数理解析研究所講究録
1135 巻 2000 年 1-14

Blow-up Criteria for Semilinear Parabolic Equations

ATSUHITO KOHDA 香田温人

Department of Mathematics, Faculty of Engineering, Tokushima University, Tokushima 770-8506 Japan

TAKASHI SUZUKI 鈴木貴

Department of Mathematics, Graduate School of Science, Osaka University, Toyonaka 560-0043 Japan

1 Introduction

Given a bounded domain $\Omega \subset \mathcal{R}^n$ with smooth boundary $\partial\Omega$, let us consider the initial boundary value problem

$$\frac{\partial u}{\partial t} - \Delta u = f(u) \quad \text{in} \quad \Omega \times (0,T), \qquad u|_{\partial\Omega} = 0, \qquad u|_{t=0} = u_0(x) \quad (1)$$

with $f \in C^1(\mathcal{R})$ standing for the nonlinearity in consideration.

If the initial value $u_0 \in C_0(\overline{\Omega})$, which means that $u_0(x)$ is continuous on $\overline{\Omega}$ and $u_0 = 0$ on $\partial\Omega$, then it holds the unique existence of the classical solution local in time $u = u(x,t) \in C(\overline{\Omega} \times [0,T)) \cap C^{2,1}(\Omega \times (0,T))$. When only $u_0 \in C(\overline{\Omega})$ is assumed, we still have the unique existence of the solution local in time $u = u(x,t) \in C(\overline{\Omega} \times (0,T)) \cap C^{2,1}(\Omega \times (0,T))$, and

$$\lim_{t\downarrow 0} \|u(\cdot,t) - u_0\|_p = 0$$

for any $1 \leq p < +\infty$. In any case, if we denote by T_b the maximal time for the existence of such a solution, then $T_b < +\infty$ implies

$$\lim_{t\uparrow T_b} \|u(\cdot,t)\|_\infty = +\infty.$$

And we call this case the blow-up of the solution. We refer to Ladyzenskaya, Solonnikov, and Ural'ceva [11], Matano [12], and Henry [8] for those fundamental facts.

The blow-up phenomena have been studied extensively; when and how they occur, and what happens after the blow-up time. The present paper is devoted to the first problem and we give a new criterion for the blow-up of the solution.

As a typical nonlinearity we think of $f(u) = \lambda_0 e^u$ with a constant $\lambda_0 > 0$. In this case if the stationary problem

$$-\Delta v = f(v) \qquad \text{in} \quad \Omega, \qquad v = 0 \qquad \text{on} \quad \partial\Omega \tag{2}$$

has a classical solution $v \in C_0(\overline{\Omega}) \cap C^2(\Omega)$ then S, the totality of its solutions, possesses the minimal element $\underline{v}$. Namely, $\underline{v} \in S$ and $v \geq \underline{v}$ on Ω for any $v \in S$.

In the pioneering work [6], H. Fujita proved the following. When a non-minimal stationary solution $\overline{v}$ of (2) exists then we have;

1. If $u_0 \leq \overline{v}$ and $u_0 \not\equiv \overline{v}$, then $T_b = +\infty$ and $\lim_{t\to+\infty} \|u(\cdot, t) - \underline{v}\|_\infty = 0$.
2. If $u_0 \geq \overline{v}$ and $u_0 \not\equiv \overline{v}$, then either $T_b < +\infty$ or

$$T_b = +\infty \qquad \text{and} \qquad \lim_{t\to+\infty} \|u(\cdot, t)\|_\infty = +\infty. \tag{3}$$

In the above arguments, the convexity of the nonlinearity f plays a crucial role, for this does not admit a triple of classical stationary solutions, $u, v, w \in S$ with $u \leq v \leq w$ and $u \not\equiv v \not\equiv w$. We call this the triple law.

