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<tr>
<th>Title</th>
<th>Smooth solutions for degenerate parabolic equations (Nonlinear evolution equation and its applications)</th>
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Kyoto University
Smooth solutions for degenerate parabolic equations

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1 Introduction

Many models have been proposed to describe the diffusion process. One for the linear Newtonian diffusion process is the well-known heat equation;
(HE)\(u_t = u_{xx}\). A typical example for the nonlinear diffusion process for the Newtonian filtration problem is known as the porous medium equation;
(PE)\(u_t = (u^\ell u_x)_x\). Here we note that (PE) is equivalent to (HE) as \(\ell = 0\). Concerning the non-Newtonian polytropic filtration problem, the following doubly nonlinear equation is proposed. (see [?, Kalashnikov]) \((DE)\ u_t = (u^\ell |u_x|^{p-2}u_x)_x\). In this note, we consider the following more generalized equation which covers all these examples.

\[
(E) \begin{cases}
\frac{\partial u}{\partial t} - (\varphi(u, u_x)u_x)_x = 0, & (x, t) \in \mathbb{R} \times [0, T], \\
u(x, 0) = u_0(x), & x \in \mathbb{R}.
\end{cases}
\]

These equations have been studied by so many people and many interesting results are obtained so far. Among them, concerning the regularity of solutions for nonlinear problems, the Hölder continuity of \(u\) has been well known. However, as for the estimate for the derivative of \(u\) itself, little is known. In our recent works [13], [14], [15] and [16], we constructed a time local solution which is smooth with respect to time and space variables. Our aim here is to investigate the structure condition on \(\varphi(u, u_x)\) which assures the existence of time local smooth solutions of (E). Our main result is given
in the next section, and the sketch of its proof is given in §3. This work is a collaboration with Prof. M. Ôtani (Waseda Univ.).

2 Main Result

Throughout of this note, we always assume the following conditions.

(A.1) \( \varphi(u, z) \in C^\infty(\mathbb{R} \times \mathbb{R}), \varphi(u, z), \varphi_z(u, z)z \geq 0, \forall (u, z) \in \mathbb{R} \times \mathbb{R}, \) and \( \varphi(0, 0) = 0. \)

(A.2) There exist functions \( g_1(u, z), g_2(u, z) \in L^\infty_{loc}(\mathbb{R} \times \mathbb{R}) \) such that
\[
\left| \frac{(\varphi_z(u, z))^2}{\varphi(u, z)} \right| \leq g_1(u, z), \quad \left| \frac{\varphi_{zz}(u, z)z}{\varphi_z(u, z)z} \right| \leq g_2(u, z).
\]

(A.3) \( u_0(x) \in \bigcap_{m=0}^\infty H^m(\mathbb{R}) \)

Then main result is stated as follows.

Theorem (Local existence for \( (E) \)) Assume (A.1) and (A.3), then there exists a number \( T_2 \in (0, T] \) depending only on \( |u_0|_{L^\infty}, |u_0x|_{L^\infty}, |D_x^2u_0|_{L^\infty} \) and \( |D_x^3u_0|_{L^\infty} \) such that (E) has a solution \( u \) belonging to \( C^\infty([0, T_2] \times \mathbb{R}) \). Furthermore if (A.2) is satisfied, then there exists a number \( T_1 \in (0, T] \) depending only on \( |u_0|_{L^\infty}, |u_0x|_{L^\infty} \) and \( |D_x^2u_0|_{L^\infty} \) such that (E) has a solution \( u \) belonging to \( C^\infty([0, T_1] \times \mathbb{R}) \).

Remark Note that (A.2) is not so restrictive because of the existence of the square power raised to the numerators in (A.2). In fact there are many functions satisfying (A.1) and (A.2) such as \( \varphi(u, z) = u^{2\ell}z^{2p} \) with \( \ell, p \in \mathbb{N} \).

3 Proof of main theorem

We here give the sketch of a proof of our main theorem. We first prepare the following approximate equations for (E):

\[
(E)^\varepsilon \left\{ \begin{array}{l}
\frac{\partial u^\varepsilon}{\partial t} - \varepsilon u^\varepsilon - \left( (\varphi(u^\varepsilon, u^\varepsilon_x)u^\varepsilon_x \right)_x = 0, \quad (x, t) \in \Omega \times [0, T], \\
u^\varepsilon(x, 0) = u_0(x), \quad x \in \Omega,
\end{array} \right.
\]
We tell roughly perturbation not for unique the following we get the solving proposition.

\[ k=0,1, \]

In order to see that \((E)^\varepsilon\) have \(C^\infty\)-solutions, we have only to show the following proposition.

**Proposition 3.1** For any \( T > 0, n \in \mathbb{N} \) and \( \varepsilon \in (0,1] \), \((E)^\varepsilon\) has a unique solution \( u^\varepsilon \) belonging to \( H^k(0,T;H^{2(n-k+1)}(\mathbb{R})) \) for all \( k = 0,1, \ldots, n \).

To prove Proposition 3.1, we reduce \((E)^\varepsilon\) to the following evolution equations in \( H_k = H^{2k}(\mathbb{R}) \):

\[
\begin{aligned}
(E)^\varepsilon \left\{ 
\begin{array}{l}
u_xt + \varepsilon u_x^\varepsilon - (\varphi(u^\varepsilon, u_x^\varepsilon))_x = 0, \quad (x,t) \in \Omega \times [0,T], \\
u(x,0) = u_0(x), \quad x \in \Omega,
\end{array}
\right.
\end{aligned}
\]

where \( \varepsilon \) is a positive parameter. In order to see that \((E)^\varepsilon\) have \( C^\infty \)-solutions, we have only to show the following proposition.

**Proposition 3.1** For any \( T > 0, n \in \mathbb{N} \) and \( \varepsilon \in (0,1] \), \((E)^\varepsilon\) has a unique solution \( u^\varepsilon \) belonging to \( H^k(0,T;H^{2(n-k+1)}(\mathbb{R})) \) for all \( k = 0,1, \ldots, n \).

