数理解析研究所講究録
1132 巻 2000 年 75-83

ON OPTIMAL CHOOSING OF ONE OF THE THREE BEST OBJECTS*

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Abstract. A full–information continuous–time best choice problem is considered. A stream of *iid* random variables (*rv*'s) with a known continuous distribution function (*df*) is observed. The observations appear according to some renewal process independent of observations. The objective is to maximize the probability of selecting of one of the three best observations when observation is perfect, one choice can be made and neither recall nor uncertainty of selection is allowed. The horizon of observation is a positive *rv* independent of observations. For the natural case of the Poisson renewal process and of exponentially distributed horizon it is shown that an optimal strategy is of barrier type.

Key words. best choice problem, optimal stopping, full information

AMS subject classifications. Primary 65L15, Secondary 60G40

1. Introduction. The following full–information best choice problem was studied by Gilbert and Mosteller (1966). A known number, N, of *iid rv*'s $X_1, X_2, \dots, X_N$ from a known continuous *df* F are observed sequentially. The objective is to maximize the probability of selecting the largest. After X_n is observed it must be either accepted (then the observation process is terminated) or rejected (then the observation is continued). Neither recall nor uncertainty of selection is allowed. The full–information best choice problems have been solved also for choosing the largest by accepting exactly once when the number of observations N is random (Porosiński (1987)), for selecting the largest with two choices allowed when N is fixed (Tamaki (1980)) and for choosing one of the two best for geometric number of observations (Porosiński and Szajowski (1990)).

The full–information best choice problem in some continuous time version was first posed by Sakaguchi (1976) and Bojdecki (1978). They independently considered a situation when observations appear according to the Poisson process and a decision about stopping must be made before a random moment T.

In this paper the full–information best choice problem with one choice is considered when one of the three best observations are counted as wins. The observations appear according to some renewal process independent of observations and decision about choosing must be made before a moment T, which is a positive rv independent of observations.

* Research supported by the grant no. KBN 341 428

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The problem is reduced to the classical optimal stopping problem for some Markov chain. The natural case of the Poisson renewal process with parameter λ and exponentially distributed T with parameter μ is examined in detail. The optimal strategy is of barrier type and depends on λ and μ only.

2. Model formulation. Assume that

(a) $X_1, X_2, X_3, \ldots$ are *iid rv*'s with a continuous *df* F, defined on the probability space $(\Omega, \mathcal{F}, P)$,

(b) $\rho_1, \rho_2, \rho_3, \ldots$ are *iid* positive *rv*'s with a continuous *df* G,

(c) T is a positive *rv* with a *df* H,

(d) the *rv*'s $X_1, X_2, \ldots, \rho_1, \rho_2, \ldots, T$ are independent.

The *rv*'s $\rho_1, \rho_2, \ldots$ may be interpreted as the length of time intervals between consecutive values of X's. The *rv* T represents the moment when the observation is terminated. Let

$$S_n = \rho_1 + \ldots + \rho_n, \; n = 1, 2, \ldots, \; S_0 = 0, \tag{1}$$

$$N(t) = \max\{n \geq 0 : S_n \leq t\}, \; t \geq 0. \tag{2}$$

So S_n is the waiting time of the nth observation and $N(t)$ is the total number of X's that appeared up to the time t. At the moment when X_n is seen, all previous values of X's and ρ's are known and moreover it is known whether the moment T follows or not i.e. the σ-field of information is

$$\mathcal{F}_n = \sigma(X_1, \ldots, X_n, \rho_1, \ldots, \rho_n, I_{\{T \geq S_1\}}, \ldots, I_{\{T \geq S_n\}}), \; n = 1, 2, \ldots,$$

where I_A denotes the indicator function of the event A.

