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Interacting Brownian Particles in Multi-Dimensions

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1. Introduction

A system of Brownian particles interacting through 2-body-potential drift terms is one of the simplest natural models that are thought to exhibit a mathematical structure of how a macroscopic evolution equation comes out from microscopic dynamics of a statistically large system regulated by conservation laws. While the model is interesting as a physical system of small particles suspended in a fluid (see [4: Part II] for physical interpretation as well as recent developments of the subject), apart from such interests in the model its hydrodynamic scaling limit has long been studied from Mathematical point of view, through which it has been well understood that under the hypothesis of the local equilibrium a non-linear diffusion equation for the limiting density of a suitably scaled distribution of particles must be derived and its diffusion coefficient be determined as a function of the density which reflects the microscopic structure of the interactions (cf., eg., [2]). The argument for the derivation is convincing but had been lacking in any mathematical vindication until Varadhan [6] gave a rigorous derivation in one-dimensional case of smooth repulsive potential. In this talk we primarily consider multidimensional models, for which the method of [6] does not (at least directly) apply: there arises a serious difficulty. Roughly speaking we can modify the method under a certain uniform bound of the space-time average of the p-th moment of a scaled empirical density, $E\int_0^T dt \int[\rho^N(\theta,t)]^p d\theta$, for some $p>3$, or something like it, but such a bound, though convincingly plausible, is difficult to verify.

2. The model and the results

Let $\mathbf{T}^d$ be the d-dimensional unit torus represented by the hyper-cube $[0,1)^d$, and $(x_1(t),...,x_N(t))$ a system of interacting Brownian particles evolving on $\mathbf{T}^d$ according to the following system of stochastic differential equations:

$$dx_i(t) = -\frac{1}{\epsilon}\sum_{j\neq i}\nabla U\left(\frac{x_i(t)-x_j(t)}{\epsilon}\right)dt + dB_i(t), \quad i=1,2,...,N.$$

Here ϵ is a small positive parameter (representing the size of the particles in a macroscopic scale), $B_1, B_2, ...$ are independent standard Brownian motions moving on $\mathbf{T}^d$ defined on some probability space $(\Omega,\mathcal{F},\mathbf{P})$; $U(x)$ is a radial function on $\mathbf{R}^d$, namely it is given in the form $U(x)=V(|x|)$. We shall suppose that V is a twice continuously differentiable function of $r>0$ that is non-increasing in a neighborhood of 0 and satisfies the following conditions:

(i) for some constant $c_o > 0$, $V(r) = 0$ for $r > c_o$;

(ii) either (a) $V \geq 0$ or (b) $\int_0^1 V(r) r^{d-1} dr = \infty$;

(iii) if V is bounded, then it can be extended to a twice continuously differentiable function of $r \geq 0$ and $V(0) := \lim_{r\to 0} V(r) > 0$; if V is unbounded, then

$$\limsup_{r\to 0} \Big[r^2 |V'(r)|^2 + r^2 |V''(r)| \Big] e^{-V(r)/2} < \infty.$$

The process $\mathbf{x}_t^N := (x_1(t), ..., x_N(t))$ is a diffusion process on $(\mathbf{T}^d)^N$ whose infinitesimal generator is given by

$$L_N = \frac{1}{2}\Delta^{(N)} - \frac{1}{\epsilon}\sum_{i=1}^{N}\sum_{j\neq i} \nabla U\left(\frac{x_i - x_j}{\epsilon}\right)\frac{\partial}{\partial x_i},$$

where $\Delta^{(N)}$ denotes the Laplace operator on $(\mathbf{T}^d)^N$ and $\partial/\partial x_i$ the gradient operator with respect to $x_i \in \mathbf{T}^d$. The process $\mathbf{x}_t^N$ is ergodic. The invariant probability law is given by

$$\nu_N(d\mathbf{x}) = \frac{1}{Z_N}\exp\Big[- \sum_{i,j(\neq)} U\left(\frac{x_i - x_j}{\epsilon}\right)\Big] dx_1 \cdots dx_N,$$

relative to which L_N is symmetric. Let ξ_t^N be the empirical distribution of the particles $x_1(t), ..., x_N(t)$, namely ξ_t^N is the counting measure on $\mathbf{T}^d$ defined by

$$\int_{\mathbf{T}^d} J(x)\xi_t^N(dx) = \epsilon^d \sum_{i=1}^{N} J(x_i(t)), \quad J \in C^\infty(\mathbf{T}^d).$$

Our main concern here is to determine the limit of ξ_t^N as $N \to \infty$ and $\epsilon \downarrow 0$ in such a way that the average density $N\epsilon^d$ remains (asymptotically) constant. The limit measure is expected to have a density (relative to the Lebesgue measure on $\mathbf{T}^d$) which solves the non-linear diffusion equation

$$\frac{\partial}{\partial t}u(\theta,t) = \frac{1}{2}\Delta P(u(\theta,t)), \qquad (\theta,t) \in \mathbf{T}^d \times (0,\infty), \tag{2.1}$$

where Δ is the Laplace operator on $\mathbf{T}^d$ and the function $P(u), u \geq 0$, is the pressure at density u in the Gibbs formulation of thermodynamics on $\mathbf{R}^d$ associated with the pair potential function $U(x)$. It may be defined as follows.

For a d-tuple $\ell = (\ell_1, ..., \ell_d)$ with positive entries $\ell_i > 0$, let $\Lambda(\ell)$ denote a hyper-interval (d-dimensional interval) $[-\ell_1, \ell_1] \times \cdots \times [-\ell_d, \ell_d]$. The canonical partition function for n particles in $\Lambda(\ell)$ with the empty-boundary condition is defined by

$$Z_{\ell,0}^0 = 1 \quad \text{and} \quad Z_{\ell,n}^0 = \int_{[\Lambda(\ell)]^n} \exp\Big\{ - \sum_{i,j(\neq)} U(q_i - q_j)\Big\}\frac{dq_1\cdots dq_n}{n!} \quad \text{for} \quad n \geq 1$$

where $\mathbf{q} = (q_1, ..., q_n) \in [\Lambda(\ell)]^n$ is an n-particle configuration. Let $\min\{\ell_1, ..., \ell_d\} \to \infty$

and $n \to \infty$ in such a way that $n/|\Lambda(\ell)| \to \rho$ $(\rho \geq 0)$. Then there exists a limit

(2.2)
$$\Phi(\rho) := \lim \frac{-1}{|\Lambda(\ell)|} \log Z^0_{\ell,n};$$

$\Phi(\rho)$, called Helmholtz' free energy, is convex and $\Phi(0+) = 0$. The pressure (or Gibbs' free energy) as a function of chemical potential λ is given by $F(\lambda) = \sup_{\rho \in \mathbf{R}}[\lambda\rho - \Phi(\rho)]$. The function Φ is differentiable and $\Phi(\rho)\rho^{-1} \to \infty$ as $\rho \to \infty$; the derivative $\Phi'(\rho)$ is necessarily non-decreasing and continuous; hence $F(\lambda)$ may be regarded as a continuous function of $\rho \geq 0$, which defines our pressure $P(\rho)$, or, what amounts to the same thing,