To prove Proposition 3.1, we reduce \((E)^\varepsilon\) to the following evolution equations in \( H_k = H^{2k}(\mathbb{R}) \):

\[
\begin{aligned}
(E)^\varepsilon \left\{ 
\begin{array}{l}
u_xt + \varepsilon u_x + \varphi(u^\varepsilon, u_x^\varepsilon)Au + \varphi_z(u^\varepsilon, u_x^\varepsilon)u_x^\varepsilon Au_x \\
= \varphi_u(u^\varepsilon, u_x^\varepsilon)(u_x^\varepsilon)^2, \quad t \in [0,T], \\
u^\varepsilon(0) = u_0.
\end{array}
\right.
\end{aligned}
\]

Here \( A \) is an operator in \( H_k \) defined by \( A = -\frac{d^2}{dx^2} \) and \( D(A) = H^{2k+2}(\mathbb{R}) \), and the inner product of \( H_k \) is given by \((u,v)_{H_k} = (u,v)_{L^2(\mathbb{R})} + (A^k u, A^k v)_{L^2(\mathbb{R})} \). In what follows we always assume \( u_0 \in D(A^{1/2}) = H^{2k+1}(\mathbb{R}) \) and for the sake of simplicity we denote \( u^\varepsilon \) by \( u \). In solving \((E)^\varepsilon\), we regard the terms \( \varphi(u^\varepsilon, u_x^\varepsilon)Au, \varphi_z(u^\varepsilon, u_x^\varepsilon)u_x^\varepsilon Au \) and \( \varphi_u(u^\varepsilon, u_x^\varepsilon)(u_x^\varepsilon)^2 \) as perturbations for \( \varepsilon Au \). We first solve the following equation with the perturbation \( \varphi(u^\varepsilon, u_x^\varepsilon)Au \):

\[
\begin{aligned}
(E)^\varepsilon_0 \left\{ 
\begin{array}{l}
u_t + \varepsilon Au + \varphi(u^\varepsilon, u_x^\varepsilon)Au = f, \quad t \in [0,T], \\
u(0) = u_0,
\end{array}
\right.
\end{aligned}
\]

where \( f \) is a given function in \( L^2(0,T;H_k) \).

The difficulty in solving this equation lies in the fact that \( \varphi(u^\varepsilon, u_x^\varepsilon)Au \) is not a small perturbation nor monotone perturbation for \( \varepsilon Au \). However, roughly speaking, \( \varphi(u^\varepsilon, u_x^\varepsilon)Au \) can be decomposed into the monotone perturbation part and the small perturbation part. In fact, we get

\[
(\varphi(u^\varepsilon, u_x^\varepsilon)Au, Au)_{H_k} = (\varphi(u^\varepsilon, u_x^\varepsilon)Au)_{L^2(\mathbb{R})} + (A^k(\varphi(u^\varepsilon, u_x^\varepsilon)Au), A^{k+1}u)_{L^2(\mathbb{R})}
\]

\[
= \int_{\mathbb{R}} \varphi(u^\varepsilon, u_x^\varepsilon)(Au)^2 \, dx + \int_{\mathbb{R}} \varphi(u^\varepsilon, u_x^\varepsilon)(A^{k+1}u)^2 \, dx + R_k
\]

\[
\geq R_k,
\]
\[
R_k = \sum_{j=1}^{2k} 2k C_j \int_{\mathbb{R}} D_x^j (\varphi(u, u_x)) \cdot D_x^{2k-j} Au \cdot A^{k+1} u dx, \quad D_x = \frac{\partial}{\partial x}.
\]

Since the most singular part of the integrand of \(R_k\) is \(D_x (\varphi(u, u_x)) \cdot D_x^{2k-1} Au \cdot A^{k+1} u\), \(R_k\) can be handled as a small perturbation for \(\varepsilon\|A^{k+1} u\|^2_{L^2}\).

To solve \((E)_0^\varepsilon\), we introduce another auxiliary equation:

\[
\lambda (E)_0^\varepsilon \left\{ \begin{array}{ll}
 u_t + \varepsilon Au + \lambda \varphi(u, u_x)Au = h + f, & t \in [0, T], \lambda \in [0, 1], \\
 u(0) = u_0.
\end{array} \right.
\]

If \(\lambda (E)_0^\varepsilon\) has a unique solution \(u^h\), we define the operator \(\lambda F_{\eta_0}\) by the following correspondence:

\[
\lambda F_{\eta_0} : h \mapsto u^h \mapsto -\eta_0 \varphi(u^h, u^h_x)Au^h, \quad \eta_0 \in \mathbb{R}.
\]

By making use of the property of \(\varphi(u, u_x)Au\) observed above, we can show the following lemma.

**Lemma 3.2** There exist a (sufficiently small) positive number \(\eta_0\) and a positive number \(R\) independent of \(\lambda\) such that \(\lambda F_{\eta_0}\) becomes a contraction mapping from \(K^T_R\) into itself, where \(K^T_R = \{v \in L^2(0, T; H_k); \|v\|_{L^2(0, T; H_k)} \leq R\}\).

It is clear that \(\lambda (E)_0^\varepsilon\) with \(\lambda = 0\) has a unique solution, so \(0 F_{\eta_0}\) is well defined. Hence \(0 F_{\eta_0}\) has a fixed point by the contraction mapping principle, which implies that \(\lambda (E)_0^\varepsilon\) with \(\lambda = \eta_0\) admits a unique solution, so \(\eta_0 F_{\eta_0}\) is well defined. Hence \(\lambda F_{\eta_0}\) with \(\lambda = 2\eta_0\) admits a unique solution. Thus repeating this procedure finite times, we can construct a unique solution of \(1(E)_0^\varepsilon\).

To solve the original approximate equation \((E)^\varepsilon\), we next introduce another mapping \(\mu S_{\eta_1}\) defined by the following correspondence:

\[
\mu S_{\eta_1} : h \mapsto u^h \mapsto -\eta_1 \varphi_z(u, u_x)u_x Au,
\]

where \(u^h\) is the unique solution of the following auxiliary equation.

\[
\lambda (E)_0^\varepsilon \left\{ \begin{array}{ll}
 u_t + \varepsilon Au + \varphi(u, u_x)Au + \mu \varphi_z(u, u_x)u_x Au = f, & t \in [0, T], \lambda \in [0, 1], \\
 u(0) = u_0.
\end{array} \right.
\]
By using much the same arguments as for $\lambda \mathcal{F}_{\eta_0}$, we can show the following Lemma 3.3.

**Lemma 3.3** There exist a (sufficiently small) positive number $\eta_1$ and a positive number $R$ independent of $\mu$ such that $\mu \mathcal{S}_{\eta}$ becomes a contraction mapping from $K^T_R$ into itself, where $K^T_R = \left\{ v \in L^2(0,T;H_k); \| v \|_{L^2(0,T,H_k)} \leq R \right\}$.

Then we can assure the existence of solutions of $\frac{1}{\mu}(E)_0$ with $\mu = 1$. Finally we introduce the mapping $\mathcal{W}$ as follows.

$$\mathcal{W} : f \mapsto u^f \mapsto \varphi(u^f, u^f_x)(u^f_x)^2,$$

where $u^f$ is the unique solution of $\frac{1}{\mu}(E)_0$. Since $\mathcal{W}$ does not involve any small parameter such as $\eta_0$ for $\lambda \mathcal{F}_{\eta_0}$ or $\eta_1$ for $\mu \mathcal{S}_{\eta}$ to control the size of the value of $\mathcal{W}$, we need the smallness of $T_0$. However, by the standard energy estimates for $(E)^\epsilon$, we can establish a priori bound for $\| A^{1/2} u(t) \|_{H_k}$, which assures that the local solution on $[0, T_0]$ can be continued up to $[0, T]$. Thus the first step is completed.