Let $\mathcal{T}$ be the set of all Markov moments with respect to the family of σ-fields $(\mathcal{F}_n)_{n=1}^{\infty}$. Let $\xi_{1,n}, \xi_{2,n}, \xi_{3,n}$ denote the largest, the second largest and the third largest value of the sequence $X_1, \ldots, X_n$, respectively ($\xi_{2,1} = \xi_{3,1} = \xi_{3,2} = -\infty$ by definition).

Consider the following problem:

(P) Find a stopping time $\tau^* \in \mathcal{T}$ such that

$$P(\tau^* \leq N(T), X_{\tau^*} \geq \xi_{3,N(T)}) = \sup_{\tau \in \mathcal{T}} P(\tau \leq N(T), X_\tau \geq \xi_{3,N(T)}).$$

In the general case we can show the following theorem.

THEOREM 2.1.

(a) Under the assumptions (a) – (d) *Problem* (P) *can be reduced to an optimal stopping problem for some Markov chain.*

(b) A solution of Problem (P) exists and has the form

$$\begin{aligned}\tau^* &= \inf\{n \in \mathbf{N} : X_n = \xi_{1,n}, (S_n, F(\xi_{1,n}), F(\xi_{2,n}), F(\xi_{3,n})) \in \Delta_1 \\ &\qquad or\ X_n = \xi_{2,n}, (S_n, F(\xi_{1,n}), F(\xi_{2,n}), F(\xi_{3,n})) \in \Delta_2 \\ &\qquad or\ X_n = \xi_{3,n}, (S_n, F(\xi_{1,n}), F(\xi_{2,n}), F(\xi_{3,n})) \in \Delta_3\},\end{aligned}$$

where $\Delta_1, \Delta_2, \Delta_3$ *are some subsets of the space* $(0, +\infty) \times [0,1] \times [0,1] \times [0,1]$ *depending on* G *and* H *only.*

Proof. Since F is known and continuous, without loss of generality it can be additionally assumed that X_n has the uniform *df* on the interval $[0,1], n \in \mathbf{N}$. Let $Z_n = 0$ for $n > N(T)$ and

$$\begin{aligned}Z_n &= P(n \le N(T), X_n \ge \xi_{3,N(T)}) \\ &= I_{\{X_n \ge \xi_{3,n}\}} \sum_{m=n}^{\infty} P(S_m \le T < S_{m+1}, (X_n = \xi_{3,n} \text{ and } X_n \text{ is the largest from } X_n, ..X_m) \\ &\qquad \text{or } (X_n = \xi_{2,n} \text{ and } X_n \text{ is at most the second largest from } X_n, ..X_m) \\ &\qquad \text{or } (X_n = \xi_{1,n} \text{ and } X_n \text{ is at most the third largest from } X_n, ..X_m) \mid \mathcal{F}_n) \\ &= I_{\{X_n \ge \xi_{3,n}\}} W_n,\end{aligned}$$

otherwise, where

$$\begin{aligned}W_n &= \sum_{k=1}^{\infty} q_k(S_n) \Big(I_{\{X_n = \xi_{3,n}\}} X_n^k + I_{\{X_n = \xi_{2,n}\}} \left(X_n^k + k X_n^{k-1}(1 - X_n) \right) \\ &\qquad + I_{\{X_n = \xi_{1,n}\}} \left(X_n^k + k X_n^{k-1}(1 - X_n) + k(k-1) X_n^{k-2}(1 - X_n)^2/2 \right) \Big),\end{aligned}$$

$$\begin{aligned}(3) \quad q_k(t) &= P(\text{exactly } k \text{ observations appear in } [t,T] \mid T > t) \\ &= \frac{1}{1 - H(t)} \int_{(t,+\infty)} \int_{[0,r-t]} (1 - G(r - t - u))\, dG^{*k}(u)\, dH(r),\end{aligned}$$

and G^{*k} stands for a *df* of S_k. Hence $EZ_\tau = P(\tau \le N(T), X_\tau \ge \xi_{3,N(T)})$.

