 2's — by double circle and player 3's — by triple circle. The vectors at the endpoints are the terminal payoffs of players, with the first components being the payoff of player 1 and so on. There are two stages in Γ. The initial stage starts in x_0. The stage 1 starts in x_7, x_8, x_9, x_{10}, x_{11}, x_{12}, x_{13}, x_{14}. $L = \{0, 1, 2\}$. For each $s \in L$ consider the game $\Gamma_s(x_0)$ and find the value $r(s)$. All possible values $r(s)$, $s \in L$, are given in Table 1.

Find the Nash equilibrium s^* of the game $\Gamma_L(x_0)$. Construct the tree $\overline{K}(x_0)$ (see in Figure 2) which satisfies the condition (2.2).

We shall say that player 1 goes Up in $x_0 \in \overline{K}(x_0)$, if he cooperates in $\Gamma_L(x_0)$ since the initial stage. Player 1 goes Down in $x_0 \in \overline{K}(x_0)$, if he does not cooperate in $\Gamma_L(x_0)$ in the initial stage. In x_{11}, x_{12},x_{13} and x_{14} of $\overline{K}(x_0)$ player 1 goes Up, if he cooperates in $\Gamma_L(x_0)$ since the stage 1. If player 1 does not cooperate $\Gamma_L(x_0)$, then he goes Down in x_{11}, x_{12},x_{13} and x_{14} of $\overline{K}(x_0)$.

Player 2 goes Up in x_1, x_2 of $\overline{K}(x_0)$, if he cooperates in $\Gamma_L(x_0)$ since the initial stage x_1, x_2 of $\overline{K}(x_0)$. If player 2 does not cooperate in $\Gamma_L(x_0)$ in the initial stage, then he goes Down in x_1, x_2 of $\overline{K}(x_0)$. Player 2 goes Up in x_9, x_{10}, x_{25}, x_{26}, x_{27}, x_{28} of $\overline{K}(x_0)$, if he cooperates in $\Gamma_L(x_0)$ since the stage 1. If player 2 does not cooperate in $\Gamma_L(x_0)$, then he goes Down in x_9, x_{10}, x_{25}, x_{26}, x_{27}, x_{28} of $\overline{K}(x_0)$.

Player 3 goes Up in x_3, x_4, x_5, x_6 of $\overline{K}(x_0)$, if he cooperates in $\Gamma_L(x_0)$ since the initial stage. If player 3 does not cooperate in $\Gamma_L(x_0)$ in the initial stage, then he goes Down in x_3, x_4, x_5, x_6 of $\overline{K}(x_0)$. Player 3 goes Up in x_8, x_{19}, x_{20}, x_{23}, x_{24}, x_{41}, x_{42}, x_{43}, x_{44} of $\overline{K}(x_0)$, if he cooperates in $\Gamma_L(x_0)$ since the stage 1. If player 3 does not cooperate in $\Gamma_L(x_0)$, then he goes Down in x_8, x_{19}, x_{20}, x_{23}, x_{24}, x_{41}, x_{42}, x_{43}, x_{44} of $\overline{K}(x_0)$.

Using the given interpretation of players' behavior, we put the values $r(s)$, $s \in L$, at the endpoints of $\overline{K}(x_0)$. Define the non–cooperative game $\overline{\Gamma}$ on $\overline{K}(x_0)$ and find the Nash equilibrium of $\overline{\Gamma}$.

There are tree Nash equilibrium in $\overline{\Gamma}$. The trajectories related to the Nash equilibrium situations are $\{x_0, \dots, x_{23}\}$, $\{x_0, \dots, x_{35}\}$ and $\{x_0, \dots, x_{45}\}$. Hence, the Nash equilibriums in $\Gamma_L(x_0)$ are $(0,2,1)$, $(0,1,2)$ and $(0,1,1)$. For all cases players get payoffs $(9, 4\frac{1}{2}, 5\frac{1}{2})$. We can see that for player 1 it is optimal (in the sense of the Nash equilibrium) not to cooperate in $\Gamma_L(x_0)$. Note, that if all players cooperate since the start of $\Gamma_L(x_0)$, then we have a usual cooperative game on $K(x_0)$. In this case, the Shapley value is $(7,7,6)$.