**Step 2 (A priori estimates)**

We apply the "$L^\infty$-energy method" introduced in ([13],[14]). Since the presence of the term $\epsilon Au$ does not disturb the following argument, it suffices to establish a priori estimates for the original equation (E).

(i) Estimate for $\| u(x,t) \|_{L^\infty(\mathbb{R} \times [0,T])}$:

Multiply $(E)^\epsilon$ with $\epsilon = 0$ by $|u|^{r-2}u$, then by the integration by parts, we get

$$\frac{1}{r} \frac{d}{dt} \| u(t) \|_{L^r(\mathbb{R})}^r = -(r - 1) \int \varphi(u, u_x)|u|^{r-2}|u_x|^2 \, dx \leq 0.$$  

Hence $\| u(t) \|_{L^r(\mathbb{R})} \leq \| u_0 \|_{L^r(\mathbb{R})}$ for all $r \in [2, \infty)$, then letting $r \to \infty$, we deduce

$$\| u(t) \|_{L^\infty(\mathbb{R})} \leq \| u_0 \|_{L^\infty(\mathbb{R})} \quad \text{for all } r \in [0, \infty) \text{ and } t \in [0,T].$$  

(2)
(ii) Estimate for $\|u_x(x, t)\|_{L^\infty(\mathbb{R} \times [0, T_0])}$:

Multiply $(E)^\epsilon$ with $\epsilon = 0$ by $-\Delta_x u = -(|u_x|^{-2}u_x)_x$, then we have

$$\frac{1}{r} \frac{d}{dt} \|D_x u(t)\|_{L^r(\mathbb{R})} + J_0 + J_1 = J_2,$$

\[ J_0 = (r - 1) \int \varphi(u, u_x)|u_x|^{r-2}(D_x^2 u)^2 \, dx \geq 0, \]
\[ J_1 = (r - 1) \int \varphi_z(u, u_x)u_x|u_x|^{r-2}(D_x^2 u)^2 \, dx \geq 0, \]
\[ J_2 = -\int \varphi_u(u, u_x)|u_x|^2|u_x|^{r-2}u_x \, dx. \]

Here

$$J_2 = -\frac{r - 1}{r + 1} \int \varphi_u(u, u_x)(|u_x|^r u_x)_x \, dx$$
$$= \frac{r - 1}{r + 1} \int \varphi_{uu}|u_x|^{r+2} \, dx + \frac{r - 1}{r + 1} \int \varphi_{uz}|u_x|^r u_x D_x^2 u \, dx$$
$$= J_2^1 + J_2^2,$$

where we get

$$J_2^1 \leq C_0 \int |u_x|^{r+2} \, dx,$$
$$J_2^2 = \frac{r - 1}{r + 1} \int \varphi_{uz}(u, u_x)|u_x|^{r+2} D_x^2 u \, dx$$
$$= \frac{r - 1}{r + 1} \int \varphi_{uz}(u, u_x)(\frac{1}{r + 2}|u_x|^{r+2})_x \, dx$$
$$\leq \frac{1}{r + 2} \int |\varphi_{uuz}(u, u_x) u_x + \varphi_{uzz}(u, u_x) D_x^2 u| |u_x|^{r+2} \, dx$$

There exists a constant $C_\epsilon$ depending on $\epsilon$ such that

$$|\varphi_{uuz}(u, u_x) u_x + \varphi_{uzz}(u, u_x) D_x^2 u| \leq C_\epsilon.$$

Hence we find

$$J_2^2 \leq \frac{C_\epsilon}{r + 2} \int |u_x|^{r+2} \, dx.$$  \hfill (4)
Thus, in view of (2), (3) and (4), we derive
\[
\|u_x(t)\|_{L^r(R)}^{-1} \frac{d}{dt} \|u_x(t)\|_{L^r(R)} \leq \left( C_0 + \frac{C_\epsilon}{r+2} \right) \int |u_x|^{r+2} \, dx.
\]
By dividing both sides by \( \|u_x(t)\|_{L^r(R)}^{-1} \), we get
\[
\frac{d}{dt} \|u_x(t)\|_{L^r(R)} \leq \left( C_0 + \frac{C_\epsilon}{r+2} \right) \|u_x\|_{L^\infty(R)}^2 \|u_x\|_{L^r(R)}.
\]
Hence letting \( r \to \infty \), we obtain
\[
\|u_x(t)\|_{L^\infty(R)} \leq \|u_{0x}\|_{L^\infty(R)} + C_\epsilon \int_0^t \|u_x(s)\|_{L^\infty(R)}^3 \, ds.
\]
Then it is easy to show that there exists a positive number \( T_0 = T_0(\|u_0\|_{L^\infty(R)}, \|u_{0x}\|_{L^\infty(R)}) \leq T \) independent of \( \epsilon \) such that
\[
\|u_x(t)\|_{L^\infty(R)} \leq 2\|u_{0x}\|_{L^\infty(R)} \quad \text{for all } t \in [0, T_0].
\]
(5)

(iii) Estimate for \( \|D_x^2 u(x, t)\|_{L^\infty(\mathbb{R} \times [0, T])} \):

We differentiate \((E)^\epsilon\) once with respect to the \( x \)-variable to obtain \((E)^\epsilon_x\), and multiply \((E)^\epsilon_x\) by \(-\Delta_x u_x = -(|D_x^2 u|^{r-2} D_x^2 u)_x\). Then we obtain
\[
\frac{1}{r} \frac{d}{dt} \|D_x^2 u(t)\|_{L^r(R)}^r = - \int D_x^2 (\varphi(u, u_x) u_x)(|D_x^2 u|^{r-2} D_x^2 u)_x \, dx
\]
\[
D_x^2 (\varphi(u, u_x) u_x) = (\varphi + \varphi_x u_x) D_x^3 u
+ a_1(u, u_x) + a_2(u, u_x) D_x^2 u + a_3(u, u_x)(D_x^2 u)^2
\]
So we have
\[
\frac{d}{dt} \|D_x^2 u(t)\|_{L^r(R)}^r + J_0 = J_1 + J_2 + J_3
\]
\[
J_0 = (r-1) \int \left( \varphi(u, u_x) + \varphi_x(u, u_x) u_x \right) |D_x^2 u|^{r-2}(D_x^3 u)^2 \, dx
\geq 0,
\]
\[
J_1 = - \int a_1(u, u_x)(|D_x^2 u|^{r-2} D_x^2 u)_x \, dx,
\]
\[
J_2 = - \int a_2(u, u_x) D_x^2 u(|D_x^2 u|^{r-2} D_x^2 u)_x \, dx,
\]
\[
J_3 = - \int a_3(u, u_x)(D_x^2 u)^2(|D_x^2 u|^{r-2} D_x^2 u)_x \, dx.
\]
By using integration by parts and (2) and (5), we get

\[ J_1 = -\int (a_{1u}(u, u_x)u_x + a_{1z}(u, u_x)D_x^2u) |D_x^2u|^{-2}D_x^2u \]
\[ = C_1 \int |D_x^2u|^{-1} |u_x| \, dx + C_2 \int |D_x^2u|^r \, dx, \quad (7) \]