(2.3)
$$P(\rho) = \Phi'(\rho)\rho - \Phi(\rho).$$

For the derivation of (2.1) it is crucial that $P(\rho) - \rho$ can be represented as a limit of averages of $-(q_i - q_j) \cdot \nabla U(q_i - q_j)/d$ over configurations $\mathbf{q} = (q_i)$ in the box $\Lambda(\ell)$ distributed according to a canonical Gibbs measure $\mu^{\omega}_{\ell,n}(d\mathbf{q})$ of n particles with a boundary configuration ω. In multi-dimensions there may occur some kind of phase transitions: we know of the validity neither of equivalence of ensembles nor of uniqueness of grand-canonical Gibbs measures, which however does not cause essential difficulty for the verification of the representation. In fact we obtain

Theorem 2.1. *For each $r > 0$, $\eta > 1$ and each pair of indices $1 \leq \alpha, \beta \leq d$*

$$\sup_{\sharp\omega \leq \exp(\ell_*)} \sup_{\rho \leq r} \int \left| \frac{1}{|\Lambda(\ell)|} \sum_{q_i, q_j \in \Lambda(\ell)} \psi_{\alpha\beta}(q_i - q_j) - [P(\rho_{\eta\ell}) - \rho_{\eta\ell}]\delta_{\alpha\beta} \right| \mu^{\omega}_{\eta\ell,n}(d\mathbf{q}) \longrightarrow 0$$

as $\ell_* := \min\{\ell_1, ..., \ell_d\} \to \infty$, where $\rho_{\eta\ell} = n/|\Lambda(\eta\ell)|$, $\psi_{\alpha\beta}(z) = -z_\beta \nabla_\alpha U(z)$.

If $d = 1$ and V is bounded, the assertion of Theorem 2.1 is proved in [6] based on the uniqueness for grand canonical Gibbs measures.

We return to the problem of the empirical measure ξ^N_t. We suppose that the diffusion process $\mathbf{x}^N_t$ starts from an initial law which has a density, denoted by f^N_0, relative to ν_N. The evolution of the process may be analytically characterized by the forward equation

$$\frac{\partial f^N_t}{\partial t} = L_N f^N_t,$$

where f^N_t is the density relative to ν_N of the law of $\mathbf{x}^N_t$. On the family of initial densities $\{f^N_0\}$ we impose the following growth condition of their entropies

(2.4)
$$\int f^N_0 \log f^N_0 \, d\nu_N = o(N^{1+2/d}) \qquad \text{as } N \to \infty.$$

We will regard ξ^N_t as a stochastic process taking values in the space of all finite measures $\mathcal{M}(\mathbf{T}^d)$, which is viewed as a metric space whose topology agrees with that of weak convergence of finite measures.

For any $T > 0$ and a non-random element $u_o \in \mathcal{M}(\mathbf{T}^d)$ we shall concern weak solutions of (2.2) on the time interval $(0, T)$ that satisfy the initial condition

$$u(\theta, t)d\theta \longrightarrow u_o(d\theta) \quad \text{as} \quad t \to 0 \tag{2.5}$$

as well as the integrability condition

$$\int_0^T dt \int_{\mathbf{T}^d} P(u(\theta, t))d\theta < \infty. \tag{2.6}$$

It is known [5] that such a solution if any is unique if $d = 1$; in the case $d \geq 2$ it is unique at least if u_o is absolutely continuous and its density is square integrable.

We put

$$\psi(r) = -rV'(r) \quad r > 0.$$

In one-dimension the method of [6] may be adapted to the case of unbounded V to deduce from Theorem 2.1 the next theorem.

Theorem 2.2. *Let $d = 1$. Suppose, in addition to (i) to (iii), that either $\psi(r) \geq 0$ for all $r > 0$; or $\int_0^1 V(r)dr = \infty$ and $\int_0^1[\psi'(r) \vee 0]dr < \infty$. (Here $a \vee b = \max\{a, b\}$.) Also suppose that (2.4) is satisfied and ξ_0^N converges in probability to a non-random element $u_o \in \mathcal{M}(\mathbf{T}^1)$. Then the random trajectory $\xi_t^N(d\theta)$, $t \geq 0$, converges in probability to a single trajectory $u(\theta, t)d\theta$, $t \geq 0$ in the topology of locally uniform convergence of continuous trajectories in $\mathcal{M}(\mathbf{T}^1)$ and the limit function $u(\theta, t)$ is a (unique) solution of the non-linear diffusion equation (2.2) satisfying (2.5) and (2.6).*

We derive a corresponding result in multi-dimensions under a hypothetical postulate. Let h be a smooth non-negative function having a compact support such that

$$\int h(\theta)d\theta = 1 \quad \text{and} \quad h(0) > 0;$$

put

$$\rho(\theta) = \rho(\theta, \mathbf{x}) = \sum_i h\left(\frac{x_i - \theta}{\epsilon}\right)$$

and

$$S(\theta) = S(\theta, \mathbf{x}) = \sum_i \sum_{j \neq i} |\psi| \left(\frac{|x_i - x_j|}{\epsilon}\right) h\left(\frac{x_i - \theta}{\epsilon}\right)$$

where $|\psi|(r) = |\psi(r)|$. Our third result, Theorem 3, reduces the problem to the following condition:

$$\sup_N \int_0^T dt \int_{\mathbf{T}^d} \mathbf{E}\left[(S(\theta, \mathbf{x}_t^N))^p + (\rho(\theta, \mathbf{x}_t^N))^3\right] d\theta < \infty \quad \text{for some } p > 3/2. \tag{H}$$

It may be noticed that if $\psi(0+) < \infty$, then $\rho(\theta) \geq C^{-1}\sqrt{S(A\theta)}$ for some positive constants A and C; if $\psi(0+) > 0$, then $\rho(\theta) \leq C\sqrt{S(\theta) + 1}$.

Theorem 2.3. *Suppose that (H) holds, (2.4) is satisfied and ξ_0^N converges in probability to a non-random element $u_o \in \mathcal{M}(\mathbf{T}^d)$ and the sequence of initial configurations $\mathbf{x}_0^N$ satisfies*

$$\lim_{M\to\infty} \sup_N \mathbf{P}\left[\int_{\mathbf{T}^d} [\rho(\theta, \mathbf{x}_0^N)]^2 d\theta > M\right] = 0. \tag{2.7}$$

Then the same conclusion as in Theorem 2 holds with obvious modification.

The conditions (H) measures the degree of non-concentration of particles in average: they would be violated only if excessively many particles accumulate in a small region. It should be verified for non-trivial initial conditions, but the present author do not know how to prove it whether V is bounded or not. If we start the process with the invariant measure ν_N, it holds that for all $p > 0$

$$\sup_N \mathbf{E}\left[\int_0^T dt \int_{\mathbf{T}^d} \Big([S(\theta, \mathbf{x}_t^N)]^p + [\rho(\theta, \mathbf{x}_t^N)]^{2p}\Big) d\theta\right] < \infty, \tag{2.8}$$

which with $p > 3/2$ of course implies (H). (2.8) is also valid for independent Brownian motions starting from initial distributions subject to a certain mild condition if ψ is replaced by any function φ such that $\int |\varphi|^p dx < \infty$. The validity of (H) is plausible for a wide class of initial distributions since the evolution law governed by L_N does not seem to develop accumulation of particles, our potential being essentially repulsive so that it must exercise a dispersing effect on the particle configurations, although we do not know of any effective argument that approves such plausibility.

