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ε-optimal controls for state constraint problems

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1 Introduction

In the study of the minimization problem of cost functionals governed by controlled dynamical systems, it is very important to find the optimal control, which minimizes the cost functional. However, in general, it is not possible to find the optimal control since the minimum might be attained by a "relaxed" (Young measure) control.

Our aim here is to find ε-optimal controls for the state constraint problem, which is a typical optimal control problem.

This work was done jointly with Prof. Hitoshi Ishii (Tokyo Metropolitan University) in [IK1].

2 Preliminaries

2.1 Notations

Let $\Omega\subset\mathbf{R}^n$ be a bounded domain and $A\subset\mathbf{R}^m$ $(m\in\mathbf{N})$ a control set. To describe the problem, we list our assumptions on given functions:

$$(A1)\quad\left\{\begin{array}{ll}(1) & f:\overline{\Omega}\times A\to\mathbf{R}\text{ and }g:\overline{\Omega}\times A\to\mathbf{R}^n\text{ are continuous,}\\ (2) & \sup\limits_{a\in A}|f(x,a)-f(y,a)|\leq O(|x-y|)\text{ for }x,y\in\overline{\Omega},\\ (3) & \sup\limits_{a\in A}\left(\|f(\cdot,a)\|_{L^\infty(\Omega)}+\|g(\cdot,a)\|_{W^{1,\infty}(\Omega)}\right)<\infty.\end{array}\right.$$

Setting the set of measurable controls,

$$\mathcal{A}=\{\alpha:[0,\infty)\to A\text{ measurable}\},$$

we denote by $X(\cdot;x,\alpha)$, for $\alpha \in \mathcal{A}$ and $x \in \overline{\Omega}$, the unique solution of

$$\begin{cases} \dfrac{dX}{dt}(t) = g(X(t), \alpha(t)) & \text{for } t > 0, \\ X(0) = x. \end{cases}$$

We define $\mathcal{A}(x)$, for $x \in \overline{\Omega}$, by the set of all $\alpha \in \mathcal{A}$ such that $X(t;x,\alpha) \in \overline{\Omega}$ for all $t \geq 0$. The **cost functional** $J_t(x,\alpha)$ upto $t \in (0,\infty]$, for $x \in \overline{\Omega}$ and $\alpha \in \mathcal{A}(x)$, is given by

$$J_t(x,\alpha) = \int_0^t e^{-s} f(X(s;x,\alpha)\alpha(s))ds.$$

The **value function** for the state constraint problem is defined by

$$V(x) = \inf_{\alpha \in \mathcal{A}(x)} J_\infty(x,\alpha).$$

For each $\varepsilon > 0$ and $x \in \overline{\Omega}$, we call $\alpha_{\varepsilon,x} \in \mathcal{A}(x)$ an **ε-optimal control** for our state constraint problem if

$$0 \leq J_\infty(x, \alpha_{\varepsilon,x}) - V(x) < \varepsilon.$$

Notice that the first inequality holds automatically.

2.2 Known results

2.2.1 The associated PDE

In the study of viscosity solution theory, it is well-known that V satisfies the Hamilton-Jacobi (HJ for short) equation **in the viscosity sense**:

$$(HJ) \qquad v(x) + \sup_{a\in A}\{-\langle g(x,a), Dv(x)\rangle - f(x,a)\} = 0 \quad \text{in } \Omega.$$

For the reader's convenience, we recall the definition: we call a function $v : \overline{\Omega} \to \mathbf{R}$ a viscosity subsolution (resp., supersolution) of (HJ) if

$$v^*(x) + \sup_{a\in A}\{-\langle g(x,a), p\rangle - f(x,a)\} \leq 0 \quad \text{for } x \in \Omega,\ p \in D^+v^*(x)$$

$$\left(\text{resp., } v_*(x) + \sup_{a\in A}\{-\langle g(x,a), p\rangle - f(x,a)\} \geq 0 \quad \text{for } x \in \Omega,\ p \in D^-v_*(x)\right).$$

We also call this v a viscosity solution of (HJ) if it is both a viscosity sub- and supersolution of (HJ).