The second statement above, in the case $u_0 \geq \overline{v}$ and $u_0 \not\equiv \overline{v}$, was refined later by Lacey [9] as follows. Let $\psi_1(x) > 0$ be the first eigenfunction of the linearized operator $-\Delta - f'(\overline{v}(x))$ around the non-minimal solution $\overline{v}$. Then for u_0 with

$$u_0 \not\equiv \overline{v} \qquad \text{and} \qquad \int_\Omega u_0 \psi_1 \geq \int_\Omega \overline{v} \psi_1$$

we have $T_b < +\infty$. In other words, the possibility (3), usually referred to as the blow-up in infinite time, is excluded, and also the initial value $u_0(x)$ may even intersect $\overline{v}(x)$ as long as the above integral inequality holds.

In this paper we will show another conditions extended in different direction. Namely, we can take v^* and v_* in place of $\underline{v}$ and $\overline{v}$, where v^* and v_* is *super*- and *sub*-solution respectively. This means that

$$-\Delta v^* \geq f(v^*) \quad \text{and} \quad -\Delta v_* \leq f(v_*) \qquad \text{in} \quad \Omega$$

and

$$v^* \geq 0 \geq v_* \qquad \text{on} \quad \partial\Omega. \tag{4}$$

Let $\lambda_1 > 0$ be the first eigenvalue of $-\Delta$.

Theorem 1 *Suppose that the nonlinearity $f \in C^1(\mathcal{R})$ is convex,*

$$\limsup_{s \to -\infty} f(s)/s < \lambda_1 < \liminf_{s \to +\infty} f(s)/s, \tag{5}$$

and

$$\int^{+\infty} \frac{ds}{f(s)} < +\infty. \tag{6}$$

Suppose, furthermore, that there exists a pair of super- and sub- solutions $v^*, v_* \in C(\overline{\Omega}) \cap C^2(\Omega)$ of (2), respectively, with

$$v^* \leq v_* \qquad \textit{and} \qquad v^* \not\equiv v_* \qquad \textit{in} \quad \Omega. \tag{7}$$

Then, for u_0 with

$$u_0 \geq v_* \quad \textit{and} \quad u_0 \not\equiv v_* \qquad \textit{in} \quad \Omega \tag{8}$$

we have $T_b < +\infty$, and actually

$$\lim_{t \uparrow T_b} \max_{\overline{\Omega}} u(\cdot, t) = +\infty. \tag{9}$$

Note that relations (4) and (7) imply $v^* = v_* = 0$ on $\partial\Omega$, or $v^*, v_* \in C_0(\overline{\Omega})$.

2 Applications

Theorem 1 provides the following blow-up criteria which have not been noticed before. Throught this section, we assume that f satisfies the assumptions of Theorem 1.

Corollary 2 *Let $f(0) \leq 0$. Suppose, furthermore, that the initial value $u_0 \in C_0(\overline{\Omega})$ is non-negative, C^2 in Ω, and*

$$-\Delta u_0 \leq f(u_0) \quad \textit{and} \quad -\Delta u_0 \not\equiv f(u_0) \qquad \textit{in} \quad \Omega. \tag{10}$$

Then we have $T_b < +\infty$.

In fact, from condition (10) and the strong maximum principle we see

$$u_t > 0, \quad -\Delta u < f(u) \qquad \text{in} \quad \Omega \times (0, T_b). \tag{11}$$

In particular, $u(\cdot, t_0) \geq u_0$ and $u(\cdot, t_0) \not\equiv u_0$ hold for $0 < t_0 < T_b$. Therefore, by Theorem 1 with $v^* = 0$ and $v_* = u_0$, regarding t_0 as the initial time, we can show the conclusion.

We note that Friedman-McLeod [5] studied the blow-up set for a rather wide class of nonlinearities, under the conditions (10) *and* $T_b < +\infty$. Above Corollary 2 provides a kind of justification for it.

On the contrary, in case of $f(0) > 0$, it may happen that $T_b = +\infty$ in spite of (10). This is the case actually shown in [6] for $f(u) = \lambda_0 e^u$. Namely, if a non-minimal stationary solution $\overline{v}(x)$ of (2) exists, then the extrapolation of $\underline{v}$ and $\overline{v}$,

$$u_0 = \theta \underline{v} + (1 - \theta)\overline{v}$$

with $\theta > 1$ satisfies (10), $u_0 > 0$ in Ω, $T_b = +\infty$, and

$$\lim_{t \to +\infty} \|u(\cdot, t) - \underline{v}\|_\infty = 0.$$

For the nonlinearity $f(u) = \lambda e^u$ we have the upper bound $\overline{\lambda} < +\infty$ of λ for which the existence of a classical solution $v(x)$ of