3. The representation of the pressure $P(\rho)$ by means of virial

In this section the proof of Theorem 2.1 is outlined in the case when $V \ge 0$, which will be assumed in the rest of this talk. (Its proof in the other case is somewhat involved.) We introduce some notations. Let c_o be the smallest positive constant such that $V(r) = 0$ if $r \ge c_o$. Given $\ell = (\ell_1, ..., \ell_d)$, $\ell_i > c_o$, we take a configuration, $\omega = (\omega_k)$ say, on the outer shell $\Lambda(\ell + \mathbf{c}_o) \setminus \Lambda(\ell)$, where $\mathbf{c}_o = (c_o, ..., c_o)$ and $\ell + \mathbf{c}_o = (\ell_1 + c_o, ..., \ell_d + c_o)$. Put

$$\mathcal{U}(\mathbf{q}) = \sum_{i,j(\neq)} U(q_i - q_j), \quad \mathcal{W}(\mathbf{q}|\omega) = 2\sum_{i,j} U(q_i - \omega_j),$$

$$\mathcal{H}^\omega(\mathbf{q}) = \mathcal{U}(\mathbf{q}) + \mathcal{W}(\mathbf{q}|\omega),$$

$$Z_n(\ell, \omega) = \int_{[\Lambda(\ell)]^n} \exp\{-\mathcal{H}^\omega(\mathbf{q})\} \frac{d\mathbf{q}}{n!} \quad \text{if} \quad n \ge 1, \quad \text{and} \quad Z_0(\ell, \omega) = 0.$$

The canonical Gibbs measure $\mu_{\ell,n}^\omega$ is then a measure on $[\Lambda(\ell)]^n$, the n-fold Cartesian product of $\Lambda(\ell)$, given by

$$\mu_{\ell,n}^\omega(d\mathbf{q}) = \frac{1}{Z_n(\ell, \omega)} \exp\{-\mathcal{H}^\omega(\mathbf{q})\} \frac{d\mathbf{q}}{n!}.$$

Throughout this section we shall write

$$\rho_\ell = n/|\Lambda(\ell)|.$$

We state several lemmas without proof.

Lemma 3.1. *Let* $\delta(\rho) = 2^{-1}(2v_d\rho)^{-1/d}$ *and*

$$C(\rho) = 2^{d+1}\rho \exp\left\{2^{d+2}\rho \int_{|x|>\delta(\rho)} U_+(x)dx\right\},$$

where $a_+ = \max\{a, 0\}$ *and* v_d *stands for the volume of d-dimensional unit ball. Then for* $n > 1$ *and* $\ell_* > 2c_o$,

$$\frac{Z_{n-1}(\ell,\omega)}{Z_n(\ell,\omega)} \leq C(\rho_\ell). \tag{3.1}$$

Lemma 3.2. *Let* c *be a positive constant such that* $\ell_i \geq c + c_o, i = 1, ..., d$, *and define*

$$m_i = \lfloor (2\ell_i - c_o)/(2c + c_o) \rfloor \quad \text{and} \quad m = m_1 \cdots m_d.$$

Then, $Z_n(\ell,\omega) \geq (2c)^{dn}$ *for* $n < m$; *and*

$$Z_n(\ell,\omega) \geq \left[\min\{Z^0_{\mathbf{c},\lfloor n/m \rfloor}, Z^0_{\mathbf{c},\lfloor n/m \rfloor + 1}\}\right]^m \quad \text{for} \quad n \geq m. \tag{3.2}$$

Here $\mathbf{c} = (c, ..., c)$ *and* $\lfloor a \rfloor$ *denotes the integral part of* a.

Let $\mathcal{N}_K = \mathcal{N}_K(\mathbf{q})$ denote the number of points q_i contained in a set K. Put

$$\mathcal{H}_K(\mathbf{q}) = \sum_{i:q_i \in K} \sum_{j \neq i} U(q_i - q_j).$$

Lemma 3.3. *There exist positive constants* A *and* B *(depending only on* U*) such that if a Borel set* K *is covered by* m *hyper-cubes of edge length* c_o *and* $\ell_* > 2c_o$, *then for* $0 \leq \gamma \leq 1$ *and for every positive integer* k,

$$\int_{\{\mathcal{N}_K = k\}} \exp\{\gamma \mathcal{H}_K(\mathbf{q})\}\mu^\omega_{\ell,n}(d\mathbf{q}) \leq \frac{(C(\rho_\ell)|K|e^{(1-\gamma)A})^k}{k!} \exp\{-(1-\gamma)m^{-1}Bk^2\},$$

where $C(\rho)$ *is the same as in Lemma 3.1 and* $|K|$ *denotes the volume of* K.

Lemma 3.4. *If a Borel set* K *is covered by* m *hyper-cubes of edge length* c_o *and* $\ell_* > 2c_o$, *then*

$$\mu^\omega_{\ell,n}(\mathcal{N}_K = k) \leq \frac{(C(\rho_\ell)|K|e^A)^k}{k!} \exp\{-m^{-1}Bk^2\}.$$

Lemma 3.5. *For* $p \geq 1$ *there exists a continuous function* $M_p(\rho)$ *of* $\rho \geq 0$ *depending* *only on* U *and* p *such that if a function* $\chi(q), q \in \mathbf{R}^d \setminus \{0\}$, *satisfies*

$$\chi(q) \leq AI(|q| < c_o)e^{U(q)/p} \tag{3.3}$$

for some positive constant A, *then for any hyper-interval* K *with all its sides* $\geq c_o$

$$\int_{[\Lambda(\ell)]^n} \Big[\sum_{i:q_i \in K} \sum_j \chi(q_i - q_j)\Big]^p \mu^{\omega}_{\ell,n}(d\mathbf{q}) \leq |K|^p A^p M_p(\rho_\ell).$$

Lemma 3.6. *There exists a continuous function* $C_1(\rho)$ *on* $[0,\infty)$ *such that for every point* ω_k *from the outer configuration* ω

$$\frac{Z_n(\ell, \omega \setminus \{\omega_k\})}{Z_n(\ell,\omega)} \leq C_1(\rho_\ell).$$

Lemma 3.7. *There exists a continuous function* $C_2(\rho)$ *of* $\rho \geq 0$ *depending only on* U *such that if* $\chi(q) \leq I(|q| < c_o)e^{U(q)}$, *then*

$$\int_{[\Lambda(\ell)]^n} \sum_{i=1}^{n} \chi(q_i - \omega_k)\mu^{\omega}_{\ell,n}(d\mathbf{q}) \leq C_2(\rho_\ell) \quad (k = 1, 2, \ldots). \tag{3.4}$$

Lemma 3.8. *If* $n, \ell_* \to \infty$ *so that* $n/|\Lambda(\ell)| \to \bar{\rho}$, *then*

$$-\frac{1}{|\Lambda(\ell)|} \log Z_n(\ell, \omega) \longrightarrow \Phi(\bar{\rho})$$

uniformly with respect to ω *and* $\bar{\rho} \leq r$, *where* r *may be an arbitrary positive constant.*