It suffices to consider Markov moments belonging to the set of *candidates*

$$\mathcal{T}_0 = \{\tau \in \mathcal{T} : \tau = n \Leftrightarrow X_n \ge \xi_{3,n}, n \in \mathbf{N}\}.$$

Now, let $\tau_k = k$ if $S_k \le T$, $\tau_k = +\infty$ if $S_k > T$, $k = 1,2,3$, and

$$\tau_{k+1} = \inf\{n : n > \tau_k, n \le N(T), X_n \ge \xi_{3,\tau_k}\}, \quad k > 3,$$

and let the *rv* R_n indicate the range of nth observation: $R_n = k$ if $X_n = \xi_{k,n}$, $k = 1,2,3$, and $R_n = 0$ otherwise. Define, for $k \in \mathbf{N}$,

$$(4) \qquad Y_k = \begin{cases} (S_{\tau_k}, \xi_{1,\tau_k}, \xi_{2,\tau_k}, \xi_{3,\tau_k}, R_{\tau_k}) & \text{if } \tau_k < +\infty, \\ \delta & \text{if } \tau_k = +\infty, \end{cases}$$

where δ is a label for the final state. $Y = (Y_k)_{k=1}^{\infty}$ is a homogeneous Markov chain with respect to $(\mathcal{F}_{\tau_k})_{k=1}^{\infty}$, with the state space

$$\mathbf{E} = [0, +\infty) \times \mathbf{A} \times \{1, 2, 3\} \cup \{\delta\}, \tag{5}$$

where $\mathbf{A} = \{(\xi_1, \xi_2, \xi_3) : 0 \leq \xi_3 \leq \xi_2 \leq \xi_1 \leq 1\}$. The transition function is

$$\begin{aligned}
(6) \qquad & p(s, a, b, c, i;\ [0, t], [0, x], a, b, 1) \\
= \ & P(S_{\tau_{k+1}} \leq t, \xi_{1,\tau_{k+1}} \leq x, \xi_{2,\tau_{k+1}} = a, \xi_{3,\tau_{k+1}} = b, R_{\tau_{k+1}} = 1 \\
& \quad | \ S_{\tau_k} = s, \xi_{1,\tau_k} = a, \xi_{2,\tau_k} = b, \xi_{3,\tau_k} = c, R_{\tau_k} = i) \\
= \ & \sum_{n=1}^{\infty} I_{\{\tau_k = n\}} \sum_{m=n+1}^{\infty} c^{m-n-1} (x - a)^+ \int_{(s,+\infty)} G^{*m-n}(t \wedge u - s) dH(u) / (1 - H(s)) \\
= \ & (x - a)^+ \sum_{k=1}^{\infty} c^{k-1} g_k(s, t),
\end{aligned}$$

where

$$g_k(s, t) = \frac{1}{1 - H(s)} \int_{(s,+\infty)} G^{*k}(t \wedge u - s) dH(u), \tag{7}$$

$y^+ = \max\{0, y\}$ and $t \wedge u = \min\{t, u\}$. Also

$$p(s, a, b, c, i;\ [0, t], a, [0, x], b, 2) = (x \wedge a - b)^+ \sum_{k=1}^{\infty} c^{k-1} g_k(s, t), \tag{8}$$

$$p(s, a, b, c, i;\ [0, t], a, b, [0, x], 3) = (x \wedge b - c)^+ \sum_{k=1}^{\infty} c^{k-1} g_k(s, t). \tag{9}$$

The state δ is absorbing and the transition function for other states can be obtained in a similar way.