$$-\Delta v = f(v) \quad \text{in} \quad \Omega, \qquad v = 0 \quad \text{on} \quad \partial\Omega \tag{12}$$

holds. For the case $\lambda > \overline{\lambda}$, [6] proved that the blow-up occurs in finite or infinite time in

$$\frac{\partial u}{\partial t} - \Delta u = f(u) \quad \text{in} \quad \Omega \times (0, T), \qquad u|_{\partial\Omega} = 0, \qquad u|_{t=0} = u_0(x), \tag{13}$$

and later [9] excluded the second possibility when $\overline{\lambda}$ lies in the spectrum. This fact holds for some class of nonlinearities including $f(u) = \lambda e^u$. This case was later studied by Bellout [3], Lacey and Tzanetis [10], and Brezis, Cazenave, Martel, and Ramiandrisoa [4]. Bellout [3] showed that the fact proven by [9] holds even when $\overline{\lambda}$ does not lie in the spectrum. On the other hand [10] showed the blow-up of infinite time may occur when $\lambda = \overline{\lambda}$. These results are refined recently by [4]. In particular it was proven the following: *If C^1 convex nonlinearity f satisfies (6), $f(0) > 0$, and $f \not\equiv f(0)$, then $\overline{\lambda} < +\infty$ follows. Furthermore, blow-up in finite time occurs in (13) whenever $\lambda > \overline{\lambda}$. and $u_0 \geq 0$.* Summing up these results, we see that blow-up in finite time always occurs for $f(u) = \lambda e^u$ with $\lambda > \overline{\lambda}$ in (13). In contrast with this, the following corollary presents a blow-up criterion for the case $\lambda \leq \overline{\lambda}$.

Corollary 3 *Suppose that the stationary problem (2) has a non-minimal degenerate solution $\overline{v}$, that is,*

$$-\Delta\psi = f'(\overline{v})\psi \qquad \text{in} \quad \Omega, \qquad \psi|_{\partial\Omega} = 0 \tag{14}$$

has a non-trivial solution $\psi \not\equiv 0$ with sign change. Then for

$$u_0 \geq \overline{v} \pm \epsilon\psi \qquad \text{and} \qquad 0 < \epsilon \ll 1 \tag{15}$$

we have $T_b < +\infty$.

In fact, we have

$$-\Delta(\overline{v} \pm \epsilon\psi) = -\Delta\overline{v} \mp \epsilon\Delta\psi = f(\overline{v}) \pm \epsilon f'(\overline{v})\psi \tag{16}$$

and

$$f(\overline{v} \pm \epsilon\psi) \geq f(\overline{v}) \pm \epsilon f'(\overline{v})\psi. \tag{17}$$

Hence $-\Delta(\overline{v} \pm \epsilon\psi) \leq f(\overline{v} \pm \epsilon\psi)$ holds and we can apply Theorem 1 with the minimal solution $v^* = \underline{v}$ and $v_* = \overline{v} + \epsilon\psi$.

Note that in Corollary 3, initial value u_0 may intersect with $\overline{v}$. Lacey [9] treats the similar case as above corollary. His blow-up criterion is, however, different from ours.

Another application is the following.

Corollary 4 *Suppose that the stationary problem (2) has the minimal degenerate solution $\underline{v}$, that is,*

$$-\Delta\psi = f'(\underline{v})\psi \qquad \text{in} \quad \Omega, \qquad \psi|_{\partial\Omega} = 0 \tag{18}$$

has a non-trivial solution $\psi > 0$. *Then for*

$$u_0 \geq \underline{v} \quad \textit{with} \quad u_0 \not\equiv \underline{v} \tag{19}$$

we have $T_b < +\infty$.

In fact, the argument to be presented at the begining of the next section reduces condition (19) to

$$u_0 \geq v_* = \underline{v} + \epsilon\psi \tag{20}$$

for some $0 < \epsilon \ll 1$. Then we obtain the conclusion with the same arguments as above.

As a direct consequence of the corollary above, we obtain the following.