\[ J_2 = -\int a_2(u, u_x)D_x^2u \cdot (|D_x^2u|^{-2}D_x^2u)_x \, dx \]
\[ = -\int a_2(u, u_x) \cdot \frac{r-1}{r} (|D_x^2u|^r)_x \, dx \]
\[ = \frac{r-1}{r} \int (a_{1u}(u, u_x)u_x + a_{1z}(u, u_x)D_x^2u) |D_x^2u|^r \, dx \]
\[ = C_3 \int |D_x^2u|^r \, dx + C_4 \int |D_x^2u|^{r+1} \, dx, \quad (8) \]

\[ J_3 = -\int a_3(u, u_x)(D_x^2u)^2 \cdot (|D_x^2u|^{-2}D_x^2u)_x \, dx \]
\[ = -\int a_3(u, u_x) \cdot \frac{r-1}{r+1} (|D_x^2u|^r D_x^2u)_x \, dx \]
\[ = \frac{r-1}{r+1} \int (a_{3u}(u, u_x)u_x + a_{3z}(u, u_x)D_x^2u) |D_x^2u|^r D_x^2u \, dx \]
\[ = C_5 \int |D_x^2u|^{r+1} \, dx + C_6 \int |D_x^2u|^{r+2} \, dx. \quad (9) \]

Then (6)–(9) give,

\[
\|D_x^2u(t)\|_{L^r(R)}^{-1} \frac{d}{dt} \|D_x^2u(t)\|_{L^r(R)} \leq C_1 \|u_x\|_{L^r(R)} \|D_x^2u\|_{L^r(R)}^{-1} \\
+ C_7 \left( 1 + \|D_x^2u\|_{L^\infty(R \times [0,T])}^2 + \|D_x^2u\|_{L^\infty(R \times [0,T])}^2 \right) \|D_x^2u\|_{L^r(R)} \, dx.
\]

Dividing both sides by \( \|D_x^2u(t)\|_{L^r(R)}^{-1} \) and letting \( r \to \infty \), we obtain

\[
\|D_x^2u(t)\|_{L^\infty(R)} \leq \|D_x^2u_0\|_{L^\infty(R)} + C_1 \|u_x\|_{L^\infty(R)} \\
+ C_7 \int_0^t \left( \|D_x^2(s)\|_{L^\infty(R)}^2 + \|D_x^2(s)\|_{L^\infty(R)}^2 + \|D_x^2(s)\|_{L^\infty(R)}^3 \right) \, ds.
\]

Then it is easy to show that there exists a positive number \( T_1 = T_1(\|u_0\|_{L^\infty(R)}, \|u_{0x}\|_{L^\infty(R)}, \|D_x^2u_0\|_{L^\infty(R)}) \leq T \) such that
\[ \|D_x^2 u(t)\|_{L^\infty(R)} \leq 2\|D_x^2 u_0\|_{L^\infty(R)} \quad \text{for all } t \in [0, T_1]. \]  

(10)

Here we note that the constants \(C_i\) \((i = 1, 2, \cdots, 7)\) depend on the initial data but not on \(\varepsilon\).

(iv) Estimate for \(\|D_x^3 u(x, t)\|_{L^\infty(R \times [0, T_1])}\):

In order to derive the a priori bound for \(\|D_x^3 u(x, t)\|_{L^\infty}\) under assumption (A.2), we rely on the following interpolation inequality:

\[ \|D_x^3 u(x, t)\|_{L^\infty} \leq \sqrt{2}\|D_x^3 u(x, t)\|_{L^2}^{1/2} \cdot \|D_x^4 u(x, t)\|_{L^2}^{1/2}. \]  

(11)

That is to say, we have only to establish a priori bounds for \(\|D_x^2 u(x, t)\|_{L^2}\) and \(\|D_x^4 u(x, t)\|_{L^2}\).

(v) Estimate for \(\sup_{0 \leq t \leq T_1} \|D_x^3 u(x, t)\|_{L^2(R)}\):

We differentiate \((E)^\varepsilon\) twice with respect to \(x\)-variable to obtain \(D_x^2(E)^\varepsilon\), and multiply \(D_x^2(E)^\varepsilon\) by \(-D_x^4 u\). Then we get

\[ \frac{1}{2} \frac{d}{dt}\|D_x^3 u(t)\|_{L^2(R)}^2 = \int D_x^3(\varphi(u, u_x)u_{x})D_x^4 u \, dx, \]

\[ D_x^3(\varphi(u, u_x)u_{x}) = (\varphi + \varphi_z u_x)D_x^4 u + (5\varphi_z + 3\varphi_{zz} u_x)D_x^2 u D_x^3 u + b_1(u, u_x, D_x^2 u) + b_2(u, u_x)D_x^3 u. \]

So we have

\[ \frac{1}{2} \frac{d}{dt}\|D_x^3 u(t)\|_{L^2(R)}^2 + J_0 = J_1 + J_2 \]

(12)

\[ J_0 = \int \left( \varphi(u, u_x) + \varphi_x(u, u_x)u_x \right)(D_x^4 u)^2 \geq 0, \]

\[ J_1 = -\int b_1(u, u_x, D_x^2 u)D_x^4 u \, dx, \]

\[ J_2 = -\int b_2(u, u_x)D_x^3 u D_x^4 u \, dx, \]

\[ J_3 = -5\int \varphi_z(u, u_x)D_x^2 u D_x^3 u D_x^4 u \, dx, \]

\[ J_4 = -3\int \varphi_{zz}(u, u_x)u_x D_x^2 u D_x^3 u D_x^4 u \, dx. \]
It is easily follows from (2), (5) and (10) that

$$J_1 = \int \left( b_{1u}(u, u_x, D_x^2 u) u_x + b_{1z}(u, u_x, D_x^2 u) D_x^3 u \right) D_x^2 u \, dx$$

$$\leq C_8 \int \left( |u_x| + |D_x^2 u| + |D_x^3 u| \right) |D_x^3 u| \, dx.$$ 

Repeating the same arguments with a priori bounds for $\|u_x\|_{L^\infty(\mathbb{R})}$ and $\|D_x^2 u\|_{L^\infty(\mathbb{R})}$, we easily obtain the following $L^2$-bound.

$$\sup_{t \in [0,T]} \|u_x(t)\|_{L^2(\mathbb{R})}, \sup_{t \in [0,T]} \|D_x^2 u(t)\|_{L^2(\mathbb{R})} \leq C_9. \quad (13)$$

Therefore we get

$$J_1 \leq 2C_9C_0 \|D_x^3 u\|_{L^2(\mathbb{R})} + C_8 \|D_x^3 u\|_{L^2(\mathbb{R})}^2. \quad (14)$$

As for $J_2$, (2), (5) and (10) yield.