$s=(s_1,s_2,s_3)$	$r(s)$	$s=(s_1,s_2,s_3)$	$r(s)$	$s=(s_1,s_2,s_3)$	$r(s)$
$(0,0,0)$	$(5,2,5)$	$(2,0,0)$	$(5,2,5)$	$(1,0,2)$	$(5\frac{1}{2},4,3\frac{1}{2})$
$(1,0,0)$	$(5,2,5)$	$(0,2,0)$	$(5,2,5)$	$(2,2,0)$	$(4\frac{1}{2},4\frac{1}{2},6)$
$(0,1,0)$	$(5,2,5)$	$(0,0,2)$	$(5,2,5)$	$(0,2,2)$	$(6\frac{1}{2},7,5\frac{1}{2})$
$(0,0,1)$	$(5,2,5)$	$(2,1,0)$	$(4\frac{1}{2},4\frac{1}{2},6)$	$(2,0,2)$	$(4\frac{1}{2},8,5\frac{1}{2}$
$(1,1,0)$	$(4\frac{1}{2},4\frac{1}{2},6)$	$(1,2,0)$	$(4\frac{1}{2},4\frac{1}{2},6)$	$(2,2,1)$	$(5,7\frac{1}{2},7\frac{1}{2})$
$(1,0,1)$	$(5\frac{1}{2},4,3\frac{1}{2})$	$(0,2,1)$	$(9,4\frac{1}{2},5\frac{1}{2})$	$(1,2,2)$	$(5\frac{1}{2},6\frac{1}{2},7)$
$(0,1,1)$	$(9,4\frac{1}{2},5\frac{1}{2})$	$(0,1,2)$	$(9,4\frac{1}{2},5\frac{1}{2})$	$(2,1,2)$	$(5,7\frac{1}{2},7\frac{1}{2})$
$(1,1,1)$	$(5\frac{1}{2},6\frac{1}{2},7)$	$(2,0,1)$	$(5\frac{1}{2},4,3\frac{1}{2})$	$(2,2,2)$	$(7,7,6)$
$(2,1,1)$	$(5\frac{1}{2},6\frac{1}{2},7)$	$(1,2,1)$	$(5\frac{1}{2},6\frac{1}{2},7)$	$(1,1,2)$	$(5\frac{1}{2},6\frac{1}{2},7)$

Table 1: Players' payoffs

We supposed that each player use the following criteria when he make decision in $\overline{\Gamma}$.

1) to maximize own payoff;

2) if criterion 1 is fulfilled, then to maximize the common payoff of all players;

3) if criteria 1, 2 are fulfilled, then to maximize the payoff of player 1 (if the player is not player 1);

4) if criteria $1, \dots, 3$ are fulfilled, then to maximize the payoff of player 2 (if the player is not player 2) and so on;

$n+2$) if criteria $1, \dots, n+1$ are fulfilled, then to maximize the payoff of player n (if the player is not player n);

$n+3$) if criteria $1, \dots, n+2$ are fulfilled, then to choose any of the remain strategies.