Proposition 5 *In Corollary 4,* $\underline{v}(x)$ *is the unique solution for the stationary problem (2).*

In fact, if there exist non-minimal solution $\overline{v}$ of (2) then we have some u_0 with $\overline{v} > u_0 > \underline{v}$ in Ω. Because $\overline{v}$ is a stationary solution, this implies $T_b = +\infty$ for $u_0 > \underline{v}$ and contradicts with Corollary 4.

The case treated in Corollary 3 or 4 occurs for $f(u) = \lambda e^u$, $\Omega = \{x \in \mathcal{R}^n \mid |x| < 1\}$, and $2 < n < 10$. See Nagasaki and Suzuki [13], for instance.

Another application of Theorem 1 is the following.

Corollary 6 *Let* $f(0) = 0$ *and* $f'(0) > \lambda_1$ *where* λ_1 *is the first eigenvalue of* $-\Delta$. *Suppose, furthermore, that the initial value* $u_0 \in C_0(\overline{\Omega}) \cap C^2(\Omega)$ *is non-negative. Then we have* $T_b < +\infty$.

In fact, let $\phi_1 > 0$ be the eigenfunction satisfying $-\Delta\phi_1 = \lambda_1\phi_1$ in Ω and $\phi_1 = 0$ on $\partial\Omega$. Because $f(u)$ is convex, the assumption $f'(0) > \lambda_1$ implies $f(s) < \lambda_1 s$ for $s < 0$. Now set $v^* = -\epsilon\phi_1$ for $\epsilon > 0$ then we have

$$-\Delta v^* = \epsilon\Delta\phi_1 = -\epsilon\lambda_1\phi_1 = \lambda_1 v^* > f(v^*). \tag{21}$$

That is, $v^* = -\epsilon\phi_1$ is a super-solution of (2), so we can apply Theorem 1 with $v^* = -\epsilon\phi_1$ and $v_* \equiv 0$ to obtain the conclusion.

Finally, we note that any interpolations and extrapolations of sub- and super-solutions are also sub- and super-solutions, respectively. For instance we have the following.

Corollary 7 *Let the stationary problem (2) has the minimal solution* $\underline{v} = v_1$ *and non-minimal solutions* v_2, v_3. *Then if* $u_0 \geq v^* = \alpha v_2 + (1-\alpha)v_3$ *for some* $\alpha > 1$ *or* $\alpha < 0$ *we have* $T_b < +\infty$.

3 Proof of Theorem 1

Let (7) and (8) hold. If we take $u_*(x,t)$ and $u^*(x,t)$ to be the solutions local in time of (1) for $u_0 = v_*$ and $u_0 = v^*$, respectively, then by the strong maximum principle and the Hopf lemma we have

$$u(\cdot, t_0) \gg u_*(\cdot, t_0) \gg u^*(\cdot, t_0)$$

for $0 < t_0 \ll 1$. This means that these functions are C^1 on $\overline{\Omega}$ and satisfy

$$u(\cdot, t_0) > u_*(\cdot, t_0) > u^*(\cdot, t_0) \qquad \text{in} \quad \Omega$$

and

$$\frac{\partial u}{\partial \nu}(\cdot, t_0) < \frac{\partial u_*}{\partial \nu}(\cdot, t_0) < \frac{\partial u^*}{\partial \nu}(\cdot, t_0) \qquad \text{on} \quad \partial\Omega,$$

where ν denotes the outer unit normal vector. Furthermore (11) holds for $v(x,t) = u_*(x,t)$, in which case we say that $u_*(\cdot, t_0)$ is a strict sub-solution of (2). Similarly, $u^*(\cdot, t_0)$ is a strict super-solution. Therefore, we may assume from the begining that v^* and v_* are strict super- and sub- solutions of (2), respectively, that $u_0(x)$, $v_*(x)$ and $v^*(x)$ are C^1 functions on $\overline{\Omega}$, and that they satisfy

$$u_0 \gg v_* \gg v^*. \tag{22}$$

From (22) we can take a constant $\theta > 1$ such that

$$u_0 \gg \theta v_* + (1-\theta)v^*. \tag{23}$$

Therefore, using the comparison principle, we can reduce the theorem to the case that u_0 is the extrapolation of v_* and v^*, that is, the right-hand side of (23) with $\theta > 1$. In this case u_0 becomes again a strict sub-solution of (2) from the convexity of f.