Lemma 3.9. *For each triplet of numbers* $r > 0$, $0 < \delta < 1$ *and* $\alpha > 0$ *there exist positive constants* η *and* L *such that if* $\rho_\ell \leq r$, $\ell_* > L$ *and* K *is a hyper-interval included in* $\Lambda(\ell)$ *and if* $|K| \geq \delta|\Lambda(\ell)|$, *then*

$$\mu^{\omega}_{\ell,n}\left(\Big|P(\rho_K) - P(\rho_\ell)\Big| > \alpha\right) < e^{-\eta|\Lambda(\ell)|}, \quad \textit{where} \quad \rho_K = |K|^{-1}\mathcal{N}_K.$$

Define

$$\psi(r) = -rV'(r),$$
$$\psi_{\alpha\beta}(z) = -z_\beta \nabla_\alpha U(z) = \frac{z_\alpha z_\beta}{|z|^2}\psi(|z|),$$
$$\Psi^\ell_{\alpha\beta} = \Psi^\ell_{\alpha\beta}(\mathbf{q}) = \sum_{i,j(\neq):q_i,q_j \in \Lambda(\ell)} \psi_{\alpha\beta}(q_i - q_j).$$

Using Lemmas 3.1 through 3.9 we now prove the following theorem, which obviously implies Theorem 2.1.

Theorem 3.1. *For each* $r > 0$ *and each pair of indices* $1 \leq \alpha, \beta \leq d$

$$\lim_{\ell_* \to \infty} \sup_{\omega} \sup_{n \leq r|\Lambda(\ell)|} \int \left|\frac{1}{|\Lambda(\ell)|}\Psi^\ell_{\alpha\beta} - [P(\rho_\ell) - \rho_\ell]\delta_{\alpha\beta}\right|^2 \mu^{\omega}_{\ell,n}(d\mathbf{q}) = 0,$$

where $\rho_\ell = n/|\Lambda(\ell)|$ and the first sup is taken over all configurations ω in $\Lambda(\ell+\mathbf{c}_o)\backslash\Lambda(\ell)$.

Let $\delta > 0$ and ω and ℓ be as above. For $1 \le s \le 1+\delta$ put

$$\omega^s = (\omega_k^s : k = 1, ..., m) \in \mathbf{R}^{d\times m}, \quad \omega_k^s = (s\omega_{k,1}, \omega_{k,2}, ..., \omega_{k,d}) \in \mathbf{R}^d,$$

$$\ell^s = (s\ell_1, \ell_2, ..., \ell_d)$$

in analogy of the proof given in [6] to the corresponding result in one-dimension. Then

$$\log Z_n(\ell^{1+\delta}, \omega^{1+\delta}) - \log Z_n(\ell, \omega) = \int_1^{1+\delta} \frac{d}{ds} \log Z_n(\ell^s, \omega^s) ds. \tag{3.5}$$

Changing variables of integration from q_i to $q_i^s = (sq_{i,1}, q_{i,2}, ..., q_{i,d})$, we have

$$Z_n(\ell^s, \omega^s) = s^n Z_{n,\ell,\omega}(s),$$

where

$$Z_{n,\ell,\omega}(s) = \int_{[\Lambda(\ell)]^n} \exp\{-\mathcal{H}^{\omega^s}(\mathbf{q}^s)\} \frac{d\mathbf{q}}{n!}.$$

Hence

$$\frac{d}{ds} \log Z_n(\ell^s, \omega^s) = \frac{n}{s} + \frac{Z'_{n,\ell,\omega}(s)}{Z_{n,\ell,\omega}(s)}. \tag{3.6}$$

Here $Z'_{n,\ell,\omega}(s)$ denotes the derivative relative to s, and is given by

$$Z'_{n,\ell,\omega}(s) = \int_{[\Lambda(\ell)]^n} \frac{1}{s} \left[\sum \psi_{11}(q_i^s - q_j^s) + 2\sum \psi_{11}(q_i^s - \omega_k^s)\right] \exp\{-\mathcal{H}^{\omega^s}(\mathbf{q}^s)\} \frac{d\mathbf{q}}{n!}.$$

(Recall $\psi_{11}(z) = -z_1 \nabla_1 U(z) = -(z_1^2/|z|)V'(|z|)$, $z \in \mathbf{R}^d$.) [At this point it may be worth making a comment. We are going to identify the limit of $|\Lambda|^{-1}\int \sum \psi_{11}(q_i - q_j) d\mu_{\ell,n}^\omega$ (as $\rho_\ell \to \bar\rho$) with $P(\bar\rho) - \bar\rho$. Changing the variable back in the integral above we get

$$\frac{Z'_{n,\ell,\omega}(s)}{Z_{n,\ell,\omega}(s)} = \frac{1}{s} \int_{[\Lambda(\ell^s)]^n} \left[\sum \psi_{11}(q_i - q_j) + 2\sum \psi_{11}(q_i - \omega_k^s)\right] \mu_{\ell^s,n}^{\omega^s}(d\mathbf{q}).$$

The identification is easily deduced from this relation together with (3.5,6) if $d = 1$ since then we have the uniqueness for grand canonical Gibbs measures (accordingly the nice ergodicity), while, with such uniqueness unabailable, we need grope some suitable device if $d > 1$.] One calculates the derivative of $s^{-1}\psi_{11}(z^s)$ relative to s to see that if

$$\chi_{11}(z) = \left[1 - \left(\frac{z_1}{|z|}\right)^2\right] \left(\frac{z_1}{|z|}\right)^2 |z| V'(|z|) - \left(\frac{z_1}{|z|}\right)^4 |z|^2 V''(|z|),$$

then

$$\frac{d}{ds}\left(\frac{1}{s}\psi_{11}(z^s)\right) = \frac{1}{s^2}\chi_{11}(z^s).$$

An elementary calculus and changing the variable back show

(3.7)

$$
\begin{aligned}
&\frac{d}{ds}\frac{Z'_{n,\ell,\omega}(s)}{Z_{n,\ell,\omega}(s)} \\
&= \frac{1}{s^2}\text{the variance of}\left\{\sum_{i,j(\neq)}\psi_{11}(q_i-q_j)+2\sum_{i,k}\psi_{11}(q_i-\omega_k^s)\right\} \\
&\qquad \text{with respect to } \ \mu_{\ell^s,n}^{\omega^s}(d\mathbf{q}) \\
&+\frac{1}{s^2}\int_{[\Lambda(\ell^s)]^n}\left\{\sum_{i,j(\neq)}\chi_{11}(q_i-q_j)+2\sum_{i,k}\chi_{11}(q_i-\omega_k^s)\right\}\mu_{\ell^s,n}^{\omega^s}(d\mathbf{q}).
\end{aligned}
$$

By the hypothesis (iii) on U we can choose a positive number A so that $|\chi_{11}(q)| \le AI(|q| < c_o)e^{U(q)}$ for every $q \in \mathbf{R}^d$. Applying Lemmas 3.5 and 3.7 we therefore obtain

$$
\begin{aligned}
&\int_{[\Lambda(\ell^s)]^n}\left\{\sum_{i,j(\neq)}|\chi_{11}(q_i-q_j)|+2\sum_{i,k}|\chi_{11}(q_i-\omega_k^s)|\right\}\mu_{\ell^s,n}^{\omega^s}(d\mathbf{q}) \\
&\le AM_1(\rho_\ell)|\Lambda(\ell)|+2AC_2(\rho_\ell)(\sharp\omega),
\end{aligned} \tag{3.8}
$$

where $\sharp\omega$ stands for the number of ω_k's (presupposed to be contained in $\Lambda(\ell+\mathbf{c}_o)\backslash\Lambda(\ell)$). Finally, by (3.5) through (3.8),