Here, we use the set of superdifferentials of v at $x \in \overline{\Omega}$ (relative to $\overline{\Omega}$):

$$D^+v(x) = \{p \in \mathbf{R}^n \mid v(y) \le v(x) + \langle p, y - x\rangle + o(|x-y|) \text{ as } y \in \overline{\Omega} \to x\},$$

and the set of subdifferentials of v at $x \in \overline{\Omega}$: $D^-v(x) = -D^+(-v)(x)$, and the upper and lower semicontinuous envelopes:

$$v^*(x) = \lim_{\varepsilon\to 0} \sup\{v(y) \mid y \in B_\varepsilon(x) \cap \overline{\Omega}\} \quad \text{and} \quad v_*(x) = -(-v)^*(x),$$

where $B_\varepsilon(x)$ denotes the standard open ball with radius $\varepsilon > 0$ and center x.

The fact that the value function is a viscosity solution of (HJ) is a direct consequence of the Dynamic Programming Principle (DPP for short):

$$V(x) = \inf_{\alpha \in \mathcal{A}(x)} \left(J_t(x,\alpha) + e^{-t} V(X(t;x,\alpha)) \right).$$

Soner [S] showed that it is a supersolution of the same equation on $\partial\Omega$;

$$v(x) + \sup_{a\in A}\{-\langle g(x,a), p\rangle - f(x,a)\} \ge 0 \quad \text{for } x \in \partial\Omega,\ p \in D^+v(x).$$

Moreover, Ishii-Koike [IK2] showed that it satisfies one more boundary condition under $(A3)$ in the next section.

$$v(x) + \sup_{a\in A(x)}\{-\langle g(x,a), p\rangle - f(x,a)\} \ge 0 \quad \text{for } x \in \partial\Omega,\ p \in D^-v(x),$$

where $A(x)$ will be given in section 3.

This result implies that the value function is continuous in $\overline{\Omega}$ while Soner [S] showed that the value function is continuous in $\overline{\Omega}$ by analyzing it directly.

Therefore, throughout this note, we will suppose that $V \in C(\overline{\Omega})$ and will not use upper and lower semicontinous envelopes.

2.2.2 ε-optimal controls

If we know that V is a C^1 function, then we can construct an ε-optimal control from the HJ equation by a classical argument, which we will essentially use in the case when the value function is merely continuous. However, we can not expect C^1 regularity for the value function in general.

On the other hand, in the literature of the viscosity solution theory, to construct ε-optimal controls, we have another approach, which is called the semi-discrete approximation.

Let us briefly recall the idea of construction of ε-optimal controls by this procedure when $\Omega = \mathbf{R}^n$ for simplicity.

Fisrt, we solve the discritized HJ equation: for $h > 0$,

$$V_h(x) + \sup_{a \in A}\{-(1-h)V_h(x + hg(x,a)) - hf(x,a)\} = 0.$$

Next, using this, we choose

$$a_h^*(x) \in \arg\max_{a \in A}\{-(1-h)V_h(x + hg(x,a)) - hf(x,a)\}.$$

We notice that

$$V_h(x) - (1-h)V_h(x + hg(x, a_h^*(x))) - hf(x, a_h^*(x)) = 0.$$

We construct a piece-wise constant ε-optimal control using this mapping $a_h^*(\cdot)$.

We refer to [BCD] (and to our argument) for the details and also for general theory of viscosity solutions of HJ equations.

2.2.3 Pontryagin's maximum principle

Using the viscosity solution theory, Barron-Jensen [BJ] showed Pontryagin's maximum principle, which is a necessary condition that the optimal controls satisfy.

Let us consider the case when $\Omega = \mathbf{R}^n$ again. To state the Pontryagin's maximum principle, we need to suppose more regulaity for given functions f and g but we shall only give a rough statement without mentioning the correct assumptions. See [BJ] for the details.

If $\alpha \in \mathcal{A}$ is the optimal control of $V(x)$ (i.e. $V(x) = J_\infty(x,\alpha)$), then

$$\begin{aligned} 0 &= \sup_{a \in A}\{V(X(t)) - \langle g(X(t),a), DV(X(t))\rangle - f(X(t),a)\} \\ & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for } t \geq 0, \\ &= V(X(t)) - \langle g(X(t),\alpha(t)), DV(X(t))\rangle - f(X(t),\alpha(t)) \end{aligned}$$

where $X(t) = X(t; x, \alpha)$.

3 Main result

Our strategy of finding ε-optimal controls is as follows: We first construct a "feedback law" $\hat{\alpha}_\varepsilon : \overline{\Omega} \to A$ from the associated HJ equation. (We note that we only use the definition of viscosity supersolutions.) We then construct a piecewise constant control $\alpha_{\varepsilon,x} \in \mathcal{A}(x)$ through $\hat{\alpha}_\varepsilon$ which approximates the value function.