$$J_2 = - \int b_2(u, u_x) D_x^3 u D_x^4 u \, dx$$

$$= - \int b_2(u, u_x) \frac{1}{2} (D_x^3 u)_x^2 \, dx$$

$$= \frac{1}{2} \int \left( b_{2u}(u, u_x) u_x + b_{1z}(u, u_x) D_x^2 u \right) (D_x^3 u)^2$$

$$= C_{10} \int \left( |u_x| + |D_x^2 u| \right) |D_x^3 u|^2 \, dx. \quad (15)$$

In order to estimate $J_3$, we note that $Z_z = \{ (u, z) \in \mathbb{R} \times \mathbb{R}; \varphi_z(u, z) \neq 0 \} \subset Z = \{ (u, z) \in \mathbb{R} \times \mathbb{R}; \varphi(u, z) \neq 0 \}$. Indeed, $\varphi(u, z) = 0$ implies $\varphi_z(u, z) = 0$, since $\varphi(u, z) \geq 0$. Then we get

$$J_3 = -5 \int_{Z_z} \varphi_z(u, u_x) D_x^2 u D_x^3 u D_x^4 u \, dx$$

$$\leq \frac{1}{4} \int_Z \varphi(u, u_x) (D_x^4 u)^2 \, dx$$

$$+ 25 \int_Z \frac{\varphi_z(u, u_x)^2}{\varphi(u, u_x)} |D_x^2 u|^2 |D_x^3 u|^2 \, dx.$$ 

Here, by assumption (A.2), $\frac{(\varphi_z(u, u_x))^2}{\varphi(u, u_x)} \leq g_1(u, u_x)$, there exists a positive constant $K_1$ depending on $\|u\|_{L^\infty(0,T;L^\infty(\mathbb{R}))}$ and $\|u_x\|_{L^\infty(0,T;L^\infty(\mathbb{R}))}$ such that
\[
\frac{(\varphi_z(u,z))^2}{\varphi(u,z)} \leq g_1(u,u_x)|_{L^\infty(R \times R)} \leq K_1.
\] (16)

Consequently we obtain
\[
J_3 \leq \frac{1}{4} J_0 + 16 K_1 (C_{11})^2 \|D_x^3 u\|_{L^2(R)}^2.
\] (17)

To estimate \(J_4\), we now note that \(Z_{zz}' = \{(u, z) \in R \times R; \varphi_{zz}(u, z)z \neq 0\} \subset Z_z' = \{(u, z) \in R \times R; \varphi_{zz}(u, z) \neq 0\}\). In fact, since \(\varphi_z(u,z)z \geq 0\), \(\varphi_z(u,z)z = 0\) implies that \(\varphi_z(u,z)z = \varphi_z(u,z)z + \varphi_z(u,z) = 0\). Then \(\varphi_z(u,z)z = 0\) or \(z = 0\) gives \(\varphi_{zz}(u, z)z = 0\). Hence we find that \(\varphi_z(u,z)z = 0\) implies \(\varphi_{zz}(u, z)z = 0\). Then we get
\[
J_4 = -3 \int_{Z_{zz}'} \varphi_{zz}(u,u_x) u_x D_x^2 u D_x^4 u \, dx
\leq \frac{1}{4} \int_{Z_z'} \varphi_z(u,u_x) u_x (D_x^4 u)^2 \, dx
+ 9 \int_{Z_z} \frac{(\varphi_{zz}(u,u_x) u_x)^2}{\varphi_z(u,u_x) u_x} \|D_x^2 u\|^2 |D_x^3 u|^2 \, dx.
\]

Here, by assumption (A.2), \(\frac{(\varphi_{zz}(u,z)z)^2}{\varphi_z(u,z)z} \leq g_2(u,u_x)\), there exists a positive constant \(K_2\) depending on \(\|u\|_{L^\infty(0,T_1;L^\infty(R))}\) and \(\|u_x\|_{L^\infty(0,T_1;L^\infty(R))}\) such that
\[
\left| \frac{(\varphi_{zz}(u,z)z)^2}{\varphi_z(u,z)z} \right|_{L^\infty(R \times R)} \leq g_2(u,u_x)|_{L^\infty(R \times R)} \leq K_2.
\] (18)

Consequently we obtain
\[
J_4 \leq \frac{1}{4} J_0 + 9 K_2 (C_{11})^2 \|D_x^3 u\|_{L^2(R)}^2.
\] (19)

Thus, in view of (12), (14) (15), (17) and (19), we derive
\[
\frac{1}{2} \frac{d}{dt} \|D_x^3 u(t)\|^2_{L^2(R)} \leq 2 C_8 C_9 \|D_x^3 u(t)\|_{L^2(R)} + \left( C_8 + C_{10} + (16 K_1 + 9 K_2)(C_{11})^2 \right) \|D_x^3 u(t)\|^2_{L^2(R)},
\]
for all \(t \in [0,T_1]\). (20)
We divide both sides of (20) by \( \|D_x^3 u(t)\|_{L^2(\mathbb{R})} \) to get,

\[
\frac{d}{dt}\|D_x^3 u(t)\|_{L^2(\mathbb{R})} \leq 2C_8C_9 + (C_8 + C_{10} + (16K_1 + 9K_2)(C_{11})^2}\|D_x^3 u(t)\|_{L^2(\mathbb{R})}
\]

Then Gronwall's inequality gives

\[
\sup_{0 \leq t \leq T} \|D_x^3 u(t)\|_{L^2(\mathbb{R})} \leq (C_{12} + \|D_x^3 u_0\|_{L^2(\mathbb{R})})e^{C_{13}T} \tag{21}
\]

Here we note that the constants appeared in this step depend on \( \|u_0\|_{L^\infty(\mathbb{R})}, \|u_0\|_{L^\infty(\mathbb{R})}, \|D_x^2 u_0\|_{L^\infty(\mathbb{R})}, K_1, K_2 \) but not on \( \varepsilon \).

(vi) Estimate for \( \sup_{0 \leq t \leq T_1} \|D_x^4 u(x,t)\|_{L^2(\mathbb{R})} \):

In order to obtain the a priori bound for \( \|D_x^4 u(t)\|_{L^2(\mathbb{R})} \), we multiply \( D_x^3(E)^5 \) by \( D_x^5 u \) and repeat much the same arguments as above to get

\[
\frac{1}{2} \frac{d}{dt}\|D_x^4 u(t)\|_{L^2(\mathbb{R})}^2 = - \int D_x^4 \varphi(u, u_x)u_x D_x^5 u \ dx,
\]

\[
D_x^4 \varphi(u, u_x)u_x = (\varphi + \varphi_x u_x)D_x^5 u + d_0(u, u_x, D_x^2 u) + d_1(u, u_x, D_x^2 u)D_x^3 u + (5\varphi_x(u, u_x) + 3\varphi_{zz} u_x)(D_x^3 u)^2 + (6\varphi_x(u, u_x) + 4\varphi_{zz}(u, u_x)u_x)D_x^2 u D_x^4 u + d_2(u, u_x)D_x^4 u.
\]