(3.9)

$$
\begin{aligned}
&\log Z_n(\ell^{1+\delta},\omega^{1+\delta})-\log Z_n(\ell,\omega) \\
&= n\log(1+\delta)+\delta f(1)+\int_1^{1+\delta}ds\int_1^s f'(t)dt \qquad \left(f(s):=\frac{Z'_{n,\ell,\omega}(s)}{Z_{n,\ell,\omega}(s)}\right) \\
&\ge n\log(1+\delta)+\delta\int_{[\Lambda(\ell)]^n}\left\{\sum\psi_{11}(q_i-q_j)+2\sum\psi(q_i-\omega_k)\right\}\mu_{\ell,n}^{\omega}(d\mathbf{q}) \\
&\quad -2^{-1}\delta^2\left[|\Lambda(\ell)|+\sharp\omega\right]C_3(\rho_\ell)
\end{aligned}
$$

for all δ small enough, where $C_3(\rho) = A(M_1(\rho) \vee 2C_2(\rho))$. In the same way we get

(3.10)

$$
\begin{aligned}
&\log Z_n(\ell^{1-\delta},\omega^{1-\delta})-\log Z_n(\ell,\omega) \\
&\ge n\log(1-\delta)-\delta\int_{[\Lambda(\ell)]^n}\left\{\sum\psi_{11}(q_i-q_j)+2\sum\psi(q_i-\omega_k)\right\}\mu_{\ell,n}^{\omega}(d\mathbf{q}) \\
&\quad -2^{-1}\delta^2\left[|\Lambda(\ell)|+\sharp\omega\right]C_3(\rho_\ell).
\end{aligned}
$$

Theorem 3.2. *Let $\bar\rho$ be a non-negative constant. Then*

$$
\frac{1}{|\Lambda(\ell)|}\int_{[\Lambda(\ell)]^n}\Psi_{11}^{\ell}(\mathbf{q})\mu_{\ell,n}^{\omega}(d\mathbf{q}) \quad\longrightarrow\quad P(\bar\rho)-\bar\rho
$$

as n, $\ell_* \to \infty$ in such a way that $n/|\Lambda(\ell)| \to \bar\rho$, provided that ω, configurations on

$\Lambda(\ell+\mathbf{c}_o)\setminus\Lambda(\ell)$, are subject to the condition that $\sharp\omega/|\Lambda(\ell)|\to 0$. For each constant $r>0$ and each positive function $\gamma(\ell)$ approaching 0 as $\ell_*\to\infty$, the convergence is uniform with respect to $(\bar\rho,\omega)$ such that $\bar\rho\le r$ and $\sharp\omega<\gamma(\ell)|\Lambda(\ell)|$.

Proof. We pass to the limit as $\ell_*\to\infty$ in (3.9). Observe that the boundary terms are negligible. In fact, an application of Lemma 3.7 with the help of the hypothesis (iii) shows

$$\left|\int_{[\Lambda(\ell)]^n}\sum_{i,k}\psi_{11}(q_i-\omega_k)\mu^{\omega}_{\ell,n}(d\mathbf{q})\right|\le \text{const}\, C_2(\rho)\sharp\omega. \tag{3.11}$$

Recalling that $|\Lambda(\ell^{1+\delta})|=(1+\delta)|\Lambda(\ell)|$, we then obtain

$$\begin{aligned}&-\frac{1}{\delta}\Big[(1+\delta)\Phi(\bar\rho/(1+\delta))-\Phi(\bar\rho)\Big]\\ &\ge \bar\rho+\limsup\frac{1}{|\Lambda(\ell)|}\int_{[\Lambda(\ell)]^n}\Psi^{\ell}_{11}(\mathbf{q})\mu^{\omega}_{\ell,n}(d\mathbf{q})+O(\delta).\end{aligned}$$

We deduce from (3.10) a similar inequality (but in the opposite direction) with δ replaced by $-\delta$ and lim sup by lim inf. By introducing the variable η determined by $(1+\delta)(1-\eta)=1$ these two inequalities may be written as

$$\begin{aligned}&\frac{1}{-\eta}\Big[\Phi((1-\eta)\bar\rho)-\Phi(\bar\rho)\Big]+o(1)\\ &\ge\Phi(\bar\rho)+\bar\rho+\limsup\frac{1}{|\Lambda(\ell)|}\int_{[\Lambda(\ell)]^n}\Psi^{\ell}_{11}(\mathbf{q})\mu^{\omega}_{\ell,n}(d\mathbf{q})\\ &\ge\Phi(\bar\rho)+\bar\rho+\liminf\frac{1}{|\Lambda(\ell)|}\int_{[\Lambda(\ell)]^n}\Psi^{\ell}_{11}(\mathbf{q})\mu^{\omega}_{\ell,n}(d\mathbf{q})\\ &\ge\frac{1}{\eta}\Big[\Phi((1+\eta)\bar\rho)-\Phi(\bar\rho)\Big]+o(1),\end{aligned}$$

where $o(1)\to 0$ as $\eta\downarrow 0$ (we have made use of the fact that Φ is continuous). Noticing that the derivative of Φ from the right is larger than or equal to that from the left since $\Phi(\rho)$ is convex, we conclude that Φ is continuously differentiable and

$$\bar\rho\Phi'(\bar\rho)=\Phi(\bar\rho)+\bar\rho+\lim_{\ell_*\to\infty}\frac{1}{|\Lambda(\ell)|}\int_{[\Lambda(\ell)]^n}\Psi^{\ell}_{11}(\mathbf{q})\mu^{\omega}_{\ell,n}(d\mathbf{q}).$$

The asserted uniformity is automatical from the very fact that there exists a limit which is independent of ω. The relation of Theorem 3.2 then follows from $\rho\Phi'(\rho)-\Phi(\rho)=F(\lambda)$. □

Theorem 3.3. *In the same sense of convergence as in Theorem 3.2,*

$$\frac{1}{|\Lambda(\ell)|}\int_{[\Lambda(\ell)]^n}\Psi^{\ell}_{\alpha\beta}(\mathbf{q})\mu^{\omega}_{\ell,n}(d\mathbf{q})\quad\longrightarrow\quad(P(\bar\rho)-\bar\rho)\delta_{\alpha\beta}. \tag{3.12}$$