If, for any $\tau \in \mathcal{T}_0$, a Markov moment σ with respect to $(\mathcal{F}_{\tau_k})_{k=1}^{\infty}$ is defined as $\sigma = k$ on the set $\{\tau = \tau_k < +\infty), k \in \mathbf{N}$, and $\sigma = +\infty$ on $\{\tau = +\infty\}$, then

$$Z_\tau = \left\{ \begin{array}{ll} W_{\tau_\sigma} & \text{if } \tau < +\infty \\ 0 & \text{if } \tau = +\infty \end{array} \right\} = f(Y_\sigma), \tag{10}$$

where $f(\delta) = 0$ ($Y_\infty = \delta$ by definition) and

$$(11)\ f(t, x, y, z, i) = \begin{cases} \sum_{k=0}^{\infty} q_k(t) \left(x^k + k x^{k-1}(1 - x) + \frac{k(k-1)}{2} x^{k-2} (1 - x)^2 \right), & i = 1, \\ \sum_{k=0}^{\infty} q_k(t) \left(y^k + k y^{k-1}(1 - y) \right), & i = 2, \\ \sum_{k=0}^{\infty} q_k(t) z^k, & i = 3. \end{cases}$$

In this way the initial Problem (P) is reduced to the problem of optimal stopping of the Markov chain Y given by (4) with the reward function f given by (11). The statement (a) is thus proved.

In order to solve an optimal stopping problem for the Markov chain Y with the reward function f, the function

$$s(t,x,y,z,i) = \sup_{\tau\in\mathcal{T}} E_{(t,x,y,z,i)} f(S_\tau, \xi_{1,\tau}, \xi_{2,\tau}, \xi_{3,\tau}, R_\tau)$$

should be calculated, where $E_{(t,x,y,z,i)}$ denotes the expectation with respect to $P_{(t,x,y,z,i)}(.) = p(t,x,y,z,i;.)$ and an optimal τ ought to be exhibited. It is known (cf. Shiryaev (1969)) that the function $s(t,x,y,z,i)$ can be obtained as the limit

$$s(t,x,y,z,i) = \lim_{k\to\infty} Q^k f(t,x,y,z,i),$$

where

$$Qf(t,x,y,z,i) = \max\{f(t,x,y,z,i), Pf(t,x,y,z,i)\} \tag{12}$$

and P is the operator defined for a bounded function $h : \mathbf{E} \to \mathbf{R}$ as

$$Ph(e) = \int_{\mathbf{E}} h(a) dP_e(a),$$

where the space $\mathbf{E}$ is given by (5). So $Ph(\delta) = 0$ and we infer from (6)–(9) that

$$\begin{aligned} Ph(t,x,y,z,i) &= \frac{1}{1-H(t)} \sum_{k=1}^{\infty} z^{k-1} \int_{(t,+\infty)} (1-H(r)) \Big(\int_z^y h(r,x,y,u,3)du \\ &+ \int_y^x h(r,x,u,y,2)du + \int_x^1 h(r,u,x,y,1)du \Big) dG^{*k}(r-t) \end{aligned} \tag{13}$$

for $i = 1,2,3$. It is well known that the Markov moment $\tau_0 = \inf\{n \in \mathbf{N} : Y_n \in \Delta\}$, where $\Delta = \{e \in \mathbf{E} : s(e) = f(e)\}$ (Δ is called the *stopping set*), is optimal if $\tau_0 < +\infty$ almost surely (*as*). The moment τ_0 is finite *as* because the chain Y attains *as* the state δ and $\delta \in \Delta$. Since by (11) and (13) both the reward function and $Pf(t,x,y,z,i)$ depend on H and G only, the stopping set depends only on H and G, as well. ∎

3. Special case. In order to solve Problem (P), successive iterations $Q^k f(t,x,y,z,i)$ of the operator Q given by (12) should be calculated. Due to the form of the operator P given by (13), it is very difficult to obtain the set Δ explicitly, even if the *df*'s G and H are fixed. Nevertheless in a natural case considered below the solution has a simple form.

Let G be the exponential *df* with the parameter λ. Thus $(N(t))_{t\in[0,+\infty)}$ is the Poisson process with the parameter λ. Moreover let the horizon T has the exponential *df* with the parameter μ.

Results for this case are summarized in the following theorem.