So we have

\[
\frac{1}{2} \frac{d}{dt}\|D_x^4 u(t)\|_{L^2(\mathbb{R})}^2 + J_0 = J_1 + J_2 + J_3 + J_4 + J_5 \tag{22}
\]

\[
J_0 = \int (\varphi(u, u_x) + \varphi_x(u, u_x)u_x)(D_x^5 u)^2 \geq 0,
\]

\[
J_1 = - \int d_0(u, u_x, D_x^2 u)D_x^5 u \ dx,
\]

\[
J_2 = - \int d_1(u, u_x, D_x^2 u)D_x^3 u D_x^5 u \ dx,
\]

\[
J_3 = -6 \int \varphi_x(u, u_x)(D_x^3 u)^2 + \frac{5}{6}D_x^2 u D_x^4 u)D_x^5 u \ dx,
\]
\[ J_4 = -4 \int \varphi_{xz} u_x \left( (D_x^3 u)^2 + \frac{3}{4} D_x^2 u T_x D_x^5 u \right) \, dx, \]
\[ J_5 = - \int d_2(u, u_x) D_x^4 u D_x^5 u \, dx. \]

The integration by parts and estimates (2), (5) and (10) yield
\[ J_1 = - \int d_0(u, u_x, D_x^2 u) D_x^5 u \, dx \]
\[ = \int \left( d_{0u}(u, u_x, D_x^2 u) u_x + d_{0x}(u, u_x) D_x u \right) \, dx \]
\[ + \int d_{0v}(u, u_x, D_x^2 u) D_x^3 u \, dx \]
\[ \leq C_{14} \int (|u_x| + |D_x^2 u| + |D_x^3 u|) |D_x^4 u| \, dx \]

Therefore, by virtue of (13) and (21), we get
\[ J_1 \leq 3C_{14}C_{15} \| D_x^4 u(t) \|_{L^2(\Omega)}. \quad (23) \]

Similarly we get
\[ J_2 = - \int d_1(u, u_x, D_x^2 u) D_x^3 u D_x^5 u \, dx \]
\[ = \int \left( d_{1u}(u, u_x, D_x^2 u) u_x + d_{1x}(u, u_x) D_x^2 u \right) \, dx \]
\[ + \int d_{1v}(u, u_x, D_x^2 u) D_x^2 u \, dx \]
\[ \leq C_{16} \int \left( 1 + |D_x^3 u| \right) |D_x^3 u||D_x^4 u| \, dx + C_{17} \int (D_x^4 u)^2 \, dx \]
\[ \leq C_{16} \| D_x^3 u \|_{L^2(\Omega)} \| D_x^4 u \|_{L^2(\Omega)} \]
\[ + C_{16} \| D_x^3 u \|_{L^2(\Omega)}^2 \| D_x^4 u \|_{L^2(\Omega)} + C_{17} \| D_x^4 u \|_{L^2(\Omega)}^2 \]

Here we note
\[ \int (D_x^3 u)^3 D_x^3 u \, dx = \int 3(D_x^3 u)^2 D_x^2 u D_x^2 u \, dx \]
\[ \leq 3 \| D_x^2 u \|_{L^\infty(\Omega)} \| D_x^4 u \|_{L^2(\Omega)} \| D_x^3 u \|_{L^4(\Omega)}^2, \]
whence follows
\[ \| D_x^3 u \|_{L^4(\Omega)}^2 \leq 3 \| D_x^2 u \|_{L^\infty(\Omega)} \| D_x^4 u \|_{L^2(\Omega)}. \quad (24) \]
Then we get

\[ J_2 \leq C_{16} \| D_x^3 u \|_{L^2(R)} \| D_x^4 u \|_{L^2(R)} + \left( 3C_{16} \| D_x^2 u \|_{L^\infty(R)} + C_{17} \right) \| D_x^4 u \|_{L^2(R)}^2. \]  

(25)

Using again the fact that \( Z_z \subset Z \), we now have

\[ J_3 = -6 \int_{Z_z} \varphi_z(u, u_x) \left( \frac{5}{6} (D_x^3 u)^2 + D_x^2 u D_x^4 u \right) D_x^5 u \, dx \]

\[ \leq \frac{1}{2} \int_{Z} \varphi(u, u_x) (D_x^5 u)^2 \, dx \]

\[ + 36 \int_{Z} \frac{\varphi_z(u, u_x)^2}{\varphi(u, u_x)} \left( |D_x^3 u|^4 + |D_x^2 u|^2 |D_x^4 u|^2 \right) \, dx. \]

Hence by (16) and (24), we get

\[ J_3 \leq \frac{1}{2} J_0 + 360K_1 \| D_x^2 u \|_{L^\infty(R)} \cdot \| D_x^4 u \|_{L^2(R)}^2 \]

\[ \leq \frac{1}{2} J_0 + C_{18} \| D_x^4 u \|_{L^2(R)}^2. \]  

(26)

In order to estimate \( J_4 \), recall the fact that \( Z_{zz} \subset Z'_z \), then we have

\[ J_4 = -4 \int \varphi_{zz} u_x \left( \frac{3}{4} (D_x^3 u)^2 + D_x^2 u D_x^4 u \right) D_x^5 u \, dx \]

\[ \leq \frac{1}{2} \int_{Z} \varphi_z(u, u_x) u_x (D_x^5 u)^2 \, dx \]

\[ + 16 \int_{Z} \frac{\varphi_{zz}(u, u_x u_x)^2}{\varphi_z(u, u_x) u_x} \left( |D_x^3 u|^4 + |D_x^2 u|^2 |D_x^4 u|^2 \right) \, dx. \]

Hence by (18) and (24), we get

\[ J_4 \leq \frac{1}{2} J_0 + 160K_2 \| D_x^2 u \|_{L^\infty(R)}^2 \| D_x^4 u \|_{L^2(R)}^2 \]

\[ \leq \frac{1}{2} J_0 + C_{19} \| D_x^4 u \|_{L^2(R)}^2. \]  

(27)

As for the last term \( J_5 \), the integration by parts gives

\[ J_5 = - \int d_z(u, u_x) D_x^4 u D_x^5 u \, dx \]
\[
\begin{align*}
&= - \int d_2(u, u_x) \cdot \frac{1}{2} \left((D_x^4 u)^2 \right)_x \, dx \\
&= \frac{1}{2} \int \left( d_{2u}(u, u_x) u_x + d_{2z}(u, u_x) D_x^2 u \right)(D_x^4 u)^2 \, dx \\
&\leq C_{20} \|D_x^4 u\|_{L^2(\mathbb{R})}^2 
\end{align*}
\]

Thus, in view of (22),(23),(25) (26), (27) and (28), we derive
\[
\frac{1}{2} \frac{d}{dt} \|D_x^4 u(t)\|_{L^2(\mathbb{R})}^2 \leq C_{21} \|D_x^4 u(t)\|_{L^2(\mathbb{R})} + C_{22} \|D_x^4 u(t)\|_{L^2(\mathbb{R})}^2, \\
\text{for all } t \in [0,T_1].
\]

Consequently, by Gronwall’s inequality, we obtain
\[
\sup_{0 \leq t \leq T} \|D_x^4 u(t)\|_{L^2(\mathbb{R})} \leq (C_{21} + \|D_x^4 u_0\|_{L^2(\mathbb{R})}) e^{C_{22} T}
\]

Here we note that the constants appeared in this step depend only on the initial data and so on but not \( \varepsilon \).