From (5), there exist constants $\mu < \lambda_1$ and $C > 0$ such that

$$f(s) > \mu s - C \qquad \text{for} \quad s \le 0.$$

Take $\lambda \ge 1$ and denote by w_λ the solution of

$$(-\Delta - \mu)w_\lambda = -\lambda C \quad \text{in} \quad \Omega, \qquad w_\lambda = 0 \quad \text{on} \quad \partial\Omega.$$

A simple calculation shows that $w_\lambda(x)$ is a sub-solution of (2). Furthermore, taking λ large enough, we have $v^* \gg w_\lambda$. Then, by the method of super-sub solutions ([1], [2], e.g.) we have a solution $\tilde{v}(x)$ of (2) satisfying $w_\lambda \leq \tilde{v} \leq v^*$. In other words, we may suppose $v^*(x)$ is a stationary solution.

Under these circumstances, because $u_0 \in C^2(\Omega) \cap C^1(\overline{\Omega})$ is a strict sub-solution of (2), $u(x,t)$ is increasing in t for each $x \in \Omega$. Therefore, we have a measurable function $v(x)$

$$\lim_{t \to +\infty} u(x,t) = v(x) \in [v^*(x), +\infty] \qquad \text{for} \quad x \in \overline{\Omega} \tag{24}$$

if we assume $T_b = +\infty$.

What we are trying to show is that this function $v(x)$ must be a (singular) stationary solution satisfying $v \gg v^*$. Because it is stable from below, the third solution, unstable from both above and below, probably exists between v and v^*. But this will violate the triple law. In the present paper, however, we do a different argument based on the parabolic dynamics. In this way, we also provides a proof of the triple law involving singular stationary solutions, avoiding technical difficulties in treating singularity.

First, from (5), we have a constant j_* such that

$$f(s) - \lambda_1 s > 0 \qquad \text{for} \quad s > j_*. \tag{25}$$

Let $\phi_1(x) > 0$ be the first eigenfunction of $-\Delta$ normalized as

$$\int_\Omega \phi_1(x)dx = 1.$$

We can deduce

$$\int_\Omega u(x,t)\phi_1(x)dx \leq j_* \qquad (t \geq 0) \tag{26}$$

if $T_b = +\infty$ holds (c.f. [7]).

In fact, the function

$$j(t) = \int_\Omega u(x,t)\phi_1(x)dx$$

satisfies

$$\frac{dj}{dt} + \lambda_1 j \geq f(j) \qquad (t \geq 0)$$

because of the convexity of f and Jensen's inequality. Therefore, if $j(0) > j_*$, then $j(t) > j_*$ by (25). Hence it holds

$$\int_{j_*}^{+\infty} \frac{dj}{f(j) - \lambda_1 j} \geq \int_0^{+\infty} dt = +\infty$$

and this contradicts (6). This means that $T_b = +\infty$ implies $j(0) \le j_*$. By translating the initial time, we get the conclusion.

In use of the monotone convergence theorem we get from (26) that

$$\int_\Omega v(x)\phi_1(x)dx \le j_* \qquad \text{and} \qquad v(x) < +\infty \quad \text{a.e.} \quad x \in \Omega. \tag{27}$$

Let $-M = \min_{\overline{\Omega}} u_0$. Then $u(x,t) \ge -M$ holds on $\overline{\Omega} \times [0, +\infty)$. Because (5) implies $f' > 0$ at $+\infty$, there exists some $\gamma \in \mathcal{R}$ such that

$$s \in [-M, +\infty) \mapsto f(s) + \gamma s \quad \text{is non-decreasing.}$$

We may suppose that $\gamma \ge 0$, so $-\Delta_\gamma \equiv -\Delta + \gamma$ is invertible. In terms of the fundamental solution $\{U_\gamma(x,y;t)\}$ of $\partial_t - \Delta_\gamma$, we obtain from Duhamel's principle that