3.1 Hypotheses, theorem

Following [IK2], we introduce the notation: for $x \in \partial\Omega$,

$$A(x) = \left\{ a \in A \;\middle|\; \begin{array}{c} \text{There is } \delta > 0 \text{ such that } B_{\delta t}(y + tg(y,a)) \subset \Omega \\ \text{for } t \in (0,\delta) \text{ and } y \in B_\delta(x) \cap \overline{\Omega} \end{array} \right\}.$$

We now suppose that

$$(A2) \qquad A(x) \neq \emptyset \quad \text{for } x \in \partial\Omega.$$

We suppose that the exterior uniform sphere condition holds;

$$(A3) \qquad \left\{ \begin{array}{c} \text{There exists } R > 0 \text{ such that for any } z \in \partial\Omega, \\ \exists x \in \Omega^c \text{ which satisfies } B_R(x) \cap \overline{\Omega} = \{z\}. \end{array} \right.$$

3.1.1 Main result

Suppose that $(A1)$, $(A2)$ and $(A3)$ hold.

For any $\varepsilon > 0$, there are a constant $\tau > 0$ and a **feedback law** $\hat{\alpha} : \overline{\Omega} \to A$ satisfying the following property: For any $x \in \overline{\Omega}$, we choose $\alpha_\varepsilon \in \mathcal{A}$ by

$$\alpha_\varepsilon(t) = \hat{\alpha}(x_k) \quad \text{for } t \in [\tau k, \tau(k+1)) \qquad (k = 0, 1, 2, \ldots),$$

where

$$x_0 = x, \text{ and } x_k = X(\tau; x_{k-1}, \hat{\alpha}(x_{k-1})) \qquad (k = 1, 2, \ldots).$$

Then, $\alpha_\varepsilon \in \mathcal{A}(x)$ is an ε-optimal control of the state constraint problem.

3.1.2 Idea of proof

To consider the state constraint problem in subdomains of Ω, we introduce

$$\Omega_\gamma = \{x \in \Omega \mid \operatorname{dist}(x, \Omega^c) > \gamma\} \quad \text{for } \gamma > 0.$$

The value function of the state constraint problem for Ω_γ is given by

$$V^\gamma(x) = \inf_{\alpha \in \mathcal{A}_\gamma(x)} J_\infty(x, \alpha) \quad \text{for } x \in \overline{\Omega}^\gamma,$$

where

$$\mathcal{A}_\gamma(x) = \{\alpha \in \mathcal{A} \mid X(t; x, \alpha) \in \overline{\Omega}_\gamma \text{ for } t \geq 0\}.$$

Under $(A2)$, we may suppose that $\mathcal{A}_\gamma(x) \neq \emptyset$ for $x \in \overline{\Omega}_\gamma$. Furthermore, in view of [S] or [IK2], we may suppose that $V^\gamma \in C(\overline{\Omega}_\gamma)$.

Since we can show that

$$\lim_{\gamma \to \infty} \sup_{x \in \overline{\Omega}_\gamma} |V^\gamma(x) - V(x)| = 0,$$

we may suppose that

$$0 \leq V^\gamma(x) - V(x) < \frac{\varepsilon}{4} \quad \text{for } x \in \overline{\Omega}_\gamma. \tag{1}$$

Now we define the inf-convolution of V^γ by

$$v_\lambda^\gamma(x) = \inf_{y \in \mathbf{R}^n} \left(V^\gamma(y) + \frac{|x-y|^2}{2\lambda} \right) \quad \text{for } \lambda > 0.$$

We shall fix $\gamma^2 = \lambda\varepsilon$.

Finally we need one more definition: for a function $u : \mathbf{R}^n \to \mathbf{R}$,

$$\overline{D}^- u(x) = \left\{ p \in \mathbf{R}^n \,\middle|\, \begin{array}{c} \text{There is } \{(x_k, p_k)\}_{k=1}^\infty \subset \mathbf{R}^n \times \mathbf{R}^n \text{ such that} \\ \lim_{k \to \infty} (x_k, p_k) = (x, p) \text{ and } p_k \in D^- u(x_k) \end{array} \right\}.$$

We note that if v is a viscosity supersolution, then

$$v(x) + \sup_{a \in A}\{-\langle g(x,a), p\rangle - f(x,a)\} \geq 0 \quad \text{for } x \in \Omega,\ p \in \overline{D}^- v(x).$$