(vii) Estimate for \( \|D^m u(x, t)\|_{L^\infty} \) (\( m \geq 4 \)):

The basic method to establish a priori bounds for \( \|D^m u(x, t)\|_{L^\infty} \) with \( m \geq 4 \) is essentially the same as that for the case \( m = 4 \). So we here show how to derive the a priori bound only for the case \( m = 4 \).

First we note that \( D_x^3 u \) gives
\[
(D_x^3 u)_t = \left( \varphi + \varphi_x u_x \right) D_x^5 u + e_1(u, u_x, D_x^2 u, D_x^3 u) + e_2(u, u_x, D_x^2 u) D_x^4 u
\]

Multiplying this by \(-\Delta \left( D_x^3 u \right) = -(|D_x^4 u|^{r-2} D_x^4 u)_x \), we obtain
\[
\frac{d}{dt} \|D_x^4 u(t)\|_{L^r(\mathbb{R})}^{r-1} \|D_x^4 u(t)\|_{L^{r}(\mathbb{R})} \\
= J_0 + J_1 + J_2,
\]

\[
J_0 = (r-1) \int \left( \varphi(u, u_x) + \varphi_x(u, u_x) u_x \right) |D_x^4 u|^{r-2} (D_x^5 u)^2 \, dx \geq 0,
\]

\[
J_1 = - \int e_1(u, u_x, D_x^2 u, D_x^3 u) (|D_x^4 u|^{r-2} D_x^4 u)_x \, dx,
\]

\[
J_2 = - \int e_2(u, u_x, D_x^2 u) D_x^4 u (|D_x^4 u|^{r-2} D_x^4 u)_x \, dx.
\]
By the integration by parts, we get

\[ J_1 = \int \left( e_{1u}(u, u_x, D_x^2 u, D_x^3 u) u_x + e_{1z}(u, u_x, D_x^2 u, D_x^3 u) D_x^2 u \
+ e_{1u}(u, u_x, D_x^2 u, D_x^3 u) D_x^3 u \
+ e_{1w}(u, u_x, D_x^2 u, D_x^3 u) D_x^4 u \right) |D_x^4 u|^{-2} D_x^4 u \, dx \]

\[ \leq C_{23} \left( |u_x| + |D_x^2 u| + |D_x^3 u| + |D_x^4 u| \right) |D_x^4 u|^{-1} \, dx. \]

\[ \leq C_{23} \left( \|u_x\|_{L^r(R)} + \|D_x^2 u\|_{L^r(R)} + \|D_x^3 u\|_{L^r(R)} \right) \|D_x^4 u\|_{L^r(R)}^{-1} 
+ C_{23} \|D_x^4 u\|_{L^r(R)}. \quad (32) \]

\[ J_2 = -\int e_2(u, u_x, D_x^2 u) \cdot \frac{r-1}{r} |D_x^4 u|^r \, dx. \]

\[ = -\frac{r-1}{r} \int e_2(u, u_x, D_x^2 u) |D_x^4 u|^r \, dx. \]

\[ = \int \left( e_{2u}(u, u_x, D_x^2 u) u_x + e_{2z}(u, u_x, D_x^2 u) D_x^2 u \
+ e_{2w}(u, u_x, D_x^2 u) D_x^3 u \right) D_x^4 u \, dx \]

\[ \leq C_{24} \left( |u_x| + |D_x^2 u| + |D_x^3 u| \right) |D_x^4 u|^{-1} \, dx. \]

\[ \leq C_{24} \left( \|u_x\|_{L^\infty(R)} + \|D_x^2 u\|_{L^\infty(R)} + \|D_x^3 u\|_{L^\infty(R)} \right) \|D_x^4 u\|_{L^r(R)}. \quad (33) \]

Since \( \|v\|_{L^r} \leq \|v\|_{L^2}^{\frac{2}{r-2}} \|v\|_{L^\infty}^{\frac{r-2}{r}} \), (31) – (33) together with (2), (5), (10) and (13) give

\[ \frac{d}{dt} \|D_x^4 u(t)\|_{L^r(R)} \leq C_{25} \left( 1 + \|D_x^4 u\|_{L^r(R)} \right). \]

Hence, by Gronwall’s inequality, we get

\[ \sup_{0 \leq t \leq T_1} \|D_x^4 u(t)\|_{L^r(R)} \leq (C_{25} T + \sup_{2 \leq r \leq \infty} \|D_x^4 u_0\|_{L^r(R)}) e^{C_{25} T}. \quad (34) \]

Here we note that the constants depend only on the initial data but not on \( \varepsilon \). Thus a priori estimates for solutions of approximate equations independent of \( \varepsilon \) are established. To complete the proof of Theorem,
we have only to apply standard arguments for convergence.

We can derive the local estimate for \( \|D^3_x u(x, t)\|_{L^\infty(\mathbb{R} \times [0, T_1])} \) without assuming (A.2) as follows.

(viii) Local estimate for \( \|D^3_x u(x, t)\|_{L^\infty(\mathbb{R} \times [0, T_1])} \):

In order to derive the a priori bound for \( \|D^3_x u(x, t)\|_{L^\infty} \), we differentiate \((E)^\varepsilon\) twice with respect to x-variable to obtain \( D^2_x(E)^\varepsilon \), and multiply \( D^2_x(E)^\varepsilon \) by \( -\Delta_x D^3_x u = -(|D^3_x u|^{-2}D^3_x u)_x \). Then we get

\[
\frac{1}{r} \frac{d}{dt} \|D^3_x u(t)\|_{L^r(\mathbb{R})} = -\int D^3_x (\varphi(u, u_x)u_x)(|D^3_x u|^{-2}D^3_x u)_x \, dx,
\]

\[
D^3_x (\varphi(u, u_x)u_x) = (\varphi + \varphi_x u_x)D^4_x u + (5\varphi_x + 3\varphi_{xx} u)D^2_x u D^3_x u + b_1(u, u_x, D^2_x u) + b_2(u, u_x)D^3_x u.
\]

So we have

\[
\frac{1}{r} \frac{d}{dt} \|D^3_x u(t)\|_{L^r(\mathbb{R})} + J_0 = J_1 + J_2
\]

\[
J_0 = \int (r - 1)(\varphi(u, u_x) + \varphi_x(u, u_x)u_x)\|D^3_x u\|^{-2}(D^4_x u)^2 \geq 0,
\]

\[
J_1 = -\int b_1(u, u_x)(|D^3_x u|^{-2}D^3_x u)_x \, dx,
\]

\[
J_2 = -\int b_2(u, u_x)D^3_x u(|D^3_x u|^{-2}D^3_x u)_x \, dx,
\]

\[
J_3 = -\int (5\varphi_x + 3\varphi_{xx} u_x)u_x D^2_x u D^3_x u(|D^3_x u|^{-2}D^3_x u)_x \, dx,
\]