$$u(x,t) = \int_\Omega U_\gamma(x,y;t)u_0(y)dy + \int_0^t ds \int_\Omega U_\gamma(x,y;s) f_\gamma\left(u(y,t-s)\right)dy, \tag{28}$$

where $f_\gamma(s) = f(s) + \gamma s$. Again by (24) we have

$$v(x) = \int_0^\infty ds \int_\Omega U_\gamma(x,y;s) f_\gamma\left(v(y)\right)dy,$$

or more precisely,

$$\begin{aligned} v(x) &= \int_0^\infty \int_\Omega U_\gamma(x,y;s)\left[f_\gamma\left(v(y)\right) - f_\gamma\left(v^*(y)\right)\right]dyds \\ &\quad + \int_0^\infty \int_\Omega U_\gamma(x,y;s) f_\gamma\left(v^*(y)\right)dyds. \end{aligned} \tag{29}$$

To see this, let $G_\gamma(x,y)$ be the Green's function of $-\Delta_\gamma$. Then

$$U_\gamma(x,y;t) > 0, \qquad \int_\Omega G_\gamma(x,y)dy < +\infty,$$

and

$$G_\gamma(x,y) = \int_0^\infty U_\gamma(x,y;s)ds \tag{30}$$

hold. In particular we have

$$U_\gamma(x,\cdot;\cdot) \in L^1\left(\Omega \times (0,+\infty)\right) \tag{31}$$

for any $x \in \Omega$.

We write the second term of the right-hand side of (28) as

$$\int_0^\infty \int_\Omega U_\gamma(x,y;s)\left[f_\gamma\left(u(y,t-s)\right) - f_\gamma\left(v^*(y)\right)\right] \chi_{[0,t]}(s) dy ds \\ + \int_0^\infty \int_\Omega U_\gamma(x,y;s) f_\gamma\left(v^*(y)\right) \chi_{[0,t]}(s) dy ds.$$

For the first term of the above representation, the monotone convergence theorem is applicable. As for the second term of the above, by (31), we can apply the dominated convergence theorem as $t \to +\infty$. So the desired consequence (29) follows because $\lim_{t\to+\infty} U_\gamma(x,y;t) = 0$ exponentially.

Now, we deduce from (29) and (30) that

$$\begin{aligned} v(x) &= \int_\Omega \int_0^\infty U_\gamma(x,y;s) ds \cdot \left[f_\gamma\left(v(y)\right) - f_\gamma\left(v^*(y)\right)\right] dy \\ &\quad + \int_\Omega \int_0^\infty U_\gamma(x,y;s) ds \cdot f_\gamma\left(v^*(y)\right) dy \\ &= \int_\Omega G_\gamma(x,y)\left[f_\gamma\left(v(y)\right) - f_\gamma\left(v^*(y)\right)\right] dy \\ &\quad + \int_\Omega G_\gamma(x,y) f_\gamma\left(v^*(y)\right) dy. \end{aligned} \tag{32}$$

Relations (27) and (32) imply

$$\delta \cdot v \in L^1(\Omega), \qquad \delta \cdot f(v) \in L^1(\Omega), \tag{33}$$

and

$$v(x) = \int_\Omega G_\gamma(x,y) f_\gamma\left(v(y)\right) dy \qquad \text{for} \quad x \in \overline{\Omega}, \tag{34}$$

where $\delta(x) = \operatorname{dist}(x, \partial\Omega)$.

In general, given a measurable function $u_0(x)$ with $v^* \le u_0 \le v$, we can construct the minimal solution for (1) via the monotone iteration (c.f. [14]). That is,

$$u_1(x,t) = \int_\Omega U_\gamma(x,y;t) u_0(y) dy + \int_0^t ds \int_\Omega U_\gamma(x,y;t-s) f_\gamma\left(v^*(y)\right) dy$$

and

$$\begin{aligned} u_{k+1}(x,t) &= \int_\Omega U_\gamma(x,y;t) u_0(y) dy \\ &\quad + \int_0^t ds \int_\Omega U_\gamma(x,y;t-s) f_\gamma\left(u_k(y,s)\right) dy. \end{aligned} \tag{35}$$