Proof. Let $T_{\alpha\beta}$ be a limit point of the left-hand side of (3.12). Let $a = (a_1, ..., a_d)$ be a unit vector in $\mathbf{R}^d$. We have $\sum_{\alpha,\beta} a_\alpha \psi_{\alpha\beta} a_\beta = -(z \cdot a)^2 |z|^{-2} \psi(|z|)$. Taking a small positive number δ, we cover $\Lambda(\ell)$ by identical and disjoint hyper-cubes of edge length $\ell_* \delta$ whose edges are parallel or perpendicular to the vector a. Theorem 3.2 may be applied to Ψ's corresponding to these hyper-cubes. The error that arises from the interaction between neighboring hyper-cubes and that from the contribution of those sitting on the border of $\Lambda(\ell)$ are estimated from above, respectively, by

$$AM_1(\rho_\ell) \times \frac{\text{the total volume of corridors}}{|\Lambda(\ell)|} \leq \text{const}\, M_1(\rho_\ell)[\delta \ell_*]^{-1}$$

and by

$$AM_1(\rho_\ell) \times \frac{\text{the total volume of hyper-cubes on the border}}{|\Lambda(\ell)|} \leq \text{const}\, M_1(\rho_\ell)\delta$$

as is deduced from Lemma 3.5 and the hypothesis (iii) on V. These bounds for errors vanish in the limit as $\ell \to \infty$ and $\delta \to 0$ in this order. We can therefore conclude that

$$\sum_{\alpha,\beta} a_\alpha T_{\alpha\beta} a_\beta = P(\bar{\rho}) - \bar{\rho},$$

proving that T is a constant times the identity matrix. The proof of Theorem 3.3 is complete. □

Proof of Theorem 3.1. The proof is carried out only in the case $\alpha = \beta = 1$ since the other case can be similarly dealt with. For a large positive integer m we partition $\Lambda(\ell - \mathbf{c}_o)$ into m^d hyper-intervals which are shifts of $\Lambda((\ell - \mathbf{c}_o)/m)$. Then we may write

$$\Psi_{11}^{\ell} = \sum_i \tau_{y(i)} \Psi_{11}^{\ell'} + R^{\ell,m}, \quad \ell' = \frac{\ell - \mathbf{c}_o}{m}$$

where $y(i)$ denotes the center of the i-th hyper-interval, τ_y the translation operator and $R^{\ell,m}$ the remainder term, which consists of the contribution to Ψ_{11}^{ℓ} of the interaction across the borders and that of the configurations on $\Lambda(\ell) \setminus \Lambda(\ell - \mathbf{c}_o)$. We are going to take limit as $\ell_* \to \infty$ and $m \to \infty$ in this order. From Lemma 3.5 it follows that

$$\int |R^{\ell,m}|^2 d\mu_{\ell,n}^{\omega} \leq CM_2(r) \left(m|\partial\Lambda(\ell)|\right)^2,$$

where $|\partial\Lambda|$ denotes the surface area of Λ and C some constant, so that the contribution of $R^{\ell,m}$ vanishes in the limit. We have to prove

$$\lim_{\ell_* \to \infty} \sup_{\omega} \sup_{n \leq r|\Lambda(\ell)|} \int \left| \frac{1}{|\Lambda(\ell)|} \sum_i \tau_{y(i)} \Psi_{11}^{\ell'} - [P(\rho_\ell) - \rho_\ell] \right|^2 d\mu_{\ell,n}^{\omega} = 0.$$

According to Lemma 3.9 we may replace $P(\rho_\ell) - \rho_\ell$ by

$$\frac{1}{m^d}\sum_i [P(\rho_{K(i)}) - \rho_{K(i)}], \quad \text{where} \quad \rho_{K(i)} = \frac{1}{|K(i)|}\mathcal{N}_{K(i)}$$

and $K(i)$ denote the i-th hyper-interval. By Lemma 3.5 again

$$\sup_{\ell,i}\sup_{\omega}\sup_{n\le r|\Lambda(\ell)|}\int\left[\left(|\Lambda(\ell')|^{-1}\tau_{y(i)}\Psi_{11}^{\ell'}\right)^2 + [P(\rho_{K(i)}) - \rho_{K(i)}]^2\right]d\mu_{\ell,n}^{\omega} < \infty.$$

Hence it suffices to show that for each m,

$$\int\left(\frac{\tau_{y(i)}\Psi_{11}^{\ell'}}{|\Lambda(\ell')|} - [P(\rho_{K(i)}) - \rho_{K(i)}]\right)\left(\frac{\tau_{y(j)}\Psi_{11}^{\ell'}}{|\Lambda(\ell')|} - [P(\rho_{K(j)}) - \rho_{K(j)}]\right)d\mu_{\ell,n}^{\omega} \to 0$$

as $\ell_* \to \infty$ uniformly in ω and $n \le r|\Lambda(\ell)|$ as well as in i, j such that $K(i)$ and $K(j)$ are separated by a distance more than c_o from each other. By the DLR equation and the Schwarz inequality this follows if we show that, uniformly with respect to i, ω and $n \le r|\Lambda(\ell)|$,

$$\int\left(\frac{1}{|\Lambda(\ell')|}\int_{[\Lambda(\ell')]^k}\Psi_{11}^{\ell'}(\mathbf{q})\mu_{\ell',k}^{\tau(-y(i))\omega'}(d\mathbf{q})\Big|_{k=\mathcal{N}_{K(i)}} - [P(\rho_{K(i)}) - \rho_{K(i)}]\right)^2 d\mu_{\ell,n}^{\omega} \to 0$$

(ω' is a random configuration in the shifted shell $\tau_{y(i)}[\Lambda(\ell'+\mathbf{c}_o)\setminus\Lambda(\ell')]$), which in turn follows from Theorem 3.2. The proof of Theorem 3.1 is complete. □

4. Local Gibbs States.

In this section we state the local equilibrium result (Theorem 4.1 below), which is essentially the same as that given in [6] for the one-dimensional model, and then a consequence of it combined with Theorem 2.1. The entropy bound (2.4) is supposed to hold in what follows.

Let h be a smooth, non-negative and radial function on $\mathbf{R}^d$ such that $h(\theta) = 0$ if $|\theta| > 1$, $\int h(\theta)d\theta = 1$ and $\theta\cdot\nabla h(\theta) < 0$ if $0 < |\theta| < 1$. Put, for $\lambda > 0$ and a configuration $\mathbf{x} = (x_1, ..., x_N) \in (\mathbf{T}^d)^N$,

$$\rho_\lambda(\theta) = \rho_\lambda(\theta;\mathbf{x}) = \sum_{i=1}^N h_\lambda\left(\frac{x_i - \theta}{\epsilon}\right) \qquad (h_\lambda(\theta) = \lambda^{-d}h(\lambda^{-1}\theta)).$$

Let $\mathbf{x}^\theta$ be the configuration viewed from $\theta \in \mathbf{T}^d$: $\mathbf{x}^\theta = (x_1 - \theta, ..., x_N - \theta)$ and define

$$\bar f^N(\mathbf{x}) = \frac{1}{T}\int_0^T dt\int_{\mathbf{T}^d} f_t^N(\mathbf{x}^\theta)d\theta.$$

Theorem 4.1. *For each $c > 0$, any limit point, as $N \to \infty$, of the law of the point process $\{\epsilon^{-1}x_i : \epsilon^{-1}x_i \in \Lambda(\mathbf{c})\}$ induced from $\bar f^N(\mathbf{x})\nu_N(d\mathbf{x})$ is a convex combination of* *canonical Gibbs measures* $\mu^{\omega}_{c,n}$ *of average particle density not greater than* $\limsup \epsilon^d N$ *over varying particle numbers* n *and boundary configurations* ω.