By virtue of (2), (5) and (10), it is easy to see that

\[
J_1 = \int (b_{1u}(u, u_x)u_x + b_{1x}(u, u_x)D^2_x u)|D^3_x u|^{-2}D^3_x u \, dx \leq C_{27} \int (|u_x| + |D^2_x u|)|D^3_x u|^{-1} \, dx \leq C_{27} \left( |u_x|_{L^r(\mathbb{R})} + |D^2_x u|_{L^r(\mathbb{R})} \right) |D^3_x u|^{-1}_{L^r(\mathbb{R})},
\]

(36)
\[ J_2 = - \int b_2(u, u_x) \cdot \frac{r - 1}{r} (|D_x^3 u|^r)_x \, dx \]
\[ = \frac{r - 1}{r} \int \left( b_{2u}(u, u_x)u_x + b_{1z}(u, u_x)D_x^2 u \right) |D_x^3 u|^r \, dx \]
\[ \leq C_{28} |D_x^3 u|_{L^r(R)}, \quad (37) \]

and
\[ J_3 = - \int \left( 5 \varphi_x + 3 \varphi_{zz} u_x \right) D_x^2 u \cdot \frac{r - 1}{r} (|D_x^3 u|^r)_x \, dx \]
\[ = \frac{5(r - 1)}{r} \int \left( \varphi_{uz}(u, u_x)u_x + \varphi_{zz}(u, u_x)D_x^2 u \right) D_x^2 u |D_x^3 u|^r \, dx \]
\[ + \frac{3(r - 1)}{r} \int \left( (\varphi_{u_{zzz}}(u, u_x)u_x + \varphi_{zzz}(u, u_x)D_x^2 u)u_x D_x^2 u \right) \]
\[ + \varphi_{zz}(u, u_x)(D_x^2 u)^2 |D_x^3 u|^r \, dx \]
\[ \leq C_{29} \int |D_x^3 u|^r \, dx + C_{30} \int |D_x^3 u|^{r+1} \, dx \]
\[ \leq C_{29} |D_x^3 u|_{L^r(R)} + C_{30} |D_x^3 u|_{L^r(R)} |D_x^3 u|_{L^r(R)}, \quad (38) \]

Thus, in view of (35), (36), (37) and (38), we derive
\[ \|D_x^3 u(t)\|_{L^r(R)}^{r-1} \frac{d}{dt} \|D_x^3 u(t)\|_{L^r(R)} \]
\[ \leq C_{27} \left( |u_x|_{L^r(R)} + |D_x^2 u|_{L^r(R)} \right) \|D_x^3 u(t)\|_{L^r(R)}^{r-1} \]
\[ + (C_{28} + C_{29} + C_{30} |D_x^3 u|_{L^r(R)}) \|D_x^3 u(t)\|_{L^r(R)} \]

By dividing both hands side by \( \|D_x^3 u(t)\|_{L^r(R)}^{r-1} \), we get
\[ \frac{d}{dt} \|D_x^3 u(t)\|_{L^r(R)} \]
\[ \leq C_{27} \left( |u_x|_{L^r(R)} + |D_x^2 u|_{L^r(R)} \right) \]
\[ + (C_{28} + C_{29} + C_{30} |D_x^3 u|_{L^r(R)}) \|D_x^3 u(t)\|_{L^r(R)} \]

Hence letting \( r \to \infty \), we can obtain
\[ \|D_x^3 u(t)\|_{L^\infty(R)} \]
\[ \leq \|D_x^3 u_0\|_{L^\infty(R)} + C_{31} \int_0^t \|D_x^3 u(s)\|_{L^\infty(R)} + \|D_x^3 u(s)\|_{L^\infty(R)}^2 \, ds \]
for all \( t \in [0, T_1] \).
Here we note that the constants in this step depend only on the initial data but not on $\varepsilon$.
Then it is easy to show that there exists a positive number $T_2 = T_2(u_0\|_{L^\infty(\mathbb{R})}, \|u_{0x}\|_{L^\infty(\mathbb{R})}, \|D_x^2u_0\|_{L^\infty(\mathbb{R})}, \|D_x^3u_0\|_{L^\infty(\mathbb{R})}) \leq T_1$ such that
\[
\|D_x^3u(t)\|_{L^\infty(\mathbb{R})} \leq 2\|D_x^3u_0\|_{L^\infty(\mathbb{R})} \quad \text{for all } t \in [0,T_2].
\]

**Concluding remarks**

(1) Our arguments here can work also for initial boundary value problems for the same generalized porous medium equations. In fact, for Neumann problems with the homogeneous Neumann condition $u(x,t) = 0$ on the boundary $\partial I$, the same assertion of Theorem holds good, provided that the initial data satisfies the compatibility condition:
\[
D_x^{2k-1}u_0(x,t) \mid_{\partial I} = 0, \quad \text{for } k = 0,1,2,\ldots
\]
and $\varphi(u,u_x)$ satisfies the additional assumption
\[
D^i_xD_x^{2j+1}\varphi(\cdot,0) = 0, \quad \text{for all } i,j = 0,1,2,\ldots.
\]

As for the Dirichlet problems with the homogeneous Dirichlet boundary condition, in order to derive the same assertion of Theorem, we need to assume the additional assumption on $\varphi(u,u_x)$ such that
\[
D_x^{2i+1}D^j_x\varphi(0,\cdot) = 0, \quad \text{for all } i,j = 0,1,2,\ldots,
\]
together with the compatibility condition on $u_0$:
\[
D_x^{2k}u_0(x,t) \mid_{\partial I} = 0, \quad \text{for all } k = 0,1,2,\ldots.
\]

(2) With slight modifications, our argument works also for the multi-dimension case. However we need much more heavy calculations than those exploited here.

(3) The following perturbation problem can be also treated within our framework.
(GP) \[
\begin{align*}
\frac{\partial u}{\partial t}(x,t) &= (\varphi(u, u_x)u_x)_x + f(u, u_x), \quad (x, t) \in \mathbb{R} \times [0, T], \\
u(x, 0) &= u_0(x), \quad x \in \mathbb{R},
\end{align*}
\]
under the following condition: There exist functions $d_1(u, z), d_2(u, z) \in L^\infty_{loc}(\mathbb{R} \times \mathbb{R})$ such that
\[
\left|\frac{f_z(u, z)}{\varphi(u, z)}\right| \leq d_1(u, z), \quad \left|\frac{f_{zz}(u, z)}{\varphi(u, z)}\right| \leq d_2(u, z).
\]

References


[6] H.Brezis, Monotonicity methods in Hilbert spaces and some applications to nonlinear partial differential equation