We can show inductively that the function

$$u_k \in C\left([0,+\infty), L^1(\Omega, \delta(x)dx)\right)$$

is well-defined, that

$$v^*(x) \leq u_k(x,t) \leq v(x) \qquad \left(x \in \overline{\Omega},\ t \geq 0\right) \tag{36}$$

holds, and that $\{u_k(x,t)\}$ is non-decreasing in k. In particlular we have

$$u^*(x,t) = \lim_{k \to +\infty} u_k(x,t) \in [v^*(x), v(x)]\,. \tag{37}$$

In fact, first note $v^*(x)$ is a classical stationary solution and hence

$$\int_\Omega U_\gamma(x,y;t)v^*(y)dy + \int_0^t ds \int_\Omega U_\gamma(x,y;t-s) f_\gamma\left(v^*(y)\right) dy = v^*(x). \tag{38}$$

We can deduce also that

$$\int_\Omega U_\gamma(x,y;t)v(y)dy + \int_0^t ds \int_\Omega U_\gamma(x,y;t-s) f_\gamma\left(v(y)\right) dy = v(x) \tag{39}$$

from (34) and the standard identity

$$\int_0^t U_\gamma(x,y;t-s)ds = G_\gamma(x,y) - \int_\Omega U_\gamma(x,z;t)G_\gamma(y,z)dz. \tag{40}$$

Now, from monotonicity of f_γ and relations (33), (38), and (39) we have inequality (36) and well-definedness of $u_k(x,t)$ in $C\left([0,+\infty), L^1(\Omega, \delta(x)dx)\right)$ inductively. Monotonicity of $u_k(x,t)$ in k follows similarly by an induction.

Returning to the case that u_0 is an extrapolation of the sub-solution v_* and the solution v^* of (2), we have

$$v^* \ll u_0 \ll u(\cdot, t_0) \leq v \tag{41}$$

for $t_0 > 0$. This allows us to take a constant $\beta \in (0,1)$ such that

$$\begin{aligned} u_0 &\leq \beta u(\cdot,t_0) + (1-\beta)v^* \\ &\leq \beta v + (1-\beta)v^* \equiv \tilde{v}. \end{aligned} \tag{42}$$

Because $v^* \le \tilde{v} \le v$, we can take the minimal solution $\tilde{u}(x,t)$ of (1) with the initial value $\tilde{v}$. Actually, this is defined by

$$\tilde{u}(x,t) = \lim_{k\to\infty} \tilde{u}_k(x,t)$$

with

$$\tilde{u}_1(x,t) = \int_\Omega U_\gamma(x,y;t)\tilde{v}(y)dy + \int_0^t ds \int_\Omega U_\gamma(x,y;t-s) f_\gamma(v^*(y))\, dy$$

and

$$\begin{aligned}\tilde{u}_{k+1}(x,t) &= \int_\Omega U_\gamma(x,y;t)\tilde{v}(y)dy \\ &\quad + \int_0^t ds \int_\Omega U_\gamma(x,y;t-s) f_\gamma(\tilde{u}_k(y,s))\, dy. \end{aligned} \tag{43}$$

Again equalities (38) and (39), and convexity and mononicity of f_γ imply

$$\tilde{u}_k(x,t) \le \beta v(x) + (1-\beta)v^*(x) \qquad (x \in \overline{\Omega},\ t \ge 0) \tag{44}$$

inductively.

We have assumed $T_b = +\infty$ and hence a classical solution $u(x,t)$ global in time exists for the initial value $u_0(x)$ given above. Therefore, $u(x,t)$ coincides with the minimal solution. Namely,

$$u(x,t) = \lim_{k\to\infty} u_k(x,t) \tag{45}$$

holds for the sequence $\{u_k(x,t)\}$ defined by (35).