Proof. The proof given for Lemma 7.5 of [6] in one dimension may be followed word for word. □

Put for $\lambda \geq 1$

$$(4.1) \qquad S^{(\lambda)}(\theta) = S^{(\lambda)}(\theta, \mathbf{x}) = \rho_{2\lambda}(\theta) + \sum_{i,j(\neq)} |\psi| \left(\frac{|x_i - x_j|}{\epsilon} \right) h_{2\lambda} \left(\frac{x_i - \theta}{\epsilon} \right)$$

and with the microscopic variables $\mathbf{q} = (q_i)$

$$\Psi^{*(\lambda)}_{\alpha\beta}(\mathbf{q}) = \sum_{i,j(\neq)} h_\lambda(q_i) \psi_{\alpha\beta}(q_i - q_j).$$

Theorem 4.2. *Let* $\Psi^{(\lambda)}_{\alpha\beta}(\mathbf{x}) = \Psi^{*(\lambda)}_{\alpha\beta}(\epsilon^{-1}\mathbf{x})$. *Then for each* $r > 0$ *and* $1 \leq \alpha, \beta \leq d$

$$\lim_{\lambda\to\infty} \limsup_{N\to\infty} E^{\bar{f}^N} \left[\left| \Psi^{(\lambda)}_{\alpha\beta} - [P(\rho_\lambda(0)) - \rho_\lambda(0)]\delta_{\alpha\beta} \right|; S^{(\lambda)}(0) \leq r \right] = 0.$$

Proof. Since the integrand is uniformly bounded due to the truncation by $S^{(\lambda)}$, it suffices, in view of Theorem 4.1, to prove that for each $r > 0$ and for some $\theta > 1$

$$(4.2) \qquad \lim_{\lambda\to\infty} \sup_{\sharp\omega < e^\lambda} \sup_{n < r\lambda^d} \int_{[\Lambda(\theta\underline{\lambda})]^n} \left| \Psi^{*(\lambda)}_{\alpha\beta}(\mathbf{q}) - [P(\rho^*_\lambda(\mathbf{q})) - \rho^*_\lambda(\mathbf{q})]\delta_{\alpha\beta} \right| \mu^{\omega}_{\theta\underline{\lambda},n}(d\mathbf{q}) = 0,$$

where $\underline{\lambda} = (\lambda, ..., \lambda)$, $\sharp\omega$ denotes the number of particles constituting ω and $\rho^*_\lambda(\mathbf{q}) := \rho_\lambda(0, \epsilon\mathbf{q}) = \sum_{i=1}^N h_\lambda(q_i)$. For very large λ the function h_λ being locally almost constant in the sense that

$$\sup_{|q|\leq c} |h_\lambda(q_i - q) - h_\lambda(q_i)| \leq C_h \frac{c}{\lambda} h_\lambda(q_i/2)$$

for $c < \lambda/4$, the relation (4.2) is easily deduced from Theorem 1.1 with the help of Lemmas 3.4 and 3.9 by partitioning the support of h_λ into small blocks as in the proof of Theorem 2.1. □

5. Strong convergence of $P(\rho_\lambda(\theta, \mathbf{x}^N_t))$.

Let $\rho_\lambda(\theta) = \rho_\lambda(\theta, \mathbf{x})$ be as in the previous section. We wish to compare the microscopic density $\rho_\lambda(\theta)$ with the macroscopic one $\rho_\lambda * h_\eta(\theta)$ where λ is taken large and η small. Under the possibility of phase transition and without knowledge of growth rate of $P(\rho)$ for large ρ we consider $f(P(\rho_\lambda)) - f(P(\rho_\lambda * h_\eta))$ for each bounded continuous f instead of $\rho_\lambda - \rho_\lambda * h_\eta$ itself.

Theorem 5.1. *Suppose that the hypothesis (H) as well as (2.4) holds. For any subsequence of* $N = 1, 2, ...$ *there exists its subsequence* $\{N'\}$ *such that the law of*

$\{\xi_t^{N'} : 0 \le t \le T\}$ is convergent and that for each bounded continuous function $\varphi(\rho)$ of the form $\varphi(\rho) = f(P(\rho))$ with some bounded continuous function f on $[0, \infty)$,

$$\lim_{\eta \downarrow 0} \limsup_{\lambda \to \infty} \limsup_{N' \to \infty} \mathbf{E} \int_0^T dt \int_{\mathbf{T}^d} \left| \varphi(\rho_\lambda(\theta, \mathbf{x}_t^{N'})) - \varphi(\rho_\lambda(\cdot, \mathbf{x}_t^{N'}) * h_\eta(\theta)) \right| d\theta = 0,$$

where $\rho_\lambda(\cdot, \mathbf{x}_t^{N'}) * h_\eta(\theta)$ denotes the convolution $\int \rho_\lambda(\theta - \theta', \mathbf{x}_t^{N'}) h_\eta(\theta') d\theta'$.

Let $G = G(\theta)$ be a smooth function on $\mathbf{T}^d$. Put

$$F(\mathbf{x}) = \int_{\mathbf{T}^d} \rho_\lambda(\theta)(G * \rho_\lambda)(\theta) d\theta, \tag{5.1}$$

where $G * \rho_\lambda(\theta) = \int_{\mathbf{T}^d} G(\theta - \theta')\rho_\lambda(\theta')d\theta'$. Then, after carrying out integration by parts, we obtain

$$\frac{\partial}{\partial x_i} F(\mathbf{x}) = 2 \int_{\mathbf{T}^d} h_\lambda\left(\frac{x_i - \theta}{\epsilon}\right) \nabla G * \rho_\lambda(\theta) d\theta$$

and

$$\begin{aligned} L_N F = & \int \rho_\lambda(\theta) \Delta G * \rho_\lambda(\theta) d\theta \\ & + \sum_{i=1}^N \iint h_\lambda\left(\frac{x_i - \theta}{\epsilon}\right) \Delta G(\theta' - \theta) h_\lambda\left(\frac{x_i - \theta'}{\epsilon}\right) d\theta d\theta' \\ & - 2 \int \rho_\lambda(\theta') d\theta' \int \nabla G(\theta' - \theta) \cdot \sum_{i,j(\neq)} h_\lambda\left(\frac{x_i - \theta}{\epsilon}\right) \frac{1}{\epsilon} \nabla U\left(\frac{x_i - x_j}{\epsilon}\right) d\theta. \end{aligned}$$

Here the domain $\mathbf{T}^d$ is omitted from the integration sign. Since $\nabla U(-x) = -\nabla U(x)$, in the last integral $2h_\lambda((x_i - \theta)/\epsilon)$ may be replaced by

$$h_\lambda\left(\frac{x_i - \theta}{\epsilon}\right) - h_\lambda\left(\frac{x_j - \theta}{\epsilon}\right) = \int_0^1 \epsilon^d \nabla(h_{\lambda\epsilon})(x_i - s(x_i - x_j) - \theta) ds \cdot (x_i - x_j).$$