We define the feedback law $\hat{\alpha}_\varepsilon : \overline{\Omega} \to A$ by

$$v_\lambda^\gamma(x) - \langle g(x, \hat{\alpha}_\varepsilon(x)), p\rangle - f(x, \hat{\alpha}_\varepsilon(x)) \geq -\frac{\varepsilon}{4} \tag{2}$$

for $x \in \overline{\Omega}_{\gamma/2}$ and $p \in \overline{D}^{-} v_\lambda^\gamma(x)$, and

$$\hat{\alpha}_\varepsilon(x) \in A(\hat{x}) \quad \text{for } x \in \overline{\Omega} \setminus \overline{\Omega}_{\gamma/2},$$

where $\hat{x} \in \partial\Omega$ is the nearest point in $\partial\Omega$ from x; $\mathrm{dist}(x, \partial\Omega) = |\hat{x} - x|$.

We note that $\emptyset \neq \overline{D}^{-} v_\lambda^\gamma(x) \subset D^{+} v_\lambda^\gamma(x)$ for all $x \in \mathbf{R}^n$ by Lemma 2.4 in [IK1]. Moreover, we note that there exists $\hat{\alpha}_\varepsilon \in A$ such that (2) holds true by the definition of viscosity supersolutions and $(A1)$.

We also note that for any Lipschitz function $X : \mathbf{R}^n \to \mathbf{R}^n$, it holds that

$$\frac{dv_\lambda^\gamma}{dt}(X(t)) = \left\langle \frac{dX}{dt}(t), p \right\rangle \tag{3}$$

provided $p \in D^{+} v_\lambda^\gamma(X(t))$ for almost all $t \geq 0$.

We recall that because of the semi-concavity of v_λ^γ, a monotonicity for superdifferentials of v_λ^γ holds (Proposition 2.3 in [IK1]);

$$\langle p - q, x - y \rangle \leq \frac{|x - y|^2}{\lambda} \quad \text{for } p \in D^{+} v_\lambda^\gamma(x) \text{ and } q \in D^{+} v_\lambda^\gamma(y). \tag{4}$$

It is easy to verify that

$$|X(t) - x| \leq t M_g, \tag{5}$$

$$\left| \frac{X(t) - x}{t} - g(x, a) \right| \leq \frac{t M_g^2}{2}, \tag{6}$$

and

$$\left| \frac{X(t) - x}{t} - g(X(t), a) \right| \leq \frac{t M_g^2}{2}, \tag{6'}$$

where $X(t) = X(t; x, a)$ and $M_g = \sup_{a \in A} \|g(\cdot, a)\|_{W^{1,\infty}}$.

We have to derive the inequality (2) even when $x \in \overline{\Omega} \setminus \overline{\Omega}_{\gamma/2}$. (To this end, we need $(A2)$ and $(A3)$.)

In fact, we obtain that

$$-\langle g(x, \hat{\alpha}_\varepsilon(x)), p \rangle \geq \sup_{\lambda,\ \gamma > 0} \|v_\lambda^\gamma\|_{L^\infty} + \sup_{a \in A} \|f(\cdot, a)\|_{L^\infty} \tag{7}$$

for $x \in \overline{\Omega} \setminus \overline{\Omega}_{\gamma/2}$ and $p \in D^{-} v_\lambda^\gamma(x)$, provided $\lambda > 0$ is small enough. Intuitively, taking $x_\lambda \in \overline{\Omega}_\gamma$ such that $p = (x - x_\lambda)/\lambda$, in view of a careful estimate in [CLSS] (Lemma 3.5 in [IK1]), we may show that x_λ is very close to $\hat{x} \in \overline{\Omega}_\gamma$, where $|x - \hat{x}| = \mathrm{dist}(x, \overline{\Omega}_\gamma)$. Since we have

$$-\langle g(x, a), x - \hat{x} \rangle \geq \theta > 0$$

for some $\theta > 0$, we get (7) for small $\lambda > 0$.

See section 3 (more precisely, Lemma 3.6) in [IK1] for the details.