To prove this, first we deduce

$$u_k(x,t) \le u(x,t)$$

inductively from $u_0 \ge v^*$, monotonicity of f_γ, and

$$u(x,t) = \int_\Omega U_\gamma(x,y;t)u_0(y)dy + \int_0^t ds \int_\Omega U_\gamma(x,y;t-s) f_\gamma(u(y,s))\, dy.$$

Therefore, the right-hand side of (45), denoted by $\hat{u}(x,t)$, satisfies

$$v^*(x) \le \hat{u}(x,t) \le u(x,t). \tag{46}$$

Letting $k \to \infty$ in (35), we obtain

$$\hat{u}(x,t) = \int_\Omega U_\gamma(x,y;t)u_0(y)dy + \int_0^t ds \int_\Omega U_\gamma(x,y;t-s)f_\gamma(\hat{u}(y,s))\,dy. \quad (47)$$

However, relations (46) and (47) imply that $\hat{u}(x,t)$ is a classical solution of (1) and hence $\hat{u}(x,t) = u(x,t)$. This means (45).

Now, monotonicity of f_γ implies

$$u_k(x,t) \le \tilde{u}_k(x,t) \qquad (x \in \Omega,\ t \ge 0)$$

inductively. Then, letting $k \to \infty$, we have

$$u(x,t) \le \tilde{u}(x,t) \le \beta v(x) + (1-\beta)v^*(x) \qquad (x \in \Omega,\ t \ge 0).$$

Therefore,

$$v(x) \le \beta v(x) + (1-\beta)v^*(x) \qquad (x \in \Omega) \quad (48)$$

by letting $t \to +\infty$.

However, $0 < \beta < 1$ so that (48) contradicts (41) with (27). □

Acknowledgements

We wish to thank professor H. Matano for his kind advices. He pointed out to us that some conditions might be unnecessary in Theorem 1 and could be improved. Especially in the condition (5),

$$\limsup_{s \to -\infty} f(s)/s < \lambda_1$$

is unnecessary.

References

1. K. Akô, On the Dirichlet problem for quasilinear elliptic differential equaitons of the second order, *J. Math. Soc. Japan* **13** (1961), 45-62.

2. H. Amann, Fixed point equations and nonlinear eigenvalue problems in ordered Banach spaces, *SIAM Rev.* **18** (1976), 620-709.

3. H. Bellout, A criterion for blow-up of solutions to a semilinear heat equations, *SIAM J. Math. Anal.* **18** (1987), 722-727.

4. H. Brezis, T. Cazenave, Y. Martel, and A. Ramiandrisoa, Blow up for $u_t - \Delta u = g(u)$ revisited, *Advances in Differential Equations* **1** (1996), 73-90.

5. A. Friedman, and B. McLeod, Blow-up of positive solutions of semilinear heat equations, *Indiana Univ. Math. J.* **34** (1985), 425-447.

6. H. Fujita, On the nonlinear equations $\Delta u + e^u = 0$ and $\partial v/\partial t = \Delta v + e^v$, *Bull. Amer. Math. Soc.* **75** (1969), 132-135.

7. H. Fujita, On some nonexistence and nonuniqueness theorems for nonlinear parabolic equations, *Proc. Symp. Pure Math.* **18**, Amer. Math. Soc. (1970), 105-113,

8. D. Henry, Geometric Theory of Semilinear Parablic Equations, *Lecture Notes in Math.* **840**, Springer, 1981.

9. A. A. Lacey, Mathematical analysis of thermal runaway for spatially inhomogeneous reactions, *SIAM J. Appl. Math.* **43** (1983), 1350-1366.

10. A. A. Lacey, and D. E. Tzanetis, Global existence and convergence to a singular steady state for a semilinear heat equation, *Proc. Roy. Soc. Edinb.* **105A** (1987), 289-305.

11. O. A. Ladyzenskaya, V. A. Solonnikov, and N. N. Ural'ceva, Linear and Quasilinear Equation of Parabolic Type, *Amer. Math. Soc.*, Providence, 1968.

12. H. Matano, Asymptotic behavior and stability of solutions of semilinear diffusion equations, *Publ. RIMS Kyoto Univ.* **12** (1979), 401-454.

13. K. Nagasaki, and T. Suzuki, Spectral and related properties about the Emden-Fowler equation $-\Delta u = \lambda e^u$ on circular domains, *Math. Ann.* **299** (1994), 1-15.

14. I. Peral, and J. L. Vazquez, On the stability or instability of the singular solution of semilinear heat equation with exponential reaction term, *Arch. Rational Meth. Anal.* **129** (1995), 201-224.