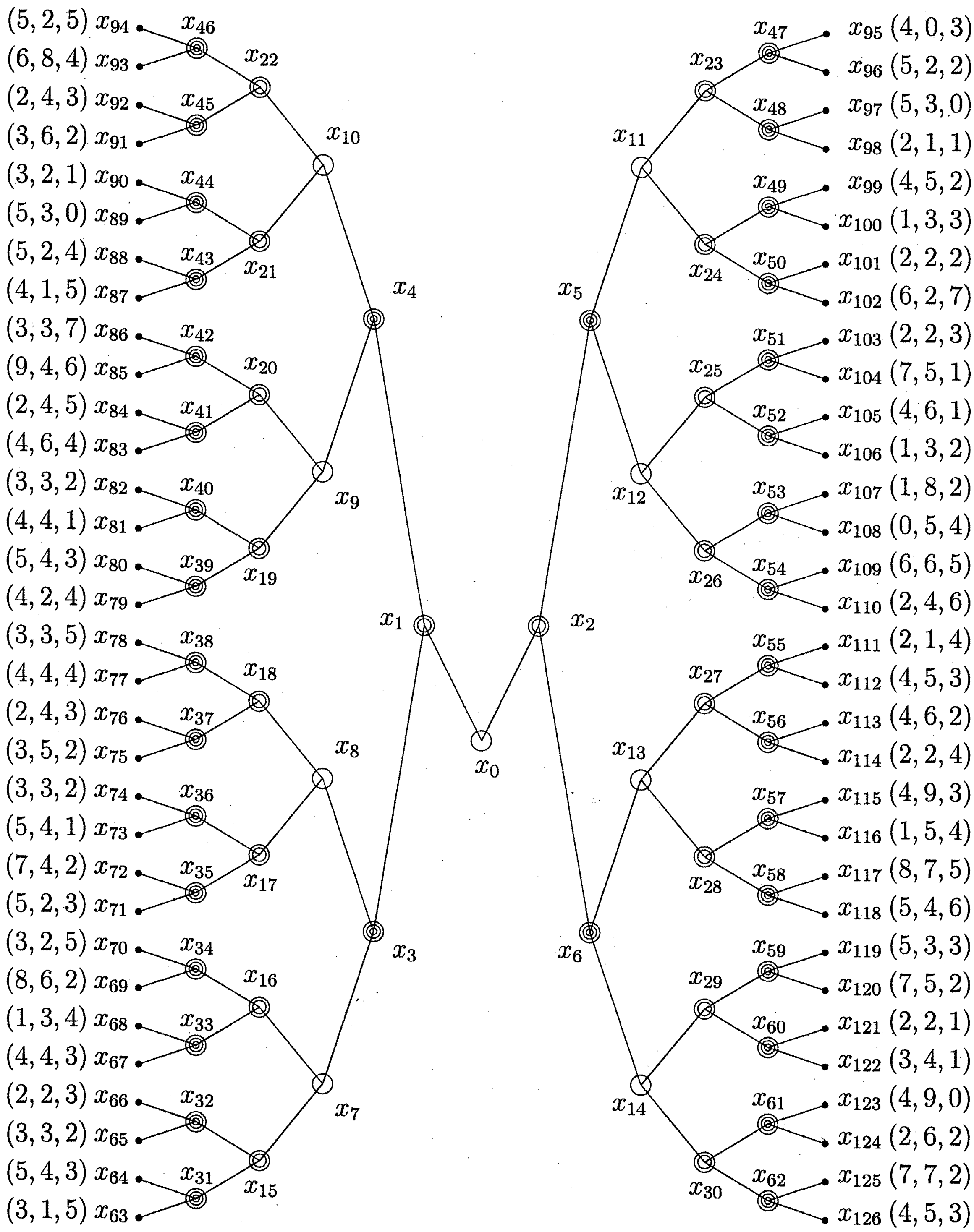

Figure 1: The game tree $K(x_0)$

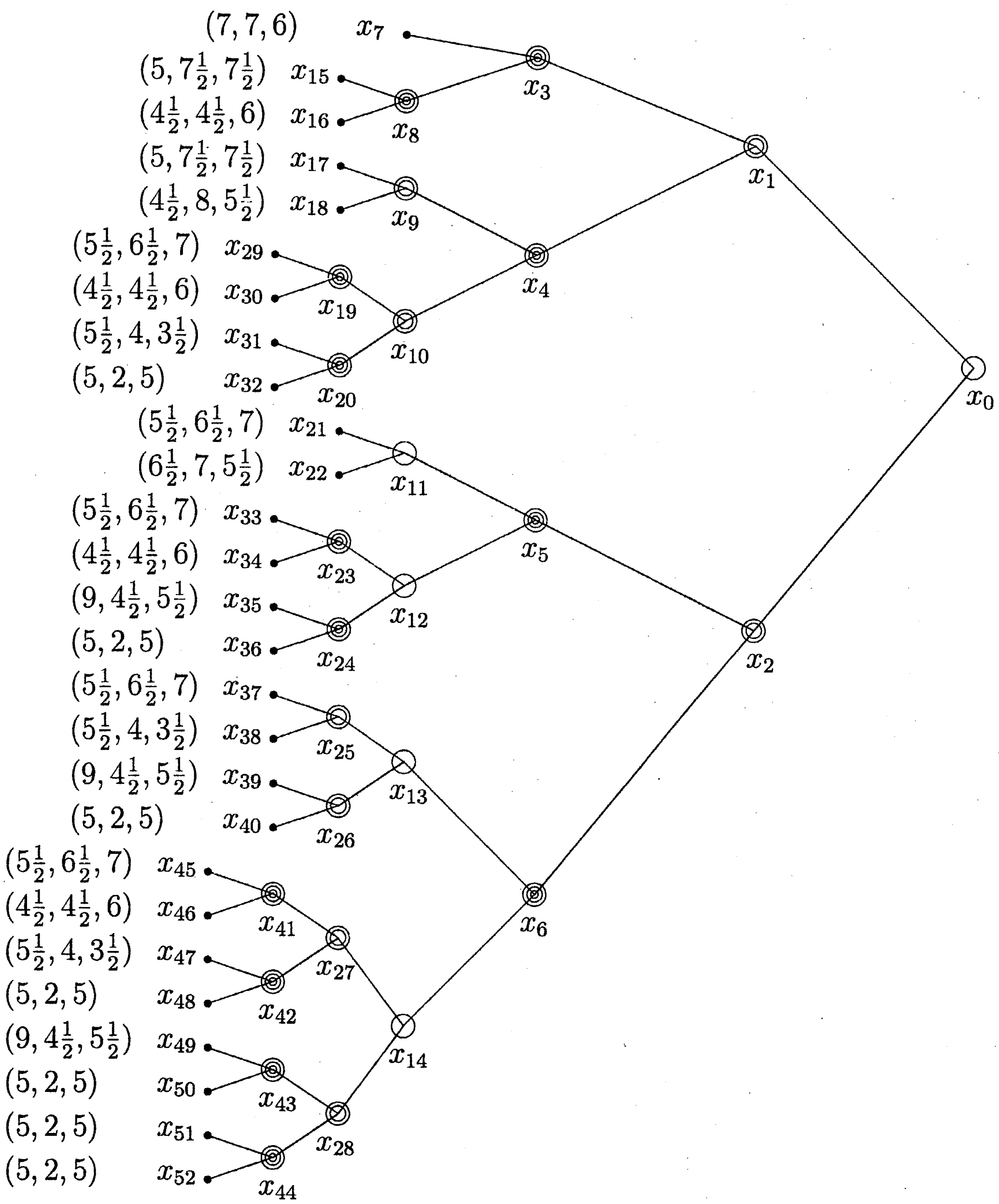

Figure 2: The game tree $\overline{K}(x_0)$

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