Hence, if $p \in \overline{D}^- v_\lambda^\gamma(x)$ and $p(t) \in \overline{D}^- v_\lambda^\gamma(X(t))$ (for almost all $t \geq 0$), then (4), (5) , (6) and (6′) yield that

$$-\langle g(x,a),p\rangle + \langle g(X(t),a),p(t)\rangle \leq \frac{tC}{\lambda} + \left\langle \frac{X(t)-x}{t}, p(t)-p \right\rangle \leq \frac{tC}{\lambda},$$

where $C > 0$ stands for the various constant independent of $\lambda, \varepsilon > 0$. Thus, setting $\tau = \varepsilon\gamma\lambda$, we have

$$\begin{aligned} -\frac{\varepsilon}{2} &\leq v_\lambda^\gamma(x) - \langle g(X(t;x,\hat{\alpha}_\varepsilon(x)),\hat{\alpha}_\varepsilon(x)),p(t)\rangle - f(x,\hat{\alpha}_\varepsilon(x)) \\ &\leq v_\lambda^\gamma(X(t;x,\hat{\alpha}_\varepsilon(x))) - \langle g(X(t;x,\hat{\alpha}_\varepsilon(x)),\hat{\alpha}_\varepsilon(x)),p(t)\rangle \\ &\quad -f(X(t;x,\hat{\alpha}_\varepsilon(x)),\hat{\alpha}_\varepsilon(x)) + \frac{\varepsilon}{4} \end{aligned}$$

for $p(t) \in \overline{D}^- v_\lambda^\gamma(X(t;x,\hat{\alpha}_\varepsilon(x)))$ (for almost all $t \in [0,\tau]$). Thus, multiplying e^{-t} and then, integrating it over $[0,\tau]$, by (3), we have

$$\begin{aligned} -\frac{3\varepsilon}{4}(1-e^{-\tau}) \leq\ & v_\lambda^\gamma(x) - e^{-\tau}v_\lambda^\gamma(X(\tau;x,\hat{\alpha}_\varepsilon(x))) \\ & - \int_0^\tau e^{-t} f(X(t;x,\hat{\alpha}_\varepsilon(x)),\hat{\alpha}_\varepsilon(x))dt. \end{aligned}$$

Finally, in view of the construction of $\alpha_\varepsilon \in \mathcal{A}(x)$, we have

$$\begin{aligned} -\frac{3\varepsilon}{4}(1-e^{-\tau}) \leq\ & v_\lambda^\gamma(x_k) - e^{-\tau}v_\lambda^\gamma(x_{k+1}) \\ & - \int_0^\tau e^{-t} f(X(t;x_k,\hat{\alpha}_\varepsilon(x_k)),\hat{\alpha}_\varepsilon(x_k))dt, \end{aligned} \tag{8$_k$}$$

where $x_0 = x$ and $x_k = X(\tau; x_{k-1}, \hat{\alpha}_\varepsilon(x_{k-1}))$ for $k = 1, 2, \ldots$ Multiplying $e^{-k\tau}$ in $(8)_k$ and then, taking the summation over $k = 0, 1, 2, \ldots$, we have

$$-\frac{3\varepsilon}{4} \leq v_\lambda^\gamma(x) - \int_0^\infty e^{-t} f(X(t;x,\alpha_\varepsilon),\alpha_\varepsilon(t))dt. \tag{9}$$

We claim that $\alpha_\varepsilon \in \mathcal{A}(x)$. Indeed, we see that $X(t;x,\hat{\alpha}_\varepsilon(x)) \in \overline{\Omega}$ for $t \in [0,\tau]$ when $x \in \overline{\Omega} \setminus \overline{\Omega}_{\gamma/2}$, since the corresponding vector field $g(\cdot, \hat{\alpha}_\varepsilon)$

pushes the state inside of $\overline{\Omega}$ for a short period. We also see that when $x \in \overline{\Omega}_{\gamma/2}$, $X(t; x, \hat{\alpha}_\varepsilon(x)) \in \overline{\Omega}$ for $t \in [0, \tau]$ by taking smaller $\tau > 0$ if necessary.

Therefore, in view of (1), we conclude that $\alpha_\varepsilon \in \mathcal{A}(x)$ is an ε-optimal control for the state constraint problem;

$$0 \leq J_\infty(x, \alpha_\varepsilon) - V(x) < \varepsilon.$$

3.2 Extensions

In a future work, we extend our results to differential games under state constraints, which was first treated in [K]. In [K], we present the formulation of the state constraint problem and give a sufficient condition to derive the comparison principle, which implies the continuity of value functions. In the future work, we shall construct ε-optimal controls and ε-optimal strategies for each player assuming a weaker condition under which it seems hard to show that the comparison principle holds.

Also, it is not hard to extend our result to the Cauchy problem (the finite horizon problem) and the Dirichlet problem (the stopping time problem).

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