THEOREM 3.1. *Under the assumptions* (a)–(d), *let the* df*'s G and H be exponential with parameters λ or μ respectively.*

(a) If $\mu/(\lambda+\mu) < \alpha$, where $\alpha \cong 0.25997$ is a unique root of the equation $6\ln u + 3u^2 - 12u + 11 = 0$ in $(0,1)$, then the solution of Problem (P) is

$$\tau^* = \inf\left\{n : F(X_n) \geq \frac{\lambda+\mu-\alpha^{-1}\mu}{\lambda}\right\}.$$

The probability $P(win)$ that, using this stopping rule, at most the third largest X is obtained is equal to $P(win) = 1-(1-\alpha)^3 \cong 0.59473$.

(b) If $\mu/(\lambda+\mu) \geq \alpha$, then $\tau^* = 1$ is a solution of (P) and

$$P(win) = \frac{\mu}{\lambda+\mu}\left(-3\ln\frac{\mu}{\lambda+\mu} - \frac{1}{2}\left(\frac{\mu}{\lambda+\mu}\right)^2 + 3\frac{\mu}{\lambda+\mu} - \frac{5}{2}\right).$$

Proof. For exponential G it is easy to see that

$$dG^{*k}(r) = \frac{\lambda^k}{(k-1)!}r^{k-1}e^{-\lambda r}dr.$$

This implies that $q_k(t)$ for exponential H given by (3) has a simple form

$$q_k(t) \quad = \quad \frac{\mu}{\lambda+\mu}\left(\frac{\lambda}{\lambda+\mu}\right)^k,$$

and the reward function (11) is transformed to

$$(14) \qquad f(t,x,y,z,i) = \begin{cases} 1-\left(1-\frac{\mu}{\lambda+\mu-\lambda x}\right)^3, & i=1, \\ 2\frac{\mu}{\lambda+\mu-\lambda y} - \left(\frac{\mu}{\lambda+\mu-\lambda y}\right)^2, & i=2, \\ \frac{\mu}{\lambda+\mu-\lambda z}, & i=3. \end{cases}$$

Since all these functions are independent of t, Pf does not depend on t and

$$(15) \quad Pf(t,x,y,z,i) \quad = \quad \left(-\frac{1}{2}\left(\frac{\mu}{\lambda+\mu-\lambda x}\right)^2 + 2\frac{\mu}{\lambda+\mu-\lambda x} + \frac{\mu}{\lambda+\mu-\lambda y} - \frac{5}{2}\right.$$
$$\left. -\ln\left(\frac{\mu}{\lambda+\mu-\lambda x}\frac{\mu}{\lambda+\mu-\lambda y}\frac{\mu}{\lambda+\mu-\lambda z}\right)\right)\frac{\mu}{\lambda+\mu-\lambda z},$$

therefore so are $s(t,x,y,i)$ and Δ. Hence from now on t will be omitted in f, Pf and s. Since the operator P given by (13) for function h independent of t has the form

$$Ph(x,y,z,i) = \frac{\lambda}{\lambda+\mu-\lambda z}\left(\int_z^y h(x,y,u,3)du + \int_y^x h(x,u,y,2)du + \int_x^1 h(u,x,y,1)du\right),$$

we obtain

$$(16) \qquad Qf(x,y,z,i) = \begin{cases} f(x,y,z,i) & \text{if } (x,y,z) \in B^i, \\ Pf(x,y,z,i) & \text{if } (x,y,z) \in \mathbf{A} - B^i, \end{cases}$$

for some sets $B^3, B^2, B^1 \subseteq \mathbf{A}$, where $B^3 \subseteq B^2 \subseteq B^1$ because $Ph(x,y,z,i)$ does not depend on i and $f(x,y,z,1) \geq f(x,y,z,2) \geq f(x,y,z,3)$. The functions (14), (15) can be written as functions of

$$u = \frac{\mu}{\lambda+\mu-\lambda x}, \quad v = \frac{\mu}{\lambda+\mu-\lambda y}, \quad w = \frac{\mu}{\lambda+\mu-\lambda z},$$