Substituting the right-hand side expression and performing integration by parts once more, we arrive at

$$\begin{aligned} L_N F = & \int_{\mathbf{T}^d} \rho_\lambda(\theta) \Delta G * \rho_\lambda(\theta) d\theta + Y^\lambda \\ & + \sum_{\alpha=1}^d \sum_{\beta=1}^d \int_{\mathbf{T}^d} \rho_\lambda(\theta') d\theta' \int_{\mathbf{T}^d} \nabla_\alpha \nabla_\beta G(\theta' - \theta) \cdot \tilde{\Psi}_{\alpha\beta}^\lambda(\theta) d\theta. \end{aligned} \tag{5.2}$$

where

$$\tilde{\Psi}_{\alpha\beta}^\lambda(\theta) = \tilde{\Psi}_{\alpha\beta}^\lambda(\theta; \mathbf{x}) = \sum_{i,j(\neq)} \int_0^1 h_\lambda\left(\frac{x_i - s(x_i - x_j) - \theta}{\epsilon}\right) ds\, \psi_{\alpha\beta}\left(\frac{x_i - x_j}{\epsilon}\right)$$

and

$$Y^\lambda = Y^{\lambda,N}(\mathbf{x}) = \sum_{i=1}^N \iint h_\lambda\left(\frac{x_i-\theta}{\epsilon}\right)\Delta G(\theta'-\theta)h_\lambda\left(\frac{x_i-\theta'}{\epsilon}\right)d\theta d\theta'.$$

The difficulty we encounter in the multidimensions is caused by the cross terms (i.e., the terms with $\alpha \neq \beta$) on the right-hand side of (5.2). Defining G by

$$G(x) = G_b(x) = \int_{\epsilon^2}^b p_t(x)dt$$

where $p_t(y-x), x, y \in \mathbf{T}^d, t > 0$, is the fundamental solution for the heat equation $\partial_t u = \frac{1}{2}\Delta u$ on $\mathbf{T}^d$, we proceed as in [5] and [6]. The actual proof is somewhat involved. It is only noted that the cross terms must vanish in the limit because of the local equilibrium once a relevant uniform integrability is established, and for the latter purpose we cannot help employing some bound of $S^{(\lambda)}(\theta)$ like (H) along with the following lemma.

Lemma 5.1. *For each $p > 1$ there exists a constant A_p independent of $b\ (\leq 1)$ and ϵ such that for any L^p-function f on $\mathbf{T}^d$ and for $1 \leq \alpha, \beta \leq d$,*

$$\|\nabla_\alpha \nabla_\beta G_b * f\|_p \leq A_p \|f\|_p$$

where $\|f\|_p = \left(\int |f|^p dx\right)^{1/p}$ $(p \geq 1)$.

6. Proof of Theorem 2.3.

The proof of Theorem 2.3 is based on Theorems 4.2 and 5.1 and the uniqueness result for weak solutions of (2.2) as stated in Section 2. It is not hard to prove that the set of random quantities $X^{N,\lambda} = (\rho^{N,\lambda}(\cdot,t), 0 \leq t \leq T)$ where $\rho^{N,\lambda}(\theta,t) := \rho_\lambda(\theta, \mathbf{x}_t^N)$ is tight as a family of measure-valued continuous processes taking values in the space of finite measures $\mathcal{M}(\mathbf{T}^d)$. Let $Q^{N,\lambda}$ be the probability law induced by $X^{N,\lambda}$ and Q any limit point of $\{Q^{N,\lambda}\}$ as $N \to \infty$ and $\lambda \to \infty$ in this order. We can show that

$$Q\left[\int_0^T dt \int_{\mathbf{T}^d} d\theta \int_0^\infty \rho(\theta,t)P(\rho(\theta,t)) < \infty\right] = 1. \tag{6.1}$$

Suppose that ξ_0^N converges to $u_o \in \mathcal{M}(\mathbf{T}^d)$. Let J be a smooth function on the torus $\mathbf{T}^d$. We write $\xi_t^N(J)$ for the integral $\int_{\mathbf{T}^d} J(\theta)\xi_t^N(d\theta)$. In view of the trivial bound

$$\left|\xi_t^N(J) - \int_{\mathbf{T}^d} J(\theta)\rho_\lambda(\theta, \mathbf{x}_t^N)d\theta\right| \leq \|\nabla J\|_\infty \lambda\epsilon,$$

it suffices to prove that the (potentially random) function $\rho(\theta,t)$ is a weak solution of (2.2) (with this $\rho(\theta,t)$ in place of $u(\theta,t)$) satisfying the initial condition (2.5) a.s.(Q) since the integrability condition (2.6) is valid by virtue of (6.1). As in the previous section (see (5.2)) we obtain

(6.2)

$$\int_{\mathbf{T}^d} J(\theta)\rho_\lambda(\theta,\mathbf{x}_t^N)d\theta - \int_{\mathbf{T}^d} J(\theta)\rho_\lambda(\theta,\mathbf{x}_0^N)d\theta$$
$$= \frac{1}{2}\int_0^t ds \sum_{\alpha=1}^d \sum_{\beta=1}^d \int \nabla_\alpha\nabla_\beta J(\theta)\left[\tilde{\Psi}^\lambda_{\alpha\beta}(\theta,\mathbf{x}_s^N) + \rho_\lambda(\theta,\mathbf{x}_s^N)\delta_{\alpha\beta}\right]d\theta + m_t$$

where

$$m_t = \int_0^t \sum_i \int_{\mathbf{T}^d} d\theta h_\lambda\left(\frac{x_i-\theta}{\epsilon}\right)\nabla J(\theta)\cdot dB_i.$$

A simple computation yields that $\mathbf{E}[|m_t|^2] \le \|\nabla J\|_\infty^2 N t\epsilon^{2d}$. We decompose the integral on the right side of (6.2) by dividing the domain of integration according as

$$\rho_\lambda(\theta,\mathbf{x}_s^N) \le M \quad \text{or} \quad \rho_\lambda(\theta,\mathbf{x}_s^N) > M. \tag{6.3}$$

The contribution of the second part can be easily shown to be negligible by using (H). For the first part in (6.3) we may replace $\tilde{\Psi}^\lambda_{\alpha\beta}(\theta,\mathbf{x}_s^N)$ by $[P(\rho_\lambda(\theta,\mathbf{x}_s^N)) - \rho_\lambda(\theta,\mathbf{x}_s^N)]\delta_{\alpha\beta}$ since the error arising by the replacement converges to zero in probability according to Theorem 4.2. Therefore we can write

$$\xi_t^N(J) - \xi_0^N(J) = \frac{1}{2}\int_0^t ds \int_{\mathbf{T}^d} \Delta J(\theta)\, P_M(\rho_\lambda(\theta,\mathbf{x}_s^N))d\theta + R^{N,\lambda,M}, \tag{6.4}$$

where $P_M(\rho) = P(\rho)g(P(\rho)/M)$ and the error term $R^{N,\lambda,M}$ converges to zero in the sense that for every $\delta > 0$

$$\lim_{M\to\infty}\limsup_{\lambda\to\infty}\limsup_{N\to\infty}\mathbf{P}\left[|R^{N,\lambda,M}| > \delta\right] = 0.$$

Now we apply Theorem 5.1 to deduce from (6.4) the relation

$$\int_{\mathbf{T}^d} J(\theta)\rho(\theta,t)d\theta - \int_{\mathbf{T}^d} J(\theta)u_o(d\theta) = \frac{1}{2}\int_0^t ds \int_{\mathbf{T}^d} \Delta J(\theta)\, P(\rho(\theta,s))d\theta \quad \text{a.s. } (Q).$$

Thus Theorem 2.3 has been proved. □

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