in the following way

$$(17) \qquad f(u,v,w,i) = \begin{cases} 1-(1-u)^3, & i=1, \\ 1-(1-v)^2, & i=2, \\ w, & i=3, \end{cases}$$

$$(18) \qquad Pf(u,v,w,i) = w\left(-\frac{1}{2}u^2+2u+v-\frac{5}{2}-\ln(uvw)\right), \; i=1,2,3$$

and the set $\mathbf{A}$ is transformed into $\{(u,v,w) : \mu/(\lambda+\mu) \leq w \leq v \leq u \leq 1\}$.

The set B^i is defined by the inequality $f(x,y,z,i) \geq Pf(x,y,z,i)$ or, equivalently, by $f(u,v,w,i) \geq Pf(u,v,w,i)$ with conditions $\mu/(\lambda+\mu) \leq w \leq v \leq u \leq 1$ This inequality, considered in a set $0 \leq w \leq v \leq u \leq 1$, is fulfilled in the set $C_i = \{(u,v,w) : f(u,v,w,i) \geq Pf(u,v,w,i)\}$.

Since the functions (17), (18) are independent of $\mu/(\lambda+\mu)$, the sets B^i in the coordinates (u,v,w) are $B^i = C_i \cap \{(u,v,w) : \mu/(\lambda+\mu) \leq w \leq v \leq u \leq 1\}$, and in the coordinates (x,y,z) their forms are similar, because the above transformation preserves monotonicity.

In order to obtain $s(x,y,z,i)$ successive iterations of $Qf(x,y,z,i)$ should be calculated. As a consequence of induction, $\Delta_3 = \{(x,y,z) : s(x,y,z,3) = f(x,y,z,3)\} = B^3$ and there exist sequences $(B^i_n)_{n=1}^{\infty}$, $i=1,2$, such that

$$(19) \quad Q^n f(x,y,z,i) = \begin{cases} f(x,y,z,i) & \text{if } (x,y,z) \in B^i_n, \\ PQ^{n-1}f(x,y,z,i) > f(x,y,z,i) & \text{if } (x,y,z) \in \mathbf{A} - B^i_n, \end{cases}$$

where $B^3 \subseteq B^i_n \subseteq B^i_{n-1}$, $n \geq 2$, $B^i_1 = B^i$. Therefore the limit $\Delta_i = \lim_{n\to\infty} B^i_n$ exists and the stopping set has the form

$$\Delta = (0,+\infty) \times (\Delta_1 \times \{1\} \cup \Delta_2 \times \{2\} \cup \Delta_3 \times \{3\}) \cup \{\delta\}.$$

We shall find Δ_1 and Δ_2 in the explicit form. To this end denote by $h(x,y,z)$ the probability that the stopping set will be reached from the state (x,y,z,i). The function $h(x,y,z)$ does not depend on range of current observation being a candidate. For $(x,y,z) \in \mathbf{A}-B^3$, if the next candidate of value r satisfies the inequality $z < r \leq y$, it does not attain the set B^3 for $z < r < z_0 = \min\{y, z(x,y)\}$, where the function $z(x,y)$ describes the boundary of B^3. If $r > y$ then the new state $(x,r,y,2)$ or $(r,x,y,1)$ does not depend on z. Thus

$$h(x,y,z) = \frac{\lambda}{\lambda+\mu-\lambda z}\left(\int_z^{z_0} h(x,y,r)dr + g(x,y)\right),$$

where $g(x,y)$ is some function independent of z. Since the partial derivative of h with respect to z is equal to 0, $h(x,y,z)$ is independent of z. Since $f(x,y,z,1)$ and $f(x,y,z,2)$ are also independent of z, $(x,y,z) \in \Delta$ implies $(x,y,r) \in \Delta$ for every $0 \leq r \leq y$ and $(x,r,y) \in \Delta$ for every $y \leq r \leq x$. Similarly $h(x,y,z)$ is independent of y and x as well. Hence $h(x,y,z) = \text{const}$ and the sets Δ_1 and Δ_2 are of the form $\{(x,y,z) \in \mathbf{A} : x \geq L\}$.

As the result of the above properties, the optimal strategy allows us to stop observation only in such a moment, when an observed candidate is the largest one so far and exceeds some level L (independent of the second largest and third largest so far). So the optimal strategy can be obtained by maximization of the probability of reaching of Δ_1.

Let the event that the objective is achieved be called a *win*. Then

$$\begin{aligned} & P(\text{stop at the moment } k \text{ \& win } \mid N(T)=n) \\ = \; & P(X_1 \leq L, \ldots, X_{k-1} \leq L, X_k > L, \text{ at most two } X\text{'s} \\ & \text{from } X_{k+1}, \ldots, X_n \text{ are greater than } X_k \mid N(T)=n) \\ = \; & L^{k-1}\int_L^1 \left(x_k^{n-k} + (n-k)x_k^{n-k-1}(1-x_k) + \frac{(n-k)(n-k-1)}{2}x_k^{n-k-2}(1-x_k)^2\right)dx_k \end{aligned}$$

for $k = 1, \ldots, n$, and

$$\begin{aligned} P(win) &= \sum_{n=1}^{\infty} \frac{\mu}{\lambda+\mu}\left(\frac{\lambda}{\lambda+\mu}\right)^n \sum_{k=1}^{n} P(\text{stop at } k \text{ \& win } \mid N(T)=n) \\ &= \frac{\mu}{\lambda+\mu-\lambda L}\left(-\frac{1}{2}\left(\frac{\mu}{\lambda+\mu-\lambda L}\right)^2 + 3\frac{\mu}{\lambda+\mu-\lambda L} - \frac{5}{2} - 3\ln\frac{\mu}{\lambda+\mu-\lambda L}\right). \end{aligned}$$

Let $\mu/(\lambda+\mu-\lambda L) = u$. The function $f(u) = u(-u^2/2 + 3u - 5/2 - 3\ln u)$ associated with $P(win)$, has a unique local extremum (maximum) in $(0,1)$ at the point $\alpha \cong 0.25997$ for which the derivative $f'(u) = -3u^2/2 + 6u - 5/2 - 3 - 3\ln u$ is equal to zero. So, since $u \in [\mu/(\lambda+\mu), 1]$, $f(u)$ attains its maximum at α if $\mu/(\lambda+\mu) < \alpha$ (equivalently if $f'(\mu/(\lambda+\mu)) > 0$) or at $\mu/(\lambda+\mu)$ if $\mu/(\lambda+\mu) \geq \alpha$. Thus

$$L^* = \frac{\lambda+\mu-\alpha^{-1}\mu}{\lambda}, \quad P(win) = f(\alpha) = 1-(1-\alpha)^3 \cong 0.59473$$

for $\mu/(\lambda+\mu) < \alpha$ while $L^* = 0$ and $P(win) = f(\mu/(\lambda+\mu))$ for $\mu/(\lambda+\mu) \geq \alpha$. ∎

In the special case when both the time intervals between observations ρ and the curtailment time T are exponentially distributed, it is interesting and quite unexpected that the probability of winning in all *natural* situations (i.e. when $\mu/(\lambda+\mu)$ is *small*, because T should not be, on the average, small in comparison with ρ) is constant. The optimal strategy does not depend on the number of preceding observations and the time that elapsed. In accordance with the optimal rule, given in Theorem 3.1, the observation should be stopped at the moment when the first *candidate* occurs which exceeds some constant barrier. The same property is possessed by the optimal strategy in selecting the largest when observations appear according to a Poisson process and T is exponential (Bojdecki (1978)) and when the number of observations N is geometric (Porosiński (1987)). This interesting fact seems to be a consequence of the *memoryless* property of the geometric and exponential *df*